

Memo: Universal Connections

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Abstract

We follow the proof of the fact [1] that any connection is obtained as the pullback of a certain canonical connection on a universal principal bundle.

1 Universal connection on the Stiefel bundle

Unless otherwise specified, all maps are assumed to be of class C^∞ .

We introduce the canonical connection, namely the Berry connection, defined on the Grassmannian. Consider the Stiefel bundle

$$U(k) \rightarrow V(N, k) \rightarrow G(N, k). \quad (1.1)$$

Here

$$G(N, k) = \frac{U(N)}{U(k) \times U(N-k)} = \{H \in \text{Mat}_{N \times N}(\mathbb{C}) | H^2 = 1, H^\dagger = H, H \sim \text{diag}(1_{N-k}, 1_k)\}, \quad (1.2)$$

$$V(N, k) = \frac{U(N)}{U(N-k)} = \{U \in \text{Mat}_{N \times k}(\mathbb{C}) | U^\dagger U = 1_k\} \quad (1.3)$$

are the Grassmannian and the Stiefel manifold, respectively. An element $H \in G(N, k)$ may be regarded as a space of ‘‘Hamiltonians’’ with k occupied states, while $\Phi \in V(N, k)$ may be regarded as a space of k occupied states. The Grassmannian is the space obtained by quotienting out the freedom of changing the frame Φ . As a connection on the Stiefel bundle, namely the Berry connection, introduce

$$\gamma_0 = \Phi^\dagger d\Phi. \quad (1.4)$$

We call this the canonical connection.

First we prove the following lemma.

Lemma 1.1 ([1]). *Let $U \subset \mathbb{R}^n$ be an open set, and let V be a relatively compact open set whose closure is contained in U , $\bar{V} \subset U$. Any $\mathfrak{u}(k)$ -valued 1-form $\alpha \in \Omega^1(U; \mathfrak{u}(k))$ on U can be written on V using a collection of $m' = (2n+1)k^2$ smooth $k \times k$ matrices $\{\phi_i\}_{i=1, \dots, m'}$, $\phi_i : V \rightarrow \text{Mat}_{k \times k}(\mathbb{C})$, as*

$$\alpha = \sum_{i=1}^{m'} \phi_i^\dagger d\phi_i, \quad \sum_{i=1}^{m'} \phi_i^\dagger \phi_i = 1_k \quad \text{on } V. \quad (1.5)$$

(Proof) We follow [1]. Expand

$$-i\alpha = \sum_{s=1}^n \sum_{r=1}^{k^2} \lambda_{sr}(x) f_r dx^s. \quad (1.6)$$

Here f_1, \dots, f_{k^2} are a suitable basis of the space of $k \times k$ Hermitian matrices, normalized so that $f_r > 0$ and $\|f_r\| = 1$, where the largest eigenvalue of f_r is 1. The norm is the matrix norm

$$\|X\| = \sup_{v \in \mathbb{C}^n, \|v\|=1} \|Xv\|. \quad (1.7)$$

Set

$$a_{sr} = \sup_{x \in \bar{V}} |\lambda_{sr}(x)| < \infty \quad (1.8)$$

where \bar{V} is a compact closed set, and define

$$\mu_{sr}(x) := \frac{1}{2}(\lambda_{sr}(x) + a_{sr} + \epsilon), \quad (1.9)$$

$$\nu_{sr}(x) := \frac{1}{2}(-\lambda_{sr}(x) + a_{sr} + \epsilon). \quad (1.10)$$

Here $\epsilon > 0$ is arbitrary. Since μ_{sr} and ν_{sr} are positive C^∞ functions, their square roots are well-defined and of class C^∞ . Choose a constant $C > 0$ such that

$$\sum_{s=1}^n (\mu_{sr} + \nu_{sr}) = \sum_{s=1}^n a_{sr} + n\epsilon < \frac{C}{k^2}. \quad (1.11)$$

The smallest eigenvalue of the Hermitian matrix

$$\frac{C}{k^2} 1_k - \sum_{s=1}^n (\mu_{sr} + \nu_{sr}) f_r \quad (1.12)$$

is positive,

$$\frac{C}{k^2} - \sum_{s=1}^n (\mu_{sr} + \nu_{sr}) > 0, \quad (1.13)$$

so this matrix is positive definite and its square root can be taken. Set

$$h_r(x) = \sqrt{\frac{1}{k^2} 1_k - \frac{1}{C} \sum_{s=1}^n (\mu_{sr}(x) + \nu_{sr}(x)) f_r}. \quad (1.14)$$

Choose $\{\phi_j\}_{j=1, \dots, nk^2}$, with some chosen ordering, as

$$C^{-1/2} \sqrt{\mu_{sr}} e^{iCx_s} \sqrt{f_r}, \quad r = 1, \dots, k^2, \quad s = 1, \dots, n. \quad (1.15)$$

Similarly, choose $\{\phi_j\}_{j=nk^2+1, \dots, 2nk^2}$ as

$$C^{-1/2} \sqrt{\nu_{sr}} e^{-iCx_s} \sqrt{f_r}, \quad r = 1, \dots, k^2, \quad s = 1, \dots, n. \quad (1.16)$$

Choose the remaining $\phi_{2nk^2+1}, \dots, \phi_{2nk^2+k^2}$ as

$$h_r, \quad r = 1, \dots, k^2. \quad (1.17)$$

Then a direct computation gives

$$\sum_{j=1}^{2nk^2+k^2} \phi_j^\dagger \phi_j = \sum_r \left(\sum_s \frac{1}{C} (\mu_{sr} + \nu_{sr}) f_r + h_r^2 \right) = 1_k, \quad (1.18)$$

$$\sum_{j=1}^{2nk^2+k^2} \phi_j^\dagger d\phi_j = \sum_{sr} \frac{1}{C} \{ \sqrt{\mu_{sr}} e^{-iCx_s} d(\sqrt{\mu_{sr}} e^{iCx_s}) + \sqrt{\nu_{sr}} e^{iCx_s} d(\sqrt{\nu_{sr}} e^{-iCx_s}) \} f_r + \sum_r h_r dh_r \quad (1.19)$$

$$= \sum_{sr} \frac{1}{C} \{ (\mu_{sr} - \nu_{sr}) iC dx_s + \frac{1}{2} (d\mu_{sr} + d\nu_{sr}) \} f_r + \sum_r h_r dh_r \quad (1.20)$$

$$= \sum_{sr} \frac{1}{C} \{ \lambda_{sr} iC dx_s + \frac{1}{2} (d\mu_{sr} + d\nu_{sr}) \} f_r + \sum_r h_r dh_r \quad (1.21)$$

$$= \alpha + \sum_r \sum_s \frac{1}{2C} d(\mu_{sr} + \nu_{sr}) f_r + \sum_r h_r dh_r. \quad (1.22)$$

Since $h_r(x)$ is a function of f_r and its matrix dependence is only through f_r , we have $[h_r(x), h_r(y)] = 0$ for all x, y . Therefore

$$h_r dh_r = dh_r h_r. \quad (1.23)$$

Thus $h_r dh_r = \frac{1}{2} dh_r^2$, and hence

$$\sum_{j=1}^{2nk^2+k^2} \phi_j^\dagger d\phi_j = \alpha + \sum_r \frac{1}{2} d \left\{ \frac{1}{C} \sum_s (\mu_{sr} + \nu_{sr}) f_r + h_r^2 \right\} \quad (1.24)$$

$$= \alpha + \sum_r \frac{1}{2} d \left(\frac{1}{k^2} 1_k \right) = \alpha. \quad \square \quad (1.25)$$

Using this lemma, we show that locally any connection is obtained as the pullback of the canonical connection γ_0 on the Stiefel bundle. Regard the matrices ϕ_j as a single frame by stacking them vertically.

Proposition 1.2. *Let $P \rightarrow X$ be a principal $U(k)$ bundle, and let γ be a connection on P . Assume $\dim X \leq n$. For any relatively compact open subset $W \subset U$ whose closure \bar{W} is contained in some coordinate neighborhood U , there exists a C^∞ bundle map*

$$\Phi : p^{-1}(W) \rightarrow V(m'', k), \quad m'' = (2n+1)k^3, \quad (1.26)$$

where $p : P \rightarrow X$ is the projection, such that $\gamma|_{p^{-1}(W)}$ is obtained as the pullback of the canonical connection γ_0 on the Stiefel bundle:

$$\gamma|_{p^{-1}(W)} = \Phi^* \gamma_0. \quad (1.27)$$

(Proof) Let $\sigma : U \rightarrow P$ be a section, and set $\alpha = \sigma^* \gamma$. By the lemma, on W there exist C^∞ $k \times k$ matrix-valued functions $\phi_1, \dots, \phi_{m'}$ satisfying

$$\sum_{j=1}^{m'} \phi_j^\dagger \phi_j = 1_k, \quad \sum_{j=1}^{m'} \phi_j^\dagger d\phi_j = \alpha, \quad m' = (2n+1)k^2. \quad (1.28)$$

Define the map Φ in the proposition by

$$\Phi(\xi) := \begin{pmatrix} \phi_1(p\xi) \\ \vdots \\ \phi_{m'}(p\xi) \end{pmatrix} \cdot s, \quad s \in U(k), \quad \xi = \sigma(p\xi)s. \quad (1.29)$$

Here s is defined by $\xi = \sigma(p\xi)s$. Then

$$\Phi(\xi)^\dagger \Phi(\xi) = s^\dagger \sum_{j=1}^{m'} \phi_j(p\xi)^\dagger \phi_j(p\xi) s = 1_k, \quad (1.30)$$

so $\Phi(\xi) \in V(m'', k)$. Since $\xi s' = \sigma(p\xi)ss'$, we have

$$\Phi(\xi \cdot s') = \Phi(\xi)s', \quad (1.31)$$

which is compatible with the $U(k)$ action, and hence Φ is a bundle map.¹

By construction, the pullback by Φ of the canonical connection on $V(m'', k)$ agrees with $\Phi^\dagger d\Phi$:

$$\Phi^\dagger d\Phi = \Phi^*(\gamma_0). \quad (1.34)$$

Furthermore, pulling this back by σ gives α :

$$\alpha = \sigma^* \Phi^* \gamma_0. \quad (1.35)$$

If γ and $\Phi^\dagger d\Phi$ are regarded as connections on $P|_{p^{-1}(W)}$ whose pullbacks by σ agree, $\sigma^* \gamma = \alpha$, then a connection 1-form is uniquely determined by a local section. Therefore $\gamma = \Phi^\dagger d\Phi$ on $P|_{p^{-1}(W)}$. \square

It remains to extend this to all of X by a partition of unity.

Theorem 1.3 (Narasimhan–Ramanan). *Let $P \rightarrow X$ be a principal $U(k)$ bundle, and let γ be a connection on P . Assume $\dim X \leq n$. There exists a C^∞ bundle map*

$$\Phi : P \rightarrow V(m, k), \quad m = (n+1)(2n+1)k^3, \quad (1.36)$$

such that

$$\gamma = \Phi^* \gamma_0. \quad (1.37)$$

(Proof) One can choose a cover $\{V_i\}_i$ of the manifold X such that (i) each \bar{V}_i is contained in some coordinate neighborhood U , and (ii) $\{V_i\}_i$ is divided into $n+1$ classes $S_a = \{V_i\}_{i \in I_a}$, $a = 1, \dots, n+1$, such that any two V_i in the same class do not intersect, $V_i \cap V_j = \emptyset$ for $i, j \in I_a$ [2]. Take an open cover $\{W_i\}_i$ satisfying $\bar{W}_i \subset V_i$ by the shrinking lemma. Let $D_a = \cup_{i \in I_a} p^{-1}(W_i)$. By the preceding proposition, there exists a C^∞ bundle map

$$\Phi_i : p^{-1}(V_i) \rightarrow V((2n+1)k^3, k) \quad (1.38)$$

such that

$$\gamma|_{p^{-1}(V_i)} = \Phi_i^* \gamma_0. \quad (1.39)$$

Let Ψ_a be the C^∞ bundle map obtained by extending these maps to D_a ; namely,

$$\Psi_a|_{p^{-1}(W_i)} = \Phi_i, \quad i \in I_a. \quad (1.40)$$

Let ζ_a be a C^∞ partition of unity subordinate to $\{D_a\}_a$ and invariant under the $U(k)$ action. Thus the support of ζ_a is contained in D_a , $\zeta_a(\xi s) = \zeta_a(\xi)$, and $\sum_{a=1}^{n+1} \zeta_a^2 = 1$. Introduce the map

$$\Phi : P \rightarrow V((n+1)(2n+1)k^3, k), \quad \Phi(\xi) = \begin{pmatrix} \zeta_1(\xi)\Psi_1(\xi) \\ \vdots \\ \zeta_{n+1}(\xi)\Psi_{n+1}(\xi) \end{pmatrix}. \quad (1.41)$$

Since

$$\Phi(\xi)^\dagger \Phi(\xi) = \sum_{a=1}^{n+1} \zeta_a(\xi)^2 \Psi_a(\xi)^\dagger \Psi_a(\xi) = \sum_{a=1}^{n+1} \zeta_a(\xi)^2 \mathbf{1}_k = \mathbf{1}_k, \quad (1.42)$$

¹The map between base spaces

$$f : W \rightarrow G(m'', k) \quad (1.32)$$

is defined so that the commutativity

$$\Phi \circ p_{\text{uni}} = p \circ f \quad (1.33)$$

holds, where $p_{\text{uni}} : V(m'', k) \rightarrow G(m'', k)$ is the projection.

this is indeed a map to the Stiefel bundle. Moreover, since $\Psi_a(\xi s) = \Psi_a(\xi)s$ and $\zeta_a(\xi s) = \zeta_a(\xi)$, we have $\Phi(\xi s) = \Phi(\xi)s$, so Φ is a bundle map. Finally,

$$\Phi^\dagger d\Phi = \sum_{a=1}^{n+1} \zeta_a \Psi_a^\dagger d(\zeta_a \Psi_a) \quad (1.43)$$

$$= \sum_{a=1}^{n+1} \zeta_a^2 \Psi_a^\dagger d\Psi_a + \sum_{a=1}^{n+1} \zeta_a d\zeta_a \Psi_a^\dagger \Psi_a \quad (1.44)$$

$$= \sum_{a=1}^{n+1} \zeta_a^2 \alpha + \sum_{a=1}^{n+1} \zeta_a d\zeta_a 1_k \quad (1.45)$$

$$= \alpha + \sum_{a=1}^{n+1} \frac{1}{2} d\zeta_a^2 1_k \quad (1.46)$$

$$= \alpha + \frac{1}{2} d\left(\sum_{a=1}^{n+1} \zeta_a^2\right) 1_k \quad (1.47)$$

$$= \alpha. \quad \square \quad (1.48)$$

2 Universal connections on principal G bundles

Theorem 2.1. *Let G be a compact Lie group, and let n be a natural number. There exist a principal G bundle B and a connection γ_1 on B with the following property: for any principal G bundle P whose base space has dimension at most n , and for any connection γ on P , there exists a bundle map $f : B \rightarrow P$ such that $\gamma = f^*\gamma_1$.*

We omit the proof. The group G can be regarded as a closed subgroup of some $U(k)$. As a remark, G may also be a discrete group, but then its Lie algebra is trivial, so the connection is zero. The group $G = \mathbb{Z}$ is not compact.

References

- [1] M. S. Narasimhan and S. Ramanan, *Existence of Universal Connections*, Vol. 83, No. 3 (Jul., 1961), pp. 563-572.
- [2] Nash, John (1956), "The imbedding problem for Riemannian manifolds", *Annals of Mathematics* 63 (1): 20-63.