The Gawędzki-Reis Construction of the WZ Term

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Abstract

This is a note on the local construction of the WZ action in [1].

1 Preliminaries from the winding number

As preparation, we show several identities related to the three-dimensional winding number. For

$$g : M_3 \to \mathrm{GL}_n(\mathbb{C}), \tag{1}$$

the winding number is given by

$$W_3[g] = \frac{1}{24\pi^2} \int_{M_3} \mathrm{tr}\,[g^{-1}dg]^3 \in \mathbb{Z}. \tag{2}$$

Let

$$H(g) = \frac{1}{12\pi} \mathrm{tr}\,[g^{-1}dg]^3. \tag{3}$$

Then the following identity holds:

$$\boxed{H(uv^{-1}) = H(u) - H(v) + \frac{1}{4\pi} d\mathrm{tr}\,[u^{-1}du v^{-1}dv].} \tag{4}$$

The proof is as follows. Put

$$\begin{aligned} g^{-1}dg &= (uv^{-1})^{-1}(duv^{-1} + udv^{-1}) = vu^{-1}duv^{-1} + vdv^{-1} = v(u^{-1}du + dv^{-1}v)v^{-1} \qquad (5)\\ &= v(u^{-1}du - v^{-1}dv)v^{-1} =: v(X - Y)v^{-1}. \qquad (6) \end{aligned}$$

Note that

$$X^2 = -dX, \quad Y^2 = -dY. \tag{7}$$

Then

$$\begin{aligned} \mathrm{tr}\,[g^{-1}dg]^3 &= \mathrm{tr}\,[X - Y]^3 = \mathrm{tr}\,[X^3 - 3X^2Y + 3XY^2 - Y^3] = \mathrm{tr}\,[X^3] - \mathrm{tr}\,[Y^3] - 3\mathrm{tr}\,[X^2Y - XY^2] \qquad (8)\\ &= \mathrm{tr}\,[X^3] - \mathrm{tr}\,[Y^3] + 3\mathrm{tr}\,[dXY - XdY] = \mathrm{tr}\,[X^3] - \mathrm{tr}\,[Y^3] + 3d\mathrm{tr}\,[XY], \qquad (9) \end{aligned}$$

which proves the claim. □

Using this and $vdv^{-1} = -dvv^{-1}$, one obtains

$$H(uv) = H(u) + H(v) - \frac{1}{4\pi} d\mathrm{tr}\,[u^{-1}dudvv^{-1}]. \tag{10}$$

Then

$$\begin{aligned}
H(hgh^{-1}) &= H(hg) - H(h) + \frac{1}{4\pi}\mathrm{dtr}\,[(hg)^{-1}d(hg)h^{-1}dh] && (11)\\
&= H(g) - \frac{1}{4\pi}\mathrm{dtr}\,[h^{-1}dhdgg^{-1}] + \frac{1}{4\pi}\mathrm{dtr}\,[(hg)^{-1}d(hg)h^{-1}dh] && (12)\\
&= H(g) + \frac{1}{4\pi}\mathrm{dtr}\,[-h^{-1}dhdgg^{-1} + g^{-1}h^{-1}(dhg+hdg)h^{-1}dh] && (13)\\
&= H(g) + \frac{1}{4\pi}\mathrm{dtr}\,[h^{-1}dhgh^{-1}dhg^{-1} + (g^{-1}dg+dgg^{-1})h^{-1}dh]. && (14)
\end{aligned}$$

Thus

$$\boxed{H(hgh^{-1}) = H(g) + \frac{1}{4\pi}\mathrm{dtr}\,[h^{-1}dhgh^{-1}dhg^{-1} + (g^{-1}dg+dgg^{-1})h^{-1}dh]. \qquad (15)}$$

2 The Gawędzki-Reis construction: the case of $SU(n)$

In this section the target space is $SU(n)$:

$$g : M_3 \to SU(n). \qquad (16)$$

One reason for this choice is that below we assume that elements of g can be diagonalized, whereas elements of $GL_n(\mathbb{C})$ can have exceptional loci. An extension to $U(n)$ is probably possible, but for the moment we follow [1] and record the construction in the $SU(n)$ case. The $U(n)$ case is discussed in [2].

Since $H(g)$ is a closed form, locally there exists a two-form B such that $H = dB$. It can be given explicitly as follows.

As preparation, define n diagonal $n \times n$ matrices by

$$\begin{aligned}
\lambda_0 &:= \mathrm{diag}(0,\dots,0), && (17)\\
\lambda_i &:= \mathrm{diag}(\underbrace{\frac{n-i}{n},\dots,\frac{n-i}{n}}_{i \text{ times}},\underbrace{\frac{-i}{n},\dots,\frac{-i}{n}}_{(n-i) \text{ times}}), \quad i = 1,\dots,n-1. && (18)
\end{aligned}$$

Also set

$$\lambda_{ij} := \lambda_j - \lambda_i. \qquad (19)$$

Define an open cover $\{O_i\}_{i=0,\dots,n}$ of $SU(n)$ by

$$O_i := \{g = \gamma e^{2\pi i\tau}\gamma^{-1} | \gamma \in SU(n), \tau = \sum_{j=0}^{n-1}\tau_j\lambda_j \text{ with } 0 \le \tau_j, \sum_{j=0}^{n-1}\tau_j = 1, \tau_i > 0\}. \qquad (20)$$

From this definition, O_i is contractible. Writing the diagonal entries of τ as τ_{ii}, the expression $g = \gamma e^{2\pi i\tau}\gamma^{-1}$ means that the eigenvalues of g are $\{e^{2\pi i\tau_{11}},\dots,e^{2\pi i\tau_{nn}}\}$. Since

$$\tau_{11} - \tau_{22} = \tau_1, \quad \tau_{22} - \tau_{33} = \tau_2, \quad \dots \quad , \tau_{n-1n-1} - \tau_{nn} = \tau_{n-1}, \quad \tau_{nn} - \tau_{11} = \tau_0 - 1, \qquad (21)$$

if the eigenvalues of $g \in SU(n)$ are written as

$$\{e^{2\pi i\sigma_1},\dots,e^{2\pi i\sigma_n}\}, \quad \sigma_1 \ge \sigma_2 \ge \dots \ge \sigma_n \ge \sigma_1 - 1, \quad \sum_{i=1}^{n}\sigma_i = 0, \qquad (22)$$

then

$$\begin{aligned}
O_i &= \{g \in SU(n) | \sigma_1 \ge \dots \ge \sigma_i > \sigma_{i+1} \ge \dots \ge \sigma_n \ge \sigma_1 - 1\}, \quad i = 1,\dots,n-1, && (23)\\
O_0 &= \{g \in SU(n) | \sigma_1 \ge \dots \ge \sigma_i \ge \sigma_{i+1} \ge \dots \ge \sigma_n > \sigma_1 - 1\}. && (24)
\end{aligned}$$

In other words, O_i is defined by the existence of a finite gap between the i-th and $(i+1)$-st eigenvalues. If there is a gap between some pair of eigenvalues σ_i, σ_{i+1}, then g belongs to O_i. Even when all eigenvalues are degenerate, for example for $e^{2\pi i/n}1_n \in SU(n)$, the element belongs to O_0. Therefore

$$SU(n) = \cup_{i=0}^{n-1} O_i \tag{25}$$

holds. For example, for $SU(2)$,

$$O_0 = \{g \in SU(2) | g \neq -1_2\}, \quad O_1 = \{g \in SU(2) | g \neq 1_2\}. \tag{26}$$

Consider the case of $SU(3)$. The three eigenvalues $e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}$ satisfy $e^{i(\theta_1+\theta_2+\theta_3)} = 1$. If all three eigenvalues are distinct, the element lies in one of the covers O_0, O_1, O_2. If two of the three eigenvalues, say $e^{i\theta}, e^{i\theta}$, are equal, then the element fails to belong to at least one of the covers O_0, O_1, O_2. More explicitly,

$$O_1 = \{g \in SU(3) | g \not\sim \mathrm{diag}(e^{i\theta}, e^{i\theta}, e^{2i\theta}), 0 < \theta < \frac{2\pi}{3}\}, \tag{27}$$

$$O_0 = \{g \in SU(3) | g \not\sim \mathrm{diag}(e^{i\theta}, e^{i\theta}, e^{2i\theta}), \frac{2\pi}{3} < \theta < \frac{4\pi}{3}\}, \tag{28}$$

$$O_2 = \{g \in SU(3) | g \not\sim \mathrm{diag}(e^{i\theta}, e^{i\theta}, e^{2i\theta}), \frac{4\pi}{3} < \theta < 2\pi\}. \tag{29}$$

For instance, if all eigenvalues lie near 1, then the element is contained in O_0.

- Can γ be defined globally on $SU(n)$? In other words, may one assume a canonical γ? If a global γ exists, then only the choice of τ can be regarded as a gauge freedom.
- As an example, consider $SU(2)$. Writing an element as $g = e^{i\alpha \boldsymbol{n}\cdot\boldsymbol{\sigma}}$, and taking $\boldsymbol{n}\cdot\boldsymbol{\sigma}\gamma(\boldsymbol{n}) = \gamma(\boldsymbol{n})\sigma_z$, we have $g = \gamma(\boldsymbol{n}) e^{i\alpha\sigma_z/2}\gamma(\boldsymbol{n})^\dagger$. Since $\gamma(\boldsymbol{n})$ cannot be chosen globally on $\boldsymbol{n} \in S^2$ (presumably), γ cannot be defined globally on all of $SU(2)$.
- If so, then in addition to τ, γ also depends on the patch O_i. On an overlap O_{ij}, one then needs a gauge transformation of γ, so the justification for using $B_{ij} = i\mathrm{tr}\,[\lambda_{ij}\gamma^{-1}d\gamma]$ must be examined.
- For $U(n)$ one cannot impose $\sum_i \sigma_i = 0$, so an extension is needed. In the $SU(n)$ case there is no one-dimensional winding number, and this is why the WZ term can be defined by considering an extension to three dimensions.

Let $g = \gamma\Lambda\gamma^{-1}$ be a diagonalization of g. Here Λ is diagonal. From (15),

$$H(\gamma\Lambda\gamma^{-1}) = d\left[\frac{1}{4\pi}\mathrm{tr}\,[\gamma^{-1}d\gamma\Lambda\gamma^{-1}d\gamma\Lambda^{-1} + (\Lambda^{-1}d\Lambda + d\Lambda\Lambda^{-1})\gamma^{-1}d\gamma]\right] \tag{30}$$

$$= d\left[\frac{1}{4\pi}\mathrm{tr}\,[\gamma^{-1}d\gamma\Lambda\gamma^{-1}d\gamma\Lambda^{-1} + 2d\log\Lambda\gamma^{-1}d\gamma]\right]. \tag{31}$$

Integrating the second term by parts gives

$$\mathrm{tr}\,[d\log\Lambda\gamma^{-1}d\gamma] \to -\mathrm{tr}\,[\log\Lambda d(\gamma^{-1}d\gamma)] = \mathrm{tr}\,[\log\Lambda(\gamma^{-1}d\gamma)^2], \tag{32}$$

which becomes well-defined once a branch of log is specified. The choice of branch corresponds to the cover O_i. Writing the branch of log on O_i as $\log_i$, we have

$$\log_i e^{2\pi i\tau} = 2\pi i(\tau - \lambda_i). \tag{33}$$

On O_i, the exact form is given by

$$H = dB_i, \tag{34}$$

$$B_i = Q + R_i, \tag{35}$$

$$Q = \frac{1}{4\pi}\mathrm{tr}\,[\gamma^{-1}d\gamma\Lambda\gamma^{-1}d\gamma\Lambda^{-1}], \tag{36}$$

$$R_i = i\mathrm{tr}\,[(\tau - \lambda_i)(\gamma^{-1}d\gamma)^2]. \tag{37}$$

As commented above, γ may not be chosen globally, and there is an ambiguity under the "gauge transformation"

$$\gamma \mapsto \gamma W, \quad W\Lambda W^{-1} = \Lambda. \tag{38}$$

We show the following. Let

$$\Lambda = \bigoplus_I e^{i\theta_I} 1_{|I|}, \quad \sum_I |I| = n, \tag{39}$$

where $|I|$ is the degeneracy. Let a basis diagonalizing $g \in SU(n)$ be

$$g u_I = u_I e^{i\theta_I}, \quad u_I = (u_{I,1}, \dots, u_{I,|I|}), \tag{40}$$

and take

$$\tau = \frac{1}{2\pi} \bigoplus_I \theta_I 1_{|I|}, \quad |\theta_I| < \pi. \tag{41}$$

The gauge transformation is

$$u_I \mapsto u_I W_I, \quad W_I \in U(|I|). \tag{42}$$

Then

$$B_0 = \frac{1}{4\pi} \mathrm{tr}\,[\gamma^{-1} d\gamma \Lambda \gamma^{-1} d\gamma \Lambda^{-1}] + i \mathrm{tr}\,[\tau \gamma^{-1} d\gamma \gamma^{-1} d\gamma] \tag{43}$$

is gauge invariant.

Proof. Write

$$A_{IJ} = u_I^\dagger du_J. \tag{44}$$

The gauge transformation acts as

$$A_{IJ} \mapsto W_I^\dagger A_{IJ} W_J + \delta_{IJ} W_I^\dagger dW_I. \tag{45}$$

We can write

$$B_0 = \sum_{I,J} \left[\frac{1}{4\pi} \mathrm{tr}\,_I [A_{IJ} e^{i\theta_J} A_{JI} e^{-i\theta_I}] + \frac{i}{2\pi} \theta_I \mathrm{tr}\,_I [A_{IJ} A_{JI}] \right] \tag{46}$$

$$= \sum_{I,J;I\neq J} \left[\frac{1}{4\pi} \mathrm{tr}\,_I [A_{IJ} e^{i\theta_J} A_{JI} e^{-i\theta_I}] + \frac{i}{2\pi} \theta_I \mathrm{tr}\,_I [A_{IJ} A_{JI}] \right]. \tag{47}$$

Here $\mathrm{tr}\,_I$ runs over the indices $1, \dots, |I|$. The contribution with $I = J$ vanishes because $\mathrm{tr}\,_I [A_{II} A_{II}] = 0$. For $I \neq J$, $\mathrm{tr}\,[A_{IJ} A_{JI}]$ is gauge invariant, proving the claim. □

The same argument applies to B_i. Thus B_i does not depend on the gauge choice of γ.

Rewriting slightly, since

$$\mathrm{tr}\,_I [A_{IJ} A_{JI}] = \mathrm{tr}\,_I u_I^\dagger du_J u_J^\dagger du_I = -\mathrm{tr}\,_I du_I^\dagger u_J u_J^\dagger du_I, \tag{48}$$

one may also write

$$B_0 = -\frac{1}{4\pi} \sum_I e^{-i\theta_I} \mathrm{tr}\,_I \left[du_I^\dagger \sum_{J,J\neq I} e^{i\theta_J} P_J du_I \right] - i \sum_I \frac{\theta_I}{2\pi} \mathrm{tr}\,_I \left[du_I^\dagger \sum_{J,J\neq I} P_J du_I \right]. \tag{49}$$

Introducing the Berry curvature

$$F_I = du_I^\dagger (1 - P_I) du_I, \tag{50}$$

the second term can be written as

$$B_0 = -\frac{1}{4\pi}\sum_I e^{-i\theta_I}\mathrm{tr}_I\left[du_I^\dagger \sum_{J,J\neq I} e^{i\theta_J}P_J du_I\right] - i\sum_I \frac{\theta_I}{2\pi}\mathrm{tr}_I F_I. \tag{51}$$

With the above preparation, we derive the local expression of the WZ term. For

$$g : M_2 \to SU(n), \tag{52}$$

choose a bounding manifold X with $\partial X = M_2$, and choose an extension $\tilde{g} : X \to SU(n)$. The WZ term is defined by

$$WZ[g] := \frac{1}{12\pi}\int_X \mathrm{tr}\,[\tilde{g}^{-1}d\tilde{g}]^3 \in \mathbb{R}/2\pi\mathbb{Z}. \tag{53}$$

Choose a cover of X in the sense of the O_i, and, by abuse of notation, denote it again by $\{O_i\}_i$. Then, again by abuse of notation,

$$WZ[g] = \sum_i \int_{O_i} dB_i = \sum_i \int_{M_2\cap O_i} B_i + \sum_{i<j}\int_{O_i\cap O_j}(B_i - B_j). \tag{54}$$

Here

$$B_i - B_j = i\mathrm{tr}\,[(\lambda_j - \lambda_i)(\gamma^{-1}d\gamma)^2] = d\alpha_{i,j}, \tag{55}$$

$$\alpha_{i,j} = i\mathrm{tr}\,[(\lambda_i - \lambda_j)\gamma^{-1}d\gamma]. \tag{56}$$

Since this is an exact form, all integrals can be written on M_2, and we obtain

$$WZ[g] = \sum_i \int_{M_2\cap O_i} B_i + \sum_{i<j}\int_{M_2\cap O_i\cap O_j}\alpha_{i,j}. \tag{57}$$

References

[1] Krzysztof Gawędzki, Nuno Reis, "WZW branes and gerbes", arXiv:hep-th/0205233.

[2] Krzysztof Gawędzki, "Bundle gerbes for topological insulators", arXiv:1512.01028.