Relation Between the Wess–Zumino Term and the Chern–Simons Term

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Abstract

By constructing a Hamiltonian on the suspension ΣM, we see that the Wess–Zumino term and the Chern–Simons term are mapped to each other without changing their values.

1 Definitions

1.1 Wess–Zumino Term

Let M be an oriented closed even-dimensional manifold of dimension $2n$. For a smooth map

$$g : M_{2n} \to G, \quad G = U(n) \text{ or } SO(n), \tag{1.1}$$

consider an extension to a $(2n+1)$-dimensional manifold:

$$\tilde{g} : X \to G, \quad \partial X = M, \quad \tilde{g}|_M = g. \tag{1.2}$$

We assume that such an extension exists. Define the Wess–Zumino term by

$$\mathrm{WZ}_{2n}[g] = 2\pi \frac{n!}{(2n+1)!(2\pi i)^{n+1}} \int_X \mathrm{tr}\,[(\tilde{g}d\tilde{g}^{-1})^{2n+1}] \tag{1.3}$$

$$= 2\pi \frac{-n!}{(2n+1)!(2\pi i)^{n+1}} \int_X \mathrm{tr}\,[(\tilde{g}^{-1}d\tilde{g})^{2n+1}]. \tag{1.4}$$

The coefficient is chosen so that, for $G = U(n)$,

$$\mathrm{WZ}[g] \in \mathbb{R}/2\pi\mathbb{Z}. \tag{1.5}$$

For example,

$$\mathrm{WZ}_2[g] = \frac{1}{12\pi} \int_X \mathrm{tr}\,[(\tilde{g}^{-1}d\tilde{g})^3]. \tag{1.6}$$

1.2 Chern–Simons Term

Let M be an oriented closed odd-dimensional manifold of dimension $2n-1$. For a smooth map

$$P : M_{2n-1} \to B, \quad B = U(n+m)/U(n) \times U(m) \text{ or } O(n+m)/O(n) \times O(m), \quad P^2 = P \text{ (projection)}, \tag{1.7}$$

consider an extension to a $2n$-dimensional manifold:

$$\tilde{P} : X \to B, \quad \partial X = M, \quad \tilde{P}|_M = P, \quad \tilde{P}^2 = \tilde{P}. \tag{1.8}$$

We assume that such an extension exists. Define the Chern–Simons term by

$$\mathrm{CS}_{2n-1}[P] = 2\pi \frac{1}{n!} \left(\frac{i}{2\pi} \right)^n \int_X \mathrm{tr}\,[\tilde{P}(d\tilde{P})^{2n}]. \tag{1.9}$$

The coefficient is chosen so that, for $B = U(n+m)/U(n) \times U(m)$,

$$\mathrm{CS}_{2n-1}[P] \in \mathbb{R}/2\pi\mathbb{Z}. \tag{1.10}$$

For example,

$$\mathrm{CS}_3[P] = -\frac{1}{4\pi} \int_X \mathrm{tr}\,[\tilde{P}(d\tilde{P})^4]. \tag{1.11}$$

2 From the Wess–Zumino Term to the Chern–Simons Term

Given a smooth map $g : M_{2n} \to G$ and an extension $\tilde{g} : X \to G$, define a flattened Hamiltonian on the suspension ΣX by

$$\tilde{H} = \cos\theta\, \sigma_z + \sin\theta\, \tilde{\Gamma}, \quad \tilde{\Gamma} = \begin{pmatrix} & \tilde{g} \\ \tilde{g}^\dagger & \end{pmatrix}, \quad \theta \in [0, \pi]. \tag{2.1}$$

It can be written as

$$\tilde{H} = \sigma_z(\cos\theta + \sin\theta\, \sigma_z \tilde{\Gamma}) = \sigma_z e^{\theta \sigma_z \tilde{\Gamma}} = e^{-\theta \sigma_z \tilde{\Gamma}} \sigma_z. \tag{2.2}$$

The projection onto the eigenvalue -1 is

$$\tilde{P} = \frac{1 - \tilde{H}}{2}. \tag{2.3}$$

The Chern–Simons term on ΣM_{2n} is

$$\mathrm{CS}_{2n+1}[P] = 2\pi \frac{1}{(n+1)!} \left(\frac{i}{2\pi} \right)^{n+1} \int_{\Sigma X} \mathrm{tr}\,[\tilde{P}(d\tilde{P})^{2n+2}] \tag{2.4}$$

$$= 2\pi \frac{1}{(n+1)!} \left(\frac{i}{2\pi} \right)^{n+1} \int_{\Sigma X} \frac{1}{2^{2n+3}} \mathrm{tr}\,[(1 - \tilde{H})(d\tilde{H})^{2n+2}]. \tag{2.5}$$

Now write

$$d\tilde{H} = \underbrace{(-\sin\theta\, \sigma_z d\theta + \cos\theta\, \tilde{\Gamma} d\theta)}_{X} + \underbrace{\sin\theta\, d\tilde{\Gamma}}_{Y} =: X + Y. \tag{2.6}$$

Using $\tilde{\Gamma}^2 = 1$, note that

$$XY = YX. \tag{2.7}$$

Then

$$\mathrm{tr}\,[(1 - \tilde{H})(d\tilde{H})^{2n+2}] = \mathrm{tr}\,[(1 - \tilde{H})(X+Y)^{2n+2}] = (2n+2)\mathrm{tr}\,[(1 - \tilde{H})XY^{2n+1}] \tag{2.8}$$

$$= (2n+2)\mathrm{tr} \left[\begin{pmatrix} 1 - \cos\theta & -\sin\theta\, \tilde{g} \\ -\sin\theta\, \tilde{g}^\dagger & 1 + \cos\theta \end{pmatrix} \begin{pmatrix} -\sin\theta & \cos\theta\, \tilde{g} \\ \cos\theta\, \tilde{g}^\dagger & \sin\theta \end{pmatrix} d\theta (\sin\theta)^{2n+1} \begin{pmatrix} & d\tilde{g} \\ d\tilde{g}^\dagger & \end{pmatrix}^{2n+1} \right] \tag{2.9}$$

$$= (2n+2)\mathrm{tr} \left[\begin{pmatrix} * & (\cos\theta - 1)\tilde{g} \\ (\cos\theta + 1)\tilde{g}^\dagger & * \end{pmatrix} d\theta (\sin\theta)^{2n+1} \begin{pmatrix} & (d\tilde{g} d\tilde{g}^\dagger)^n d\tilde{g} \\ (d\tilde{g}^\dagger d\tilde{g})^n d\tilde{g}^\dagger & \end{pmatrix} \right] \tag{2.10}$$

$$= (2n+2)(\sin\theta)^{2n+1} d\theta\, \mathrm{tr}\,[(\cos\theta - 1)\tilde{g}(d\tilde{g}^\dagger d\tilde{g})^n d\tilde{g}^\dagger + (\cos\theta + 1)\tilde{g}^\dagger (d\tilde{g} d\tilde{g}^\dagger)^n d\tilde{g}] \tag{2.11}$$

$$= (2n+2)(\sin\theta)^{2n+1} d\theta\, \mathrm{tr}\,[(\cos\theta - 1)(-1)^{n+1}(\tilde{g}^\dagger d\tilde{g})^{2n+1} + (\cos\theta + 1)(-1)^n (\tilde{g}^\dagger d\tilde{g})^{2n+1}] \tag{2.12}$$

$$= (-1)^n 2(2n+2)(\sin\theta)^{2n+1} d\theta\, \mathrm{tr}\,[(\tilde{g}^\dagger d\tilde{g})^{2n+1}]. \tag{2.13}$$

Therefore

$$\mathrm{CS}_{2n+1}[P] = 2\pi \frac{1}{(n+1)!}\left(\frac{i}{2\pi}\right)^{n+1}\int_{\Sigma X}\frac{1}{2^{2n+3}}\mathrm{tr}\,[(1-\tilde{H})(d\tilde{H})^{2n+2}] \tag{2.14}$$

$$= 2\pi \frac{1}{(n+1)!}\left(\frac{i}{2\pi}\right)^{n+1}\int_{\Sigma X}\frac{1}{2^{2n+3}}(-1)^n 2(2n+2)(\sin\theta)^{2n+1}d\theta\,\mathrm{tr}\,[(\tilde{g}^\dagger d\tilde{g})^{2n+1}]. \tag{2.15}$$

Since

$$\int_0^\pi (\sin\theta)^{2n+1}d\theta = \frac{2(2n)!!}{(2n+1)!!} = \frac{2(2^n n!)^2}{(2n+1)!} = \frac{2^{2n+1}(n!)^2}{(2n+1)!}, \tag{2.16}$$

we obtain

$$\mathrm{CS}_{2n+1}[P] = 2\pi \frac{1}{(n+1)!}\left(\frac{i}{2\pi}\right)^{n+1}\int_{X}\frac{1}{2^{2n+3}}(-1)^n 2(2n+2)\frac{2^{2n+1}(n!)^2}{(2n+1)!}\mathrm{tr}\,[(\tilde{g}^\dagger d\tilde{g})^{2n+1}] \tag{2.17}$$

$$= 2\pi\left(\frac{i}{2\pi}\right)^{n+1}\int_X (-1)^n\frac{n!}{(2n+1)!}\mathrm{tr}\,[(\tilde{g}^\dagger d\tilde{g})^{2n+1}] \tag{2.18}$$

$$= 2\pi\left(\frac{i}{2\pi}\right)^{n+1}\int_X (-1)^{n+1}\frac{n!}{(2n+1)!}\mathrm{tr}\,[(\tilde{g} d\tilde{g}^\dagger)^{2n+1}] \tag{2.19}$$

$$= 2\pi\frac{n!}{(2n+1)!(2\pi i)^{n+1}}\int_X \mathrm{tr}\,[(\tilde{g} d\tilde{g}^\dagger)^{2n+1}] \tag{2.20}$$

$$= \mathrm{WZ}_{2n}[g]. \tag{2.21}$$

Thus, under suspension, the Wess–Zumino term is mapped directly to the Chern–Simons term without changing its value.

3 From the Chern–Simons Term to the Wess–Zumino Term

Given a smooth map $P : M_{2n-1} \to B$ and an extension $\tilde{P} : X \to B$, define a unitary matrix on the suspension ΣX by

$$\tilde{g} = e^{i\theta\tilde{H}} = \cos\theta + i\sin\theta\,\tilde{H}, \quad \tilde{H} = 1-2\tilde{P}, \quad \theta\in[0,\pi]. \tag{3.1}$$

The Wess–Zumino term on ΣM_{2n-1} is

$$\mathrm{WZ}_{2n}[g] = 2\pi\frac{n!}{(2n+1)!(2\pi i)^{n+1}}\int_{\Sigma X}\mathrm{tr}\,[(\tilde{g}d\tilde{g}^{-1})^{2n+1}] \tag{3.2}$$

$$= 2\pi\frac{n!}{(2n+1)!(2\pi i)^{n+1}}\int_{\Sigma X} -\mathrm{tr}\left[i^{2n+1}(2n+1)(\sin\theta)^{2n}d\theta\,\tilde{H}(d\tilde{H})^{2n} + i^{2n+1}(\sin\theta)^{2n+1}(d\tilde{H})^{2n+1}e^{i\theta\tilde{H}}\right]. \tag{3.3}$$

Here I used the calculation in App. A. The term $(d\tilde{H})^{2n+1}$ vanishes for dimensional reasons. Using

$$\int_0^\pi (\sin\theta)^{2n}d\theta = \frac{\pi(2n-1)!!}{(2n)!!} = \frac{\pi(2n)!}{2^{2n}n!n!}, \tag{3.4}$$

we find

$$\mathrm{WZ}_{2n}[g] = 2\pi\frac{n!}{(2n+1)!(2\pi i)^{n+1}}\int_{\Sigma X} -\mathrm{tr}\,[i^{2n+1}(2n+1)(\sin\theta)^{2n}d\theta\,\tilde{H}(d\tilde{H})^{2n}] \tag{3.5}$$

$$= 2\pi\frac{n!}{(2n+1)!(2\pi i)^{n+1}}\int_X -i^{2n+1}(2n+1)\frac{\pi(2n)!}{2^{2n}n!n!}\mathrm{tr}\,[\tilde{H}(d\tilde{H})^{2n}] \tag{3.6}$$

$$= 2\pi\frac{n!}{(2n+1)!(2\pi i)^{n+1}}\int_X -i^{2n+1}(2n+1)\frac{\pi(2n)!}{n!n!}\mathrm{tr}\,[(1-2\tilde{P})(d\tilde{P})^{2n}] \tag{3.7}$$

$$= 2\pi\frac{-i^{2n+1}\pi}{n!(2\pi i)^{n+1}}\int_X \mathrm{tr}\,[(1-2\tilde{P})(d\tilde{P})^{2n}]. \tag{3.8}$$

Here

$$\int_X \mathrm{tr}\,[(d\tilde{P})^{2n}] = \int_X d\,\mathrm{tr}\,[\tilde{P}(d\tilde{P})^{2n-1}] = \int_{M_{2n-1}} \mathrm{tr}\,[P(dP)^{2n-1}], \tag{3.9}$$

but

$$\mathrm{tr}\,[P(dP)^{2n-1}] = \mathrm{tr}\,[(dP)^{2n-1}(1-P)] = \mathrm{tr}\,[(1-P)(dP)^{2n-1}], \tag{3.10}$$

and therefore

$$\mathrm{tr}\,[P(dP)^{2n-1}] = \frac{1}{2} d\,\mathrm{tr}\,[P(dP)^{2n-2}]. \tag{3.11}$$

This is a total derivative term and can be dropped. Hence

$$\begin{aligned} \mathrm{WZ}_{2n}[g] &= 2\pi \frac{2i^{2n+1}\pi}{n!(2\pi i)^{n+1}} \int_X \mathrm{tr}\,[\tilde{P}(d\tilde{P})^{2n}] && (3.12)\\ &= 2\pi \frac{i^n}{n!(2\pi)^n} \int_X \mathrm{tr}\,[\tilde{P}(d\tilde{P})^{2n}] && (3.13)\\ &= \mathrm{CS}_{2n-1}[P]. && (3.14) \end{aligned}$$

Thus, under suspension, the Chern–Simons term is also mapped to the Wess–Zumino term without changing its value.

A Calculation of $(g^\dagger dg)^n$

Since this calculation is often useful, I record the computation of

$$(g^{-1}dg)^n \tag{A.1}$$

in the case

$$g = e^{i\theta H} = \cos\theta + i\sin\theta H, \quad H: \text{ Hermitian}, \quad H^2 = 1, \quad \partial_\theta H = 0. \tag{A.2}$$

We have

$$dg = -\sin\theta\, d\theta + i\cos\theta\, d\theta H + i\sin\theta\, dH = e^{i(\theta+\frac{\pi}{2})H} + i\sin\theta\, dH, \tag{A.3}$$

and

$$g^{-1}dg = iHd\theta + i\sin\theta e^{-i\theta H} dH. \tag{A.4}$$

Using

$$dH\, H = -H\, dH, \tag{A.5}$$

this can also be written as

$$g^{-1}dg = iHd\theta + i\sin\theta\, dHe^{i\theta H}. \tag{A.6}$$

Put

$$X = iHd\theta, \quad Y = i\sin\theta\, dHe^{i\theta H}. \tag{A.7}$$

Note that

$$XY = YX. \tag{A.8}$$

Also using $X^{n\geq 2} = 0$, we find

$$\begin{aligned} (g^{-1}dg)^n &= nXY^{n-1} + Y^n && (A.9)\\ &= niHd\theta(i\sin\theta\, dHe^{i\theta H})^{n-1} + (i\sin\theta\, dHe^{i\theta H})^n && (A.10)\\ &= \begin{cases} i^n n(\sin\theta)^{n-1} d\theta\, H(dH)^{n-1} + i^n(\sin\theta)^n (dH)^n e^{i\theta H} & (n \in \text{odd}),\\ i^n n(\sin\theta)^{n-1} d\theta\, H(dH)^{n-1} e^{i\theta H} + i^n(\sin\theta)^n (dH)^n & (n \in \text{even}). \end{cases} && (A.11) \end{aligned}$$