Note: On Wall's Structure Theorem

Ken Shiozaki

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Abstract

I follow the proof of Wall's structure theorem (Theorem 1 in [1]).

1 Preliminaries

For an algebra A, a subspace I is called a left (right) ideal if $aI \subset I$ ($Ia \subset I$) for every $a \in A$. If it is both a left ideal and a right ideal, it is called a two-sided ideal. An algebra A is simple if A has no nontrivial ideal, that is, no ideal other than A or 0. Let A be a simple algebra.

(Aii) The center $Z(A) = \{z \in A \mid za = az \text{ for all } a \in A\}$ of A is a field.

(*proof.*) We show the existence of inverses. For $z \in Z(A)$, zA is a two-sided ideal of A. On the other hand, since A is simple, $zA = A$. In particular, there exists $a \in A$ such that $za = 1$.

An algebra A is central if its center $Z(A)$ consists of k-multiples of 1, namely $Z(A) = k$. Let A be a central simple algebra.

(Bv) For a simple subalgebra B of A, its centralizer $C = \{a \in A \mid ab = ba \text{ for all } b \in B\}$ is simple. Moreover, the centralizer of C is B, and $A \cong B \otimes_k C$.

2 Some properties of graded algebras

Let A be an algebra over a field k. We say that A is graded if $A = A_0 \oplus A_1$ (direct sum) and $A_i A_j \subset A_{i+j}$ are satisfied.[1] For $a_i \in A_i$, we have $a_i = 1 \cdot a_i = a_i \cdot 1$, and therefore $A_i \subset A_0 A_i$ and $A_i \subset A_i A_0$ also hold. Thus note that $A_0 A_i = A_i A_0 = A_i$. The only possible failure of equality is $A_1^2 \neq A_0$. A subspace B of A is graded if B is the direct sum of $B \cap A_i$.[2]

If I is an ideal of A_0, then $A_0 I \subset I$ and $I A_0 \subset I$. On the other hand, since $1 \in A_0$, we also have $I \subset A_0 I$ and $I \subset I A_0$. Thus note that $A_0 I = I A_0 = I$.

- The center $Z(A)$ is a graded subspace of A.

[1] Recalling the definition of a direct sum, this means that $A = A_0 \cup A_1$ and $A_0 \cap A_1 = \emptyset$. In other words, the decomposition into two vector spaces is unique.

[2] Since B is a subspace of the graded algebra A, the uniqueness of the decomposition follows from $A = A_0 \oplus A_1$. The meaningful condition is that "B is the direct sum of $B \cap A_i$." Consider an example in a graded vector space. Let $V = \mathbb{C} \oplus \mathbb{C}$. An example of a vector subspace of V that is not a graded vector subspace is $W = \mathbb{C}(1,1)$. Indeed, $W \cap \mathbb{C}(1,0) = \mathbb{C}(1,0)$ and $W \cap \mathbb{C}(0,1)$, but $W \neq (W \cap \mathbb{C}(1,0)) \oplus (W \cap \mathbb{C}(0,1)) = V$.

(*proof.*) If $z, z' \in Z(A)$, then for $a \in A$ we have $a(z+z') = az + az' = za + z'a = (z+z')a$. Therefore $z + z' \in Z(A)$, so $Z(A)$ is a subspace. To show that it is graded, it is enough to show $Z(A) = (Z(A) \cap A_0) \oplus (Z(A) \cap A_1)$. That is, when $z = z_0 + z_1$, $z_i \in A_i$, is the unique decomposition, it is enough to show that $z_i \in Z(A)$. Let $x \in A_0$ or $x \in A_1$. Since $z = z_0 + z_1 \in Z(A)$, $x(z_0 + z_1) = (z_0 + z_1)x$. Thus $(xz_0 - z_0x) + (xz_1 - z_1x) = 0$. The first and second terms have different degrees, so both z_0 and z_1 are elements of the center $Z(A)$. //

- A graded algebra A is central if $Z(A) \cap A_0$ consists of k-multiples of 1.[3]
- A graded algebra A is simple if A has no graded two-sided ideal other than 0 and A, that is, no proper graded two-sided ideal.

Here a graded two-sided ideal means a two-sided ideal I of A that is also a graded subspace.

As an example, consider the case $A = \mathbb{C}\sigma_0 \oplus \mathbb{C}\sigma_3$. First, forget the grading and find the ideals.[4] The condition for being a left ideal is as follows. Writing $I = \mathbb{C}(\sum_{i=0,1,2,3} x_i\sigma_i)$, the condition $AI \subset I$ gives, for $a\sigma_0 + b\sigma_3 \in A$,

$$(a\sigma_0 + b\sigma_3)(x_0\sigma_0 + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3) = (ax_0 + bx_3)\sigma_0 + (ax_1 - ibx_2)\sigma_1 + (ax_2 + ibx_1)\sigma_2 + (ax_3 + bx_0)\sigma_3.$$

Hence, for arbitrary $a, b \in \mathbb{C}$, we must have $ax_1 - ibx_2 = ax_2 + ibx_1 = 0$. Also, under this condition, $(ax_0 + bx_3)x_3 = (ax_3 + bx_0)x_0$, that is, the condition is $x_3^2 = x_0^2$. Thus the left ideals are $I_\pm = \mathbb{C}(\sigma_0 \pm \sigma_3)$. Since $I_\pm$ are also right ideals, in the end these $I_\pm$ are the only two-sided ideals. On the other hand, $I_\pm \cap \mathbb{C}\sigma_0 = \mathbb{C}\sigma_0$ and $I_\pm \cap \mathbb{C}\sigma_3 = \mathbb{C}\sigma_3$, so $I_\pm$ are not graded two-sided ideals. Therefore A is simple as a graded algebra. The center of A is A itself, so A is not central when the grading is forgotten. However, since $A \cap A_0 = A_0 = \mathbb{C}\mathbf{1}$, A is central as a graded algebra.

In the following, let A be a graded simple algebra. Also, when we simply write "ideal," it will mean a two-sided ideal.

Lemma 2.1. $A_1^2 = A_0$. *Moreover, if I is a proper ideal of A_0, then $I + A_1IA_1 = A_0$ and $A_1I + IA_1 = A_1$.*

(*proof.*) By definition, $A_1^2 \subset A_0$. The equality $A_1^2 = A_0$ means that every element $x_0 \in A_0$ can be written as $x_0 = x_1x_1'$ with $x_1, x_1' \in A_1$. Suppose $A_1^2 \neq A_0$. Then $A_1^2 + A_1$ is a graded subspace,[5] and $A_1^2 + A_1 \neq A$. We show that $A_1^2 + A_1$ is a two-sided ideal. This follows from

$$A_0(A_1A_1 + A_1) \subset A_1A_1 + A_1, \qquad A_1(A_1A_1 + A_1) \subset A_0A_1 + A_1A_1 \subset A_1 + A_1A_1.$$

The right ideal condition is analogous.

We prove the second half. We have $A_0I \subset I$ and $IA_0 \subset I$. Then

$$A_0(I + A_1IA_1 + A_1I + IA_1) \subset I + A_1IA_1 + A_1I + IA_1$$

and

$$A_1(I + A_1IA_1 + A_1I + IA_1) \subset A_1I + A_0IA_1 + A_0I + A_1IA_1 \subset A_1I + IA_1 + I + A_1IA_1.$$

The right action is similar. Therefore $I + A_1IA_1 + A_1I + IA_1$ is a two-sided ideal. Moreover,

$$I + A_1IA_1 + A_1I + IA_1 = (I + A_1IA_1) \oplus (A_1I + IA_1)$$

is a graded subgroup, so $I + A_1IA_1 + A_1I + IA_1$ is a graded two-sided ideal. By the graded simplicity assumption, $I + A_1IA_1 + A_1I + IA_1 = A$. This gives the claim. //

[3] Note that $k \in A_0$.

[4] If I is a graded left (right) ideal, then I is an ungraded left (right) ideal, so first find the ungraded left (right) ideals.

[5] The sum of vector spaces $V + W$ is defined by $V + W = \{v + w \mid v \in V,\ w \in W\}$.

Lemma 2.2. *If J is a proper ungraded ideal of A, then the projections $\pi_i : J \to A_i$ are isomorphisms.*

(*proof.*) By assumption, $AJ \subset J$ and $JA \subset J$, and J is neither 0 nor A itself.[6] First note the difference between $J \cap A_0$ and $\pi_0(J)$. The former is an intersection of sets. The latter is defined as follows: for $a \in J \subset A$, decompose $a = a_0 + a_1$, $a_i \in A_i$, and set $\pi_i(a) = a_i$.

$J \cap A_0$ is an ideal of A_0. Indeed, for $x_0 \in J \cap A_0$ and $a_0 \in A_0$, since $a_0, x_0 \in A_0$, we have $a_0x_0 \in A_0$. Also, since $x_0 \in J$, the assumption gives $a_0x_0 \in J$. Thus $a_0x_0 \in J \cap A_0$. The right action is similar.

$\pi_0(J)$ is an ideal of A_0. First, for $a_0 \in A_0$ and $x = x_0 + x_1 \in J$, note that $\pi_0(a_0x) = \pi_0(a_0x_0 + a_0x_1) = a_0x_0 = a_0\pi_0(x)$. Thus, for $a_0 \in A_0$, $a_0\pi_0(x) = \pi_0(a_0x) \in \pi_0(J)$, and hence $A_0\pi_0(J) \subset \pi_0(J)$. The right action is similar.

If $J \cap A_0 = \pi_0(J)$, then J is a graded subspace. Let us show this. Since $\pi_0(J) \oplus \pi_1(J)$ is a graded subspace of A, it is enough to show $J \cap A_1 = \pi_1(J)$. For $x_1 \in J \cap A_1$, we have $x_1 = \pi_1(x_1)$, so $J \cap A_1 \subset \pi_1(J)$ holds independently of the assumption $J \cap A_0 = \pi_0(J)$. Conversely, if $x_1 \in \pi_1(J)$, then there exists $x \in J$ with $x = x_0 + x_1$, $x_0 \in A_0$. But $x_0 = \pi_0(x) \in J \cap A_0 \subset J$. Therefore $x_1 = x - x_0 \in J$, so $x_1 \in J \cap A_1$. Thus $\pi_1(J) \subset J \cap A_1$. Hence, if $J \cap A_0 = \pi_0(J)$, then J becomes a nontrivial graded two-sided ideal, contradicting the simplicity of A. Therefore $J \cap A_0 = \pi_0(J)$ is impossible.

Note that $A_1JA_1 \subset J$ and $A_1A_0A_1 \subset A_0$. It follows that $A_1(J \cap A_0)A_1 \subset J \cap A_0$. Also, $A_1\pi_0(J)A_1 \ni a_1\pi_0(x)a_1 = \pi_0(a_1xa_1)$. Since $A_1JA_1 \subset J$, we have $\pi_0(a_1xa_1) \in \pi_0(J)$, and therefore $A_1\pi_0(J)A_1 \subset \pi_0(J)$. Thus both $I = J \cap A_0$ and $I = \pi_0(J)$ satisfy $A_1IA_1 \subset I$. Therefore $I + A_1IA_1 = I$. By the second half of Lemma 2.1, if I is a proper ideal of A_0, then $I + A_1IA_1 = A_0$. Hence neither $J \cap A_0$ nor $\pi_0(J)$ can be a proper ideal of A_0. Thus each is either 0 or A_0. Since $J \cap A_0 = A_0$ and $\pi_0(J) = 0$ is impossible, and since $J \cap A_0 \neq \pi_0(J)$, we get $J \cap A_0 = 0$ and $\pi_0(J) = A_0$. Thus $\pi_0 : J \to A_0$ is surjective. Also, since $\operatorname{Ker} \pi_1 = \{x \in J \mid x = x_0 \in A_0\} = J \cap A_0$, π_1 is injective.

We show $J \cap A_1 = 0$. Since J is an ideal of A, $A_0(J \cap A_1) \subset J \cap A_1$. But noting $1 \cdot A_0$, we also have $J \cap A_1 \subset A_0(J \cap A_1)$. Therefore

$$J \cap A_1 = A_0(J \cap A_1) = A_1^2(J \cap A_1) \subset A_1(J \cap A_0) = 0.$$

Thus π_0 is injective.

We show $\pi_1(J) = A_1$. It suffices to show $A_1 \subset \pi_1(J)$. Since $\pi_0(J) = A_0$, there exists $x \in J$ such that $\pi_0(x) = 1$. Then for $a_1 \in A_1$,

$$a_1 = 1 \cdot a_1 = \pi_0(x)a_1 = \pi_1(xa_1) \in \pi_1(J).$$

Thus π_1 is surjective. //

Lemma 2.3. *Either A is simple as an ungraded algebra, or A_0 is simple and $A_1 = A_0u$ for some $u \in Z(A) \cap A_1$ satisfying $u^2 = 1$.*

(*proof.*) In the following, assume that A is not simple as an ungraded algebra.

There exists a proper ideal J of A, and Lemma 2.2 applies. The maps $\pi_i : J \to A_i$ are isomorphisms. Set $u = \pi_1\pi_0^{-1}(1 \in A_0) \in A_1$.[7] Since $\pi_0^{-1}(1), \pi_1^{-1}(u) \in J$, by linearity of J it contains the element $1 + u$.

[6] Note that J is not a graded ideal. A decomposition such as $J = J_0 \oplus J_1$ cannot be performed.

[7] In the case $A = \mathbb{C}\sigma_1 \oplus \mathbb{C}\sigma_3$, for example,

$$J = \mathbb{C}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and the isomorphisms are given by

$$\pi_0, \pi_1 : \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \mapsto 2x\sigma_0,\ 2x\sigma_3,$$

respectively. In this case, $u = \sigma_3$.

By Lemma 2.2, $J \cong A_0, A_1$, and an element $x \in J$ is determined by either component $x_i \in A_i$ in its decomposition $x = x_0 + x_1$.

Since $AJ \subset J$, we have $u(1+u) = u^2 + u \in J$, and hence $u^2 = 1$.

Let $a \in A$ be arbitrary, and write $a = a_0 + a_1$, $a_i \in A_i$. For $a_0 \in A_0$, $a_0(1+u) = a_0 + a_0 u \in J$ and $(1+u)a_0 = a_0 + ua_0 \in J$, so $ua_0 = a_0 u$. For $a_1 \in A_1$, $a_1(1+u) = a_1 u + a_1 \in J$ and $(1+u)a_1 = ua_1 + a_1 \in J$, so $ua_1 = a_1 u$. Thus $au = ua$, so u is central. Therefore $u \in Z(A) \cap A_1$.

We show $A_1 = A_0 u$. The inclusion $A_0 u \subset A_1$ is obvious. The other inclusion follows from $A_1 = A_1 \cdot 1 = A_1 u^2 = (A_1 u)u \subset A_0 u$.

Finally, we show that A_0 is simple. Let I be an ideal of A_0. Using $A_0 I = I A_0 = I$, we get

$$A_1 I A_1 = A_0 u I A_0 u = A_0 u I u = A_0 I u^2 = I.$$

Therefore $I + A_1 I A_1 = I$. By Lemma 2.1, I is not a proper ideal of A_0. //

In the following, let A be a graded central simple algebra.

Lemma 2.4. *Exactly one of A and A_0 is an ungraded central simple algebra.*

(*proof.*) Suppose A is central, so $Z(A) = k$. By Lemma 2.3, if A were not simple, there would exist $u \in Z(A) \cap A_1$, a contradiction. Thus if A is central, then A is simple.

Suppose A is not central, namely $Z(A) \neq k$.

Since A is graded central, $Z(A) \cap A_0 = k$. Therefore there exists a nonzero subspace $V = Z(A) \cap A_1$. We have $V^2 \subset Z(A) \cap A_0 = k$. If there existed $v \in V \subset Z(A)$ with $v^2 = 0$, then $Z(A)$ would not be a field. By (Aii), A would not be simple, and hence by Lemma 2.3, A_0 would be simple.

The remaining case is that for every $u \in V$, $u^2 = a \neq 0 \in k$. In this case, $A_1 = A_1 a = A_1 u^2 \subset A_0 u$ and $A_0 u \subset A_1$, so $A_1 = A_0 u$. We show that A_0 is central. Since $A_1 = A_0 u$, an arbitrary element $a \in A$ is decomposed as $a = a_0 + a_0' u$, with $a_0, a_0' \in A_0$. Since u is central in A, any element $z \in Z(A_0)$ commutes with every element of A, and hence $z \in k$. Therefore $Z(A_0) = k$. We show that A_0 is simple. If I is a nontrivial ideal of A_0, then $I + Iu$ is a nontrivial graded ideal of A, contradicting that A is graded simple. Therefore A_0 is simple.

Finally, assume for contradiction that both A and A_0 are central simple. Let

$$B = \{a \in A \mid aa_0 = a_0 a \text{ for all } a_0 \in A_0\}$$

be the centralizer of A_0.

References

[1] C. T. C. Wall, *Graded Brauer groups*, J. Reine Angew. Math. 213 (1964), 187–199.