Weyl's Inequality

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Let A be an $n \times n$ Hermitian matrix. We order the eigenvalues of A in decreasing order:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n. \tag{1}$$

For Hermitian matrices A and B, we would like to know inequalities relating the eigenvalues $\{\lambda_i(A+B)\}_i$ to $\{\lambda_i(A)\}_i$ and $\{\lambda_i(B)\}_i$.

For example, what can be said about the largest eigenvalue? Let v_i^X denote a normalized eigenvector of a matrix X with eigenvalue $\lambda = \lambda_i(X)$. Then

$$\begin{aligned}\lambda_1(A+B) &= (v_1^{A+B}, (A+B)v_1^{A+B}) = (v_1^{A+B}, Av_1^{A+B}) + (v_1^{A+B}, Bv_1^{A+B}) \qquad (2)\\ &\leq (v_1^A, Av_1^A) + (v_1^B, Bv_1^B) = \lambda_1(A) + \lambda_1(B). \qquad (3)\end{aligned}$$

More generally, the following holds.

Weyl's inequality.

$$\lambda_{i+j-1}(A+B) \leq \lambda_i(A) + \lambda_j(B) \leq \lambda_{i+j-n}(A+B). \tag{4}$$

(Proof [1]) By the min-max theorem,

$$\begin{aligned}\lambda_{i+j-1}(A+B) &= \max_{V, \dim V = i+j-1} \min_{x \in V, ||x||=1} (x, (A+B)x), \qquad (5)\\ \lambda_i(A) &= \min_{V, \dim V = n-i+1} \max_{x \in V, ||x||=1} (x, Ax), \qquad (6)\\ \lambda_j(B) &= \min_{V, \dim V = n-j+1} \max_{x \in V, ||x||=1} (x, Bx). \qquad (7)\end{aligned}$$

Thus there exist subspaces V_{A+B}, V_A, V_B such that

$$\begin{aligned}\lambda_{i+j-1}(A+B) &= \min_{x \in V_{A+B}, ||x||=1} (x, (A+B)x), \quad \dim V_{A+B} = i+j-1, \qquad (8)\\ \lambda_i(A) &= \max_{x \in V_A, ||x||=1} (x, Ax), \quad \dim V_A = n-i+1, \qquad (9)\\ \lambda_j(B) &= \max_{x \in V_B, ||x||=1} (x, Bx), \quad \dim V_B = n-j+1. \qquad (10)\end{aligned}$$

Then

$$\dim(V_A \cap V_B) = \dim V_A + \dim V_B - \dim(V_A \cup V_B) \geq \dim V_A + \dim V_B - n = n - i - j + 2, \tag{11}$$

and

$$\begin{aligned}\dim(V_{A+B} \cap (V_A \cap V_B)) &= \dim V_{A+B} + \dim(V_A \cap V_B) - \dim(V_{A+B} \cup (V_A \cap V_B)) \qquad (12)\\ &\geq (i+j-1) + (n-i-j+2) - n = 1. \qquad (13)\end{aligned}$$

Hence we can choose a nonzero vector

$$x_0 \in V_{A+B} \cap V_A \cap V_B. \tag{14}$$

It follows that

$$\lambda_{i+j-1}(A+B) \leq (x_0, (A+B)x_0) = (x_0, Ax_0) + (x_0, Bx_0) \leq \lambda_i(A) + \lambda_j(B), \tag{15}$$

which proves the left inequality. The right inequality follows by applying the left inequality to the negatives of the matrices. ☐

As a corollary, we obtain the following.

Weyl's inequality for singular values.

$$\sigma_{i+j-1}(A+B) \leq \sigma_i(A) + \sigma_j(B). \tag{16}$$

(Proof) The singular values of A are the nonnegative eigenvalues of the Hermitian matrix

$$\tilde{A} = \begin{pmatrix} & A \\ A^\dagger & \end{pmatrix}. \tag{17}$$

Define

$$\lambda_i = \sigma_i \quad i = 1, \ldots, n, \quad \lambda_{n+1} = -\sigma_{n-i+1} \quad i = 1, \ldots, n. \tag{18}$$

Then Weyl's inequality gives

$$\lambda_{i+j-1}(\tilde{A} + \tilde{B}) \leq \lambda_i(\tilde{A}) + \lambda_j(\tilde{B}) \leq \lambda_{i+j-2n}(\tilde{A} + \tilde{B}). \tag{19}$$

Restricting to $i + j - 1 \leq n$ gives the claim. ☐

References

[1] Wikipedia, *Weyl's inequality*, url.