

On Computing Homomorphisms of \mathbb{Z} -Modules

Ken Shiozaki

June 1, 2026

1 Computing $\text{im } f$ and $\text{ker } f$

Let A and B be \mathbb{Z} -modules. For a homomorphism

$$f : A \rightarrow B, \quad (1)$$

we want to compute $\text{ker } f$ and $\text{im } f$.

First write A and B as quotients of torsion-free groups:

$$A = \tilde{A}/P_A, \quad B = \tilde{B}/P_B. \quad (2)$$

For the given f , choose a lift $\tilde{f} : \tilde{A} \rightarrow \tilde{B}$ so that the induced homomorphism on quotient groups $\tilde{f} : \tilde{A}/P_A \rightarrow \tilde{B}/P_B$ agrees with f , namely

$$[\tilde{f}(\tilde{a})]_{P_B} = f([\tilde{a}]_{P_A}), \quad \forall \tilde{a} \in \tilde{A}. \quad (3)$$

Here $[\tilde{x}]_P$ denotes \tilde{x} modulo P . For the quotient homomorphism to be well-defined, \tilde{f} must satisfy

$$\tilde{f}(P_A) \subset P_B. \quad (4)$$

Equivalently, \tilde{f} fits into the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & P_A & \longrightarrow & \tilde{A} & \longrightarrow & A \longrightarrow 0 \\ & & \tilde{f}|_{P_A} \downarrow & & \tilde{f} \downarrow & & f \downarrow \\ 0 & \longrightarrow & P_B & \longrightarrow & \tilde{B} & \longrightarrow & B \longrightarrow 0. \end{array} \quad (5)$$

The lift \tilde{f} is not unique.

If bases of A, B and the matrix representation of f are given, the lift can be constructed as follows. Suppose

$$A = \bigoplus_{i=1}^N \mathbb{Z}_{p_i} a_i, \quad B = \bigoplus_{j=1}^M \mathbb{Z}_{q_j} b_j, \quad (6)$$

where $\mathbb{Z}_\infty = \mathbb{Z}$. With respect to these bases, write elements as integer column vectors $\vec{x} = (x_1, \dots, x_N)^T \in \mathbb{Z}^{\oplus N}$ and $\vec{y} = (y_1, \dots, y_M)^T \in \mathbb{Z}^{\oplus M}$:

$$x = \sum_j x_j a_j, \quad y = \sum_i y_i b_i. \quad (7)$$

Note that $p_i a_i = 0$ in A and $q_j b_j = 0$ in B . A matrix representation M of $f : A \rightarrow B$ is an integer matrix satisfying

$$f([\vec{x}]_{P_A}) = [M\vec{x}]_{P_B}, \quad \forall \vec{x} \in A. \quad (8)$$

Well-definedness requires

$$MP_A = \{M\vec{x} \mid \vec{x} \in P_A\} \subset P_B. \quad (9)$$

Once such an integer matrix M is fixed, the lift is

$$\tilde{f}(\vec{x}) = M\vec{x}, \quad \vec{x} \in \tilde{A}. \quad (10)$$

In practice, if one fixes an integer matrix M such that

$$f(a_1, \dots, a_N) = (b_1, \dots, b_M)M, \quad (11)$$

then the condition $0 = f(p_j a_j) = \sum_i b_i M_{ij}$ automatically implies $MP_A \subset P_B$.

1.1 $\text{im } f$

We have

$$\text{im } f = f(A) = \tilde{f}(\tilde{A}) \pmod{P_B}. \quad (12)$$

As a quotient group this is

$$\tilde{f}(\tilde{A}) / (\tilde{f}(\tilde{A}) \cap P_B). \quad (13)$$

By the second isomorphism theorem,

$$\text{im } f = (\tilde{f}(\tilde{A}) + P_B) / P_B. \quad (14)$$

Here

$$\tilde{f}(\tilde{A}) + P_B = \{x + y \in \tilde{B} \mid x \in \tilde{f}(\tilde{A}), y \in P_B\}. \quad (15)$$

In computing $\text{im } f$, the torsion information in A constrains possible maps f , but it does not affect the computation of $\tilde{f}(\tilde{A}) + P_B$. The generators $f(a_i)$ of $\text{im } f$ are determined independently of whether a_i is a free or torsion generator.

1.2 $\ker f$

Similarly,

$$\ker f = \{a \in A \mid f(a) = 0\} \quad (16)$$

$$= \{\tilde{a} \in \tilde{A} \mid \tilde{f}(\tilde{a}) \in P_B\} \pmod{P_A} \quad (17)$$

$$= \tilde{f}^{-1}(P_B) \pmod{P_A}. \quad (18)$$

Since $\tilde{f}(P_A) \subset P_B$, one has $P_A \subset \tilde{f}^{-1}(P_B)$, and therefore

$$\ker f = \tilde{f}^{-1}(P_B) / P_A. \quad (19)$$

The numerator is

$$\tilde{f}^{-1}(P_B) = \{\tilde{a} \in \tilde{A} \mid \tilde{f}(\tilde{a}) \in P_B\} = \{\tilde{a} \in \tilde{A} \mid \exists \tilde{b} \in P_B \text{ such that } \tilde{f}(\tilde{a}) + \tilde{b} = 0\}. \quad (20)$$

Introduce

$$\tilde{f} \oplus \text{Id}_{P_B} : \tilde{A} \oplus P_B \rightarrow \tilde{B}, \quad (\tilde{a}, \tilde{b}) \mapsto \tilde{f}(\tilde{a}) + \tilde{b}. \quad (21)$$

Then

$$\{\tilde{a} \in \tilde{A} \mid \tilde{f}(\tilde{a}) \in P_B\} = \ker(\tilde{f} \oplus \text{Id}_{P_B})|_{\tilde{A}}, \quad (22)$$

where the right-hand side denotes the projection of $\ker(\tilde{f} \oplus \text{Id}_{P_B})$ onto \tilde{A} . Thus

$$\ker f = \ker(\tilde{f} \oplus \text{Id}_{P_B})|_{\tilde{A}} / P_A. \quad (23)$$

1.3 Application: Computing $\ker g / \text{im } f$

For a sequence of homomorphisms of \mathbb{Z} -modules

$$A \xrightarrow{f} B \xrightarrow{g} C \quad (24)$$

with $\text{im } f \subset \ker g$, the quotient $\ker g / \text{im } f$ is defined. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_A & \longrightarrow & \tilde{A} & \longrightarrow & A & \longrightarrow & 0 \\ & & \tilde{f} \downarrow & & \tilde{f} \downarrow & & f \downarrow & & \\ 0 & \longrightarrow & P_B & \longrightarrow & \tilde{B} & \longrightarrow & B & \longrightarrow & 0 \\ & & \tilde{g} \downarrow & & \tilde{g} \downarrow & & g \downarrow & & \\ 0 & \longrightarrow & P_C & \longrightarrow & \tilde{C} & \longrightarrow & C & \longrightarrow & 0. \end{array} \quad (25)$$

As above, by expressing f and g in suitable bases and regarding them as integer matrices, we obtain commuting lifts \tilde{f} and \tilde{g} .

The original note contains an unfinished placeholder at this point. Using the formulae above, one obtains

$$\ker g / \text{im } f = \frac{\ker(\tilde{g} \oplus \text{Id}_{P_C})|_{\tilde{B}} / P_B}{(\tilde{f}(\tilde{A}) + P_B) / P_B}. \quad (26)$$

By the third isomorphism theorem, this becomes

$$\ker g / \text{im } f = \frac{\ker(\tilde{g} \oplus \text{Id}_{P_C})|_{\tilde{B}}}{\tilde{f}(\tilde{A}) + P_B}, \quad (27)$$

or equivalently

$$\ker g / \text{im } f = \frac{\tilde{g}^{-1}(P_C)}{\tilde{f}(\tilde{A}) + P_B}. \quad (28)$$

Thus the calculation reduces to a quotient of two subgroups of the torsion-free group \tilde{B} . Since this is a quotient of torsion-free abelian groups, it can be computed by expanding a basis of $\tilde{f}(\tilde{A}) + P_B$ in a basis of $\ker(\tilde{g} \oplus \text{Id}_{P_C})|_{\tilde{B}}$, for instance using a pseudoinverse, and then applying Smith normal form to the resulting matrix.

Finally, let us directly check $\tilde{f}(\tilde{A}) + P_B \subset \tilde{g}^{-1}(P_C)$. For $\tilde{a} \in \tilde{A}$ and $b \in P_B$,

$$\tilde{g}(\tilde{f}(\tilde{a}) + b) = \tilde{g} \circ \tilde{f}(\tilde{a}) + \tilde{g}(b). \quad (29)$$

Since $g \circ f = 0$ and the diagram commutes, $\tilde{g} \circ \tilde{f}(\tilde{A}) \subset P_C$. Also, by commutativity of the diagram, $\tilde{g}(P_B) \subset P_C$.