On $c_1(E) \equiv w_2(E_{\mathbb{R}})$

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1 Preliminaries

1.1 $U(n) \to SO(2n)$

For $u \in U(n)$, write

$$u = a + ib, \quad a = (u + u^*)/2, \quad b = (u - u^*)/(2i). \tag{1}$$

Then the map is given by

$$u \mapsto R(u) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \tag{2}$$

One can check that this is a homomorphism, $R(uv) = R(u)R(v)$. It is also injective, because if $R(u) = 1_{2n}$, then $a = 1_n$ and $b = 0$. Moreover, diagonalizing $u = \gamma \Lambda \gamma^\dagger$, with $\Lambda = \mathrm{diag}(e^{i\theta_1}, \dots)$, and noting that $R(\gamma^\dagger) = R(\gamma)^\top$ follows from $R(\gamma)R(\gamma^\dagger) = 1_{2n}$, we have

$$\det R(u) = \det R(\gamma)^2 \det R(\Lambda) = \prod_i \det e^{-i\sigma_y \theta_i} = 1. \tag{3}$$

1.2 $SO(2n) \to Spin(2n)$

The rotation matrix in the xy plane is given by

$$e^{-i\sigma_y \theta} = e^{\theta L} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad L = -i\sigma_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4}$$

In general, an $SO(n)$ matrix R can be represented as

$$R = e^{\sum_{i<j} \theta_{ij} L_{ij}}, \quad [L_{ij}]_{kl} = -\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}. \tag{5}$$

[1] Using this representation, the map $SO(n) \to Spin(n)$ is given by

$$R \mapsto \tilde{R} = \pm e^{\sum_{ij} \theta_{ij} \Sigma_{ij}}, \quad \Sigma_{ij} = \frac{[\gamma_i, \gamma_j]}{-4}, \quad \{\gamma_i, \gamma_j\} = 2\delta_{ij}. \tag{6}$$

1.3 Lifts and uniqueness

Let $p : C \to X$ be a covering map. A continuous surjection $p : C \to X$ is called a covering map if, for every $x \in X$, there exists an open neighborhood U of x such that the inverse image $p^{-1}(U)$ is a union of mutually disjoint open subsets of C, each of which is mapped homeomorphically onto U by p.

Let Y be a path-connected, and locally connected, space, and consider a continuous map $f : Y \to X$. Fix one point $c \in C$ that lies "over $f(y)$," namely such that $p(c) = f(y)$. We want to know whether there exists a lift of f, that is, a continuous map

$$\tilde{f} : Y \to C \tag{7}$$

[1] For a method to determine the $n(n-1)/2$ real parameters θ_{ij} from a given $R \in SO(n)$, see https://www2.yukawa.kyoto-u.ac.jp/~ken.shiozaki/doc/Compute_Homotopy_SO.pdf.

satisfying

$$p \circ \tilde{f} = f, \quad \tilde{f}(y) = c. \tag{8}$$

For example, suppose C is the universal cover, hence simply connected, and suppose $\pi_1(Y, y)$ is nonzero and its image under f is not killed. If one follows points along a nontrivial loop in $\pi_1(Y, y)$, then $\tilde{f}$ does not return to $\tilde{f}(z) = c$ and therefore cannot be continuous.

Write the homomorphism between fundamental groups induced by f as $f_* : \pi_1(Y, y) \to \pi_1(X, f(y))$.

- A lift $\tilde{f}$ exists if and only if

$$f_*(\pi_1(Y, y)) \subset p_*(\pi_1(C, c)) \tag{9}$$

  holds. Moreover, if the lift $\tilde{f}$ exists, it is unique.

  In particular, if C is the universal cover, so that C is simply connected, a lift $\tilde{f}$ exists if and only if

$$f_*(\pi_1(Y, y)) = 0 \tag{10}$$

  holds. In that case, $\tilde{f}$ is unique.

We postpone the proof, but the intuition is as follows. Since the image $f(Y) \subset X$ is simply connected, there are no nontrivial loops in $f(Y)$, and the statement follows from existence and uniqueness of path lifting. Existence and uniqueness of path lifting follow from the existence, for any open set U of X, of a homeomorphism between U and each sheet of $p^{-1}(U)$.

The same statement also holds for Lie groups. Let G be a Lie group and H a path-connected Lie group. Let $\tilde{G}$ be the universal covering group of G, and let $p : \tilde{G} \to G$ be a covering homomorphism. Let $f : H \to G$ be a continuous homomorphism and suppose $f_*[\pi_1[H]] = 0$. Then there exists a continuous lift $\tilde{f} : H \to \tilde{G}$, and it can be chosen so that $\tilde{f}(e) = e$. Since p is a homomorphism, we have $p(\tilde{f}(hh')) = p(\tilde{f}(h)\tilde{f}(h'))$. By the path-connectedness of H, there exists a path $h(t \in [0,1])$ with $h(0) = e$ and $h(1) = h$. In a neighborhood of $t = 0$, the local homeomorphism property of the surjection p gives $\tilde{f}(h(t)h') = \tilde{f}(h(t))\tilde{f}(h')$. Covering the path $h(t)$ by such patches from $t = 0$ to $t = 1$, one obtains $\tilde{f}(h(t)h') = \tilde{f}(h(t))\tilde{f}(h')$ for all $t \in [0, 1]$.

1.4 Direct sums and lifts

Let $SO(n) \times SO(m) \to SO(n+m)$ be the natural embedding that places matrices in block-diagonal form. Given independent lifts

$$SO(n) \ni R_1 \mapsto \tilde{R}_1 \in Spin(n), \tag{11}$$
$$SO(m) \ni R_2 \mapsto \tilde{R}_2 \in Spin(m), \tag{12}$$

does this determine a lift $SO(n+m) \to Spin(n+m)$? As a warning, even if we write

$$\begin{pmatrix} R_1 & \\ & R_2 \end{pmatrix} \mapsto \text{``} \begin{pmatrix} \tilde{R}_1 & \\ & \tilde{R}_2 \end{pmatrix} \text{''} \tag{13}$$

the right-hand side has no meaning as an element of the group $Spin(n+m)$.

Let us analyze this using the representation (4). Write rotations by θ_{12} in the x_1x_2 plane and by θ_{34} in the x_3x_4 plane as $SO(2)$ matrices

$$R_1 = e^{-i\sigma_y \theta_{12}}, \tag{14}$$
$$R_2 = e^{-i\sigma_y \theta_{34}}. \tag{15}$$

Their direct sum is written as an $SO(4)$ matrix

$$R = \begin{pmatrix} e^{-i\sigma_y \theta_{12}} & \\ & e^{-i\sigma_y \theta_{34}} \end{pmatrix}. \tag{16}$$

This can also be written as

$$R = e^{\theta_{12}L_{12}+\theta_{34}L_{34}} = e^{\theta_{12}L_{12}}e^{\theta_{34}L_{34}}. \tag{17}$$

This corresponds to the notation

$$R = \begin{pmatrix} R_1 & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & R_2 \end{pmatrix}. \tag{18}$$

The lift to $Spin(4)$ is given by

$$\tilde{R} = e^{\theta_{12}\Sigma_{12}+\theta_{34}\Sigma_{34}}. \tag{19}$$

Concretely, for example, if we set

$$(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (\sigma_x\tau_z, \sigma_y\tau_z, \tau_x, \tau_y), \tag{20}$$

then

$$\Sigma_{12} = -i\sigma_z/2, \quad \Sigma_{34} = -i\tau_z/2, \tag{21}$$

and therefore

$$\tilde{R} = e^{-i\theta_{12}\sigma_z/2}e^{-i\theta_{34}\tau_z/2}. \tag{22}$$

Notice that the sign ambiguities of the lifts $\tilde{R}_1$ and $\tilde{R}_2$ enter $\tilde{R}$ through their product. The correct relation is not a direct sum but the product

$$\tilde{R} = \widetilde{i_1(R_1)}\widetilde{i_2(R_2)}. \tag{23}$$

Here i_1 and i_2 denote the embeddings $SO(2) \to SO(4)$, respectively. More generally, for the lifts

$$SO(n) \ni R_1 = e^{\sum_{1\le i<j\le n}\theta_{ij}L_{ij}} \to \tilde{R}_1 = \pm e^{\sum_{1\le i<j\le n}\theta_{ij}\Sigma_{ij}} \in Spin(n), \tag{24}$$

$$SO(m) \ni R_2 = e^{\sum_{n+1\le i<j\le n+m}\theta_{ij}L_{ij}} \to \tilde{R}_2 = \pm e^{\sum_{n+1\le i<j\le n+m}\theta_{ij}\Sigma_{ij}} \in Spin(m), \tag{25}$$

the lift of $(R_1, R_2) \mapsto (R_1 \oplus R_2) \to i_1(R_1)i_2(R_2) \in SO(n+m)$ is given by

$$R \mapsto (\pm)(\pm)\tilde{R}_1\tilde{R}_2. \tag{26}$$

2 $c_1 \equiv w_2$

Let $E \to X$ be a complex vector bundle of rank n. Choose a good covering $\{U_i\}_i$ of X. Then E is specified by the data of transition functions

$$t_{ij} : U_{ij} \to U(n), \quad t_{ij}t_{jk}t_{ki} = 1_n. \tag{27}$$

Write

$$e^{i\theta_{ij}} = \det t_{ij}, \tag{28}$$

and choose one lift $U(1) \to \mathbb{R}$, $\theta_{ij} \mapsto \tilde{\theta}_{ij}$. By the cocycle condition,

$$(\delta\tilde{\theta})_{ijk} = \tilde{\theta}_{ij} + \tilde{\theta}_{jk} + \tilde{\theta}_{ki} \in 2\pi\mathbb{Z}. \tag{29}$$

Define

$$c_1(E) = [\delta\tilde{\theta}/(2\pi)]. \tag{30}$$

Define the real vector bundle $E_{\mathbb{R}}$ of rank $2n$ by the transition-function data

$$R(t_{ij}) : U_{ij} \to SO(2n). \tag{31}$$

Taking lifts on U_{ij},

$$R(t_{ij}) \mapsto \tilde{R}(t_{ij}) \in Spin(2n), \tag{32}$$

determines $s_{ijk} \in \{0,1\}$ by

$$\tilde{R}(t_{ij})\tilde{R}(t_{jk})\tilde{R}(t_{ki}) = e^{\pi i s_{ijk}}\mathbf{1}. \tag{33}$$

Then

$$w_2(E_{\mathbb{R}}) = [s]. \tag{34}$$

The claim is

$$c_1(E) \equiv w_2(E_{\mathbb{R}}) \mod 2. \tag{35}$$

(Proof) Define the homomorphism

$$f : U(n) \to SO(2n+2) \tag{36}$$

by

$$t \mapsto f(t) = \begin{pmatrix} R(t) & \\ & e^{-i\sigma_y \log\det t} \end{pmatrix}. \tag{37}$$

[2] Take lifts of $R(t_{ij})$ and $e^{-i\sigma_y \log\det t_{ij}}$ from the target embedding space $SO(2n+2)$. Concretely, these are given by

$$R_1(t_{ij}) = e^{\sum_{1\le i<j\le 2n}\theta_{ij}L_{ij}} \to \widetilde{R_1(t_{ij})} = e^{\sum_{1\le i<j\le 2n}\theta_{ij}\Sigma_{ij}}, \tag{38}$$

$$R_2(t_{ij}) = e^{\theta_{2n+1,2n+2}L_{2n+1,2n+2}} \to \widetilde{R_2(t_{ij})} = e^{\tilde{\theta}_{2n+1,2n+2}\Sigma_{2n+1,2n+2}}, \quad e^{i\theta_{2n+1,2n+2}} = \det t_{ij}. \tag{39}$$

Thus the lift

$$f(t_{ij}) = R(t_{ij}) \oplus e^{-i\sigma_y \log\det t_{ij}} \to \widetilde{f(t_{ij})} = \widetilde{R_1(t_{ij})}\widetilde{R_2(t_{ij})} : U_{ij} \to Spin(2n+2) \tag{40}$$

is determined. By construction,

$$\widetilde{f(t_{ij})}\widetilde{f(t_{jk})}\widetilde{f(t_{ki})} = e^{\pi i s_{ijk}} e^{i(\delta\tilde{\theta})_{ijk}/2}\mathbf{1} \tag{41}$$

holds. On the other hand, since $f_*(\pi_1(U(n))) = 0$ [3] there exists a lift

$$\tilde{f} : U(n) \to Spin(2n+2). \tag{42}$$

By the homomorphism property of $\tilde{f}$,

$$\tilde{f}(t_{ij})\tilde{f}(t_{jk})\tilde{f}(t_{ki}) = \tilde{f}(t_{ij}t_{jk}t_{ki}) = \tilde{f}(1_n) = \mathbf{1} \tag{43}$$

holds. Therefore, the 2-cocycle $e^{\pi i s_{ijk}} e^{i(\delta\tilde{\theta})_{ijk}/2}$ defines the trivial cohomology class. □

Remark: In general,

$$c_i(E) \equiv w_{2i}(E_{\mathbb{R}}), \quad i = 1, 2, \dots \tag{44}$$

holds.

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[2] https://www.youtube.com/watch?v=CjW2vRwR6Vo&list=PLESK1T0IJLino5qgZcx1ONliXtvUcoG9B&index=6

[3] $R_* : \pi_1(U(n)) \to \pi_1(SO(2n))$ is the map $1 \mapsto 1$. The image of a nontrivial loop in $U(n)$ can be contracted inside $SO(2n+2)$ by using off-diagonal components.