# On Chiral Indices and Winding Numbers

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[1]

In general, there is no way to compare chirality between different irreducible representations. For example, a "total chirality" may not be definable. We summarize this point.

Let $G = G_0 + \Gamma G_0$. For an irreducible representation $\alpha$ of $G_0$, assume that $\Gamma\alpha \sim \alpha$. General representation theory then gives irreducible representations $\alpha\pm$ of $G_0 + \Gamma G_0$ satisfying

$$\chi^{\alpha\pm}_{g\in G_0} = \chi^{\alpha}_g, \quad \chi^{\alpha-}_{g\in \Gamma G_0} = -\chi^{\alpha+}_g. \tag{1}$$

There is no canonical way to choose which of $\alpha+$ and $\alpha-$ is the positive one.

## 1 Chiral Index

Once $\alpha+$ and $\alpha-$ are fixed, the chiral index is defined by

$$\#(\alpha+ \text{ irreps}) - \#(\alpha- \text{ irreps}) = \frac{1}{|G_0+\Gamma G_0|}\sum_{g\in G_0+\Gamma G_0} (\chi^{\alpha+}_g - \chi^{\alpha-}_g)^* \mathrm{tr}\,[u_g] \tag{2}$$

$$= \frac{1}{|G_0|}\sum_{g\in \Gamma G_0} (\chi^{\alpha+}_g)^* \mathrm{tr}\,[u_g]. \tag{3}$$

Here $u_g$ is the representation matrix, and

$$P_{\alpha\pm} = \frac{1}{|G_0+\Gamma G_0|}\sum_{g\in G_0+\Gamma G_0} (\chi^{\alpha\pm}_g)^* u_g \tag{4}$$

is the projection onto the irreducible representation $\alpha\pm$.

Suppose another irreducible representation $\beta$ of $G_0$ satisfies $\Gamma\beta \sim \beta$. Choosing $\beta+$ from the irreducible representations of $G_0+\Gamma G_0$ defines the positive chirality in the $\beta$ sector. This choice is in general unrelated to the positive chirality $\alpha+$ in the $\alpha$ sector.

Now let $H \subset G$ be a subgroup and suppose $H$ contains the chiral symmetry. Write $H = H_0 + \Gamma H_0$. For an irreducible representation $\gamma$ of $H_0$, assume $\Gamma\gamma \sim \gamma$. Then the positive chirality in the $\gamma$ sector is fixed by choosing $\gamma+$ from the irreducible representations of $H_0 + \Gamma H_0$.

Since $\alpha+$ and $\beta+$ are irreducible representations of $G_0 + \Gamma G_0$, they can be paired with the irreducible representations $\gamma\pm$ of the subgroup $H_0 + \Gamma H_0$. The change of the chiral index of $\alpha+$ after restricting to $H$ is

$$(\gamma+, \alpha+) - (\gamma-, \alpha+) = \frac{1}{|H_0|}\sum_{g\in \Gamma H_0} (\chi^{\gamma+}_g)^* \chi^{\alpha+}_g. \tag{5}$$

When this change is nonzero, one can compare the positive chiralities of $\alpha+$ and $\beta+$ through the change in chiral index.

An example where "total chirality" cannot be defined is $G = \mathbb{Z}_4 = \{e, \sigma^2\} + \sigma\{e, \sigma^2\}$. In this case there is no meaningful subgroup of $G$ that contains the chiral operator.

[1] Motivation: In the first differential $d_1^{p,-n}$ of the AHSS, for odd $n$ one must double the group elements because of the chiral operator. Can one somehow compute the first differential without increasing the number of group elements? The conclusion is that, in general, there is no way to compare chirality between different sectors, so doubling the group elements is unavoidable.

## 2 Winding Number

Let a Hamiltonian $H(\boldsymbol{k})$ depending on a $(2n-1)$-dimensional parameter space have the following $G_0+\Gamma G_0$ symmetry:

$$u_g H(\boldsymbol{k}) u_g^{-1} = \begin{cases} H(\boldsymbol{k}) & (g \in G_0), \\ -H(\boldsymbol{k}) & (g \in \Gamma G_0). \end{cases} \tag{6}$$

For an irreducible representation $\alpha$ of $G_0$, if the transported representation satisfies $\Gamma\alpha \sim \alpha$, then one can define an integer-valued winding number. Fix an extension $\alpha\pm$ of $\alpha$ to $G_0 + \Gamma G_0$. The winding number is

$$W^{\alpha}_{2n-1} = \frac{n!}{(2\pi i)^n (2n)!} \int \mathrm{tr}\,[(H^{-1}dH)^{2n-1}(P_{\alpha+} - P_{\alpha-})] \tag{7}$$

$$= \frac{n!}{(2\pi i)^n (2n)!} \int \frac{1}{|G_0|} \sum_{g\in\Gamma G_0} (\chi_g^{\alpha+})^* \mathrm{tr}\,[(H^{-1}dH)^{2n-1} u_g] \in \mathbb{Z}. \tag{8}$$

Notice that, in this definition, $W^{\alpha}_{2n-1}$ can take any integer value, independently of the dimension of the representation $\alpha$.

The simplest example is $G_0 + \Gamma G_0 = \{e\} + \{\gamma\}$. If the factor system is such that $u_\gamma^2 = 1$, the representation has two extensions, $\chi_\gamma = \pm 1$. Setting $\chi_\gamma^{\pm} = \pm 1$, the winding number becomes

$$W^{\alpha}_{2n-1} = \frac{n!}{(2\pi i)^n (2n)!} \int \mathrm{tr}\,[(H^{-1}dH)^{2n-1} u_\gamma], \tag{9}$$

which agrees with the usual definition.

As a nontrivial example, take

$$G_0 + \Gamma G_0 = \{e, m_x, m_y, c_z\} + \{\gamma, m_x\gamma, m_y\gamma, c_z\gamma\}, \tag{10}$$

with

$$u_{m_x}^2 = u_{m_y}^2 = u_{c_z}^2 = -1, \tag{11}$$

$$u_{m_x} u_{m_y} = -u_{m_y} u_{m_x}, \tag{12}$$

$$u_\gamma u_{m_x} = -u_{m_x} u_\gamma, \quad u_\gamma u_{m_y} = u_{m_y} u_\gamma, \tag{13}$$

$$u_\gamma^2 = 1, \quad u_{m_y\gamma} = u_{m_y} u_\gamma. \tag{14}$$

There is a unique irreducible representation of $G_0$, with character

(15)

| | $e$ | $m_x$ | $m_y$ | $c_z$ |
|---|---|---|---|---|
| $\chi_g$ | 2 | 0 | 0 | 0. |

For its extensions to $G_0 + \Gamma G_0$, note that $u_{m_y\gamma}$ commutes with all other elements and that $u_{m_y\gamma}^2 = u_{m_y}^2 u_\gamma^2 = -1$. Thus

(16)

| | $e$ | $m_x$ | $m_y$ | $c_z$ | $\gamma$ | $m_x\gamma$ | $m_y\gamma$ | $c_z\gamma$ |
|---|---|---|---|---|---|---|---|---|
| $\chi_g^{\pm}$ | 2 | 0 | 0 | 0 | 0 | 0 | $\pm 2i$ | 0. |

The winding number is therefore

$$W^{\alpha}_{2n-1} = \frac{n!}{(2\pi i)^n (2n)!} \int \frac{1}{4} \sum_{g\in\{\gamma, m_x\gamma, m_y\gamma, c_z\gamma\}} (\chi_g^{+})^* \mathrm{tr}\,[(H^{-1}dH)^{2n-1} u_g] \tag{17}$$

$$= \frac{n!}{(2\pi i)^n (2n)!} \int \frac{1}{4} (2i)^* \mathrm{tr}\,[(H^{-1}dH)^{2n-1} u_{m_y\gamma}]. \tag{18}$$

A one-dimensional model is

$$H(k) = \sin k\, \tau_x + \cos k\, \tau_y, \tag{19}$$

$$u_{m_x} = i\sigma_x, \quad u_{m_y} = i\sigma_y, \quad u_\gamma = \sigma_y \tau_z. \tag{20}$$

Then $u_{m_y\gamma} = i\tau_z$, and

$$W_1 = \frac{1}{4\pi i} \oint \frac{-i}{2} \mathrm{tr}\,[(\sin k\, \tau_x + \cos k\, \tau_y)(\cos k\, \tau_x - \sin k\, \tau_y)(i\tau_z)]\, dk = -1. \tag{21}$$