Notes: Cubical Singular Chain Complexes

Ken Shiozaki

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Abstract

Following [1, 2], we summarize cubical singular chain complexes, the subdivision operator, cup products, and related material. These notes are preparation for reading [3].

1 Cubical Singular Chain Complexes

Let $I = [0,1]$.

- A singular n-cube is a continuous map

$$T : I^n \to X \quad (n \geq 0) \tag{1.1}$$

.

- Let $Q_n(X)$ be the free abelian group generated by the set of singular n-cubes. Thus an element of $Q_n(X)$ can be written uniquely as a finite sum $\sum_i a_i T_n$.
- A singular n-cube $T : I^n \to X$ is called degenerate if, for some $i \in \{1, \dots, n\}$, $T(x_1, \dots, x_n)$ is independent of x_i. (In other words, its dimension drops inside X.)
- Let $D_n(X) \subset Q_n(X)$ be the free abelian group generated by degenerate singular n-cubes.
- The quotient group $C_n(X) := Q_n(X)/D_n(X)$ is called the normalized cubical singular n-chain group.

We introduce the boundary operator. For a singular n-cube $T : I^n \to X$, define the front and back face maps $A_i, B_i : I^{n-1} \to X$ by

$$A_i T(x_1, \dots, x_{n-1}) := T(x_1, \dots, x_{i-1}, 0, x_i, \dots, x_{n-1}), \tag{1.2}$$
$$B_i T(x_1, \dots, x_{n-1}) := T(x_1, \dots, x_{i-1}, 1, x_i, \dots, x_{n-1}), \tag{1.3}$$

. That is, we take the i-th boundary faces $x_i = 0, 1$. Clearly, for $1 \leq i < j \leq n$,

$$A_i A_j = A_{j-1} A_i, \quad B_i B_j = B_{j-1} B_i, \quad A_i B_j = B_{j-1} A_i, \quad B_i A_j = A_{j-1} B_i \tag{1.4}$$

hold. Define the boundary operator as follows:

$$\partial_n : Q_n(X) \to Q_{n-1}(X), \quad \partial_n(T) = \sum_{i=1}^{n} (-1)^i [A_i T - B_i T], \quad n > 0, \tag{1.5}$$
$$\partial_{n \leq 0} = 0. \tag{1.6}$$

Equivalently,

$$\partial_{n>0} = \sum_{i=1}^{n} (-1)^i [A_i - B_i], \quad \partial_{n\leq 0} = 0. \tag{1.7}$$

A direct calculation gives

$$\partial_{n-1}\partial_n = \partial_{n-1}[\sum_{j=1}^{n} (-1)^j (A_j - B_j)] = \sum_{i=1}^{n-1}\sum_{j=1}^{n} (-1)^i (-1)^j (A_i A_j - B_i A_j - A_i B_j + B_i B_j). \tag{1.8}$$

Using the properties above, when $n > 1$,

$$\begin{aligned}
\sum_{i=1}^{n-1}\sum_{j=1}^{n} (-1)^i (-1)^j A_i A_j &= \sum_{i\leq i<j\leq n} (-1)^i (-1)^j A_i A_j + \sum_{1\leq j\leq i\leq n-1} (-1)^i (-1)^j A_i A_j && (1.9)\\
&= \sum_{i\leq i<j\leq n} (-1)^i (-1)^j A_{j-1} A_i + \sum_{1\leq j\leq i\leq n-1} (-1)^i (-1)^j A_i A_j && (1.10)\\
&= \sum_{i\leq j<i\leq n} (-1)^j (-1)^i A_{i-1} A_j + \sum_{1\leq j\leq i\leq n-1} (-1)^i (-1)^j A_i A_j && (1.11)\\
&= \sum_{i\leq j<i+1\leq n} (-1)^j (-1)^{i+1} A_i A_j + \sum_{1\leq j\leq i\leq n-1} (-1)^i (-1)^j A_i A_j && (1.12)\\
&= 0. && (1.13)
\end{aligned}$$

Also, if $T \in D_n(X)$, that is, if T is independent of x_j for some j, then $(A_i - B_i)T$ is 0 when $i = j$, is independent of x_j when $j < i$, and is independent of x_{j-1} when $i < j$. Hence $\partial_n T \in D_{n-1}(X)$. In summary,

$$\begin{aligned}
&\partial_{n-1}\partial_n = 0 \quad n > 1, && (1.14)\\
&\partial_n D_n(X) \subset D_{n-1}(X) \quad n > 0. && (1.15)
\end{aligned}$$

In particular, the latter implies that

$$\partial_n : C_n(X) \to C_{n-1}(X) \quad n > 0 \tag{1.16}$$

is well-defined.

$$\begin{aligned}
Z_n(X) &= \ker[\partial_n : C_n(X) \to C_{n-1}(X)], && (1.17)\\
B_n(X) &= \mathrm{im}\,[\partial_{n+1} : C_{n+1}(X) \to C_n(X)], && (1.18)\\
H_n(X) &= Z_n(X)/B_n(X), && (1.19)
\end{aligned}$$

.

Define the augmentation map $\epsilon : C_n(X) \to \mathbb{Z}$ by

$$\epsilon : C_0(X) \to \mathbb{Z}, \quad \sum_i a_i x_i \mapsto \sum_{a_i}, \quad \epsilon|_{C_{n>0}(X)} = 0, \tag{1.20}$$

. Then

$$\epsilon\partial = 0. \tag{1.21}$$

Indeed, only the case $n = 1$ is nontrivial, and

$$\epsilon\partial_1 T(x) = \epsilon([T(0)] - [T(1)]) = 1 - 1 = 0 \tag{1.22}$$

.

- Let $S_*(X)$ be the ordinary singular chain complex. There is a natural chain-homotopy equivalence $S_*(X) \to C_*(X)$; see Problems 9 and 10 in Chapter 5 of [2].
- If one uses the cubical singular chain group $Q_n(X)$ itself, rather than the normalized cubical singular chain complex $C_*(X)$, degenerate cubes remain and the homology group $H'_n(X) := \frac{Z'_n(X)=\ker[\partial:Q_n(X)\to Q_{n-1}(X)]}{B'_n(X)=\mathrm{im}\,[\partial:Q_{n+1}(X)\to Q_n(X)]}$ differs from the usual one. For example, for a one-point space, for every $n \in \mathbb{Z}$ the map $T : I^n \to \{pt\}$ is constant and $\partial T = 0$; hence $Z'_n(\{pt\}) = \mathbb{Z}$ and, since $B'_n(\{pt\}) = 0$, one gets $H'_n(\{pt\}) = \mathbb{Z}$.
- In [3], the normalized cubical singular chain complex is used instead of the ordinary singular chain complex.

2 The Subdivision Operator

We divide the unit cube I^n into 2^n pieces. In formulas this is as follows. Let $\mathcal{E}_n$ denote the set of the 2^n vertices of I^n. Thus $\mathcal{E}_n = \{0,1\}^{\times n}$. For a singular n-cube $T : I^n \to X$ and $e \in \mathcal{E}_n$, introduce

$$F_eT : I^n \to X, \quad (F_eT)(x) := T(\frac{1}{2}(x+e)), \tag{2.1}$$

. For example,

$$(F_{(0,0)}T)(x_1,x_2) = T(x_1/2, x_2/2), \tag{2.2}$$
$$(F_{(0,1)}T)(x_1,x_2) = T(x_1/2, x_2/2+1/2), \tag{2.3}$$
$$(F_{(1,0)}T)(x_1,x_2) = T(x_1/2+1/2, x_2/2), \tag{2.4}$$
$$(F_{(1,1)}T)(x_1,x_2) = T(x_1/2+1/2, x_2/2+1/2). \tag{2.5}$$

Define $Sd_n : Q_n(X) \to Q_n(X)$ by

$$Sd_{n>0}(T) := \sum_{e\in\mathcal{E}_n} F_eT, \tag{2.6}$$
$$Sd_{n=0} = id, \tag{2.7}$$

. In other words, we divide the domain of the map $T : I^n \to X$ into 2^n equal pieces and then map them by T. Since this only subdivides the map, clearly

$$Sd_n(D_n(X)) \subset D_n(X) \tag{2.8}$$

holds. Therefore

- $sd_n : C_n(X) \to C_n(X)$,

is induced. The following also holds:

- $\partial_n Sd_n = Sd_{n-1}\partial_n$.

We omit the proof. Since ∂_n takes the oriented boundary in the naive sense, it should be intuitively clear that the boundary operator ∂ commutes with subdivision. Together with the above, we get

$$\partial_n sd_n = sd_{n-1}\partial_n \tag{2.9}$$

. Thus sd is a chain map.

- If $u \in C_0(X) = Q_0(X)$, that is, if $u = \sum_i a_i x_i$, then since $Sd_0 = sd_0 = 1$,

$$\epsilon(Sd_0(u)) = \epsilon(u). \tag{2.10}$$

Theorem 2.1. *$sd = (sd_n)$ is chain homotopic to 1. That is, there are homomorphisms*

$$\phi_n : C_n(X) \to C_{n+1}(X) \tag{2.11}$$

such that

$$sd_n - 1 = \partial_{n+1}\phi_n + \phi_{n-1}\partial_n \tag{2.12}$$

.

(Proof.) We construct ϕ_n. First introduce

$$\eta_0, \eta_1 : I^2 \to I^1, \tag{2.13}$$

$$\eta_0(x_1, x_2) = \frac{x_1}{2 - x_2}, \tag{2.14}$$

$$\eta_1(x_1, x_2) = \frac{x_1 + 1 - x_1 x_2}{2 - x_2}, \tag{2.15}$$

. [1] These maps are designed to satisfy

$$\eta_0(0, y) = 0, \quad \eta_0(1, y) = \frac{1}{2 - y} = \eta_1(0, y), \quad \eta_0(x, 0) = \frac{x}{2} \quad \eta_0(x, 1) = x, \tag{2.16}$$

$$\eta_1(0, y) = \frac{1}{2 - y} = \eta_0(1, y), \quad \eta_1(1, y) = 1, \quad \eta_1(x, 0) = \frac{x + 1}{2}, \quad \eta_1(x, 1) = 1. \tag{2.17}$$

$n \geq 1$, for $e = (e_1, \ldots, e_n) \in \mathcal{E}_n$ and $T : I^n \to X$, introduce

$$G_e(T) : I^{n+1} \to X, \tag{2.18}$$

$$(G_e(T))(x_1, \ldots, x_{n+1}) := T\left(\eta_{e_1}(x_1, x_{n+1}), \eta_{e_2}(x_2, x_{n+1}), \ldots, \eta_{e_n}(x_n, x_{n+1})\right) \tag{2.19}$$

. The following hold.

- G_e sends degenerate singular cubes to degenerate singular cubes.

Indeed, in the expression

$$G_e T(x_1, \ldots, x_{n+1}) = T\left(\eta_{e_1}(x_1, x_{n+1}), \eta_{e_2}(x_2, x_{n+1}), \ldots, \eta_{e_n}(x_n, x_{n+1})\right) \tag{2.20}$$

if T is independent of x_j, then $G_e T$ is also independent of x_j. [2]

- $A_{n+1} G_e(T) = F_e(T)$.

This follows from $\eta_0(x, 0) = \frac{x}{2}$ and $\eta_1(x, 0) = \frac{x+1}{2}$.

- $B_{n+1} G_e(T) = \begin{cases} T & (e = (0, \ldots, 0)), \\ \text{is a degenerate cube} & (\text{else}). \end{cases}$

This follows from $\eta_0(x, 1) = x$ and $\eta_1(x, 1) = 1$.

- Suppose two bit strings $e, e' \in \mathcal{E}_n$ satisfy $e_j = 1, e'_j = 0$ for some $j \leq n$, and $e_i = e'_i$ for all $i \neq j$. Then

$$A_j G_e(T) = B_j G_{e'}(T). \tag{2.21}$$

[1] This η_1 is the definition in Appendix 2 of [1].

[2] Moreover, for $T : I^n \to X$, the variable dependence of $G_e T$ passes through T, so the $(n+1)$-dimensional volume of $G_e T(\Delta^n) \subset X$ is zero.

Indeed,

$$
\begin{aligned}
&(A_j G_e T)(x_1, \ldots, x_n) && (2.22)\\
&= (G_e T)(x_1, \ldots, x_{j-1}, 0, x_j, \ldots, x_n) && (2.23)\\
&= T(\eta_{e_1}(x_1, x_n), \ldots, \eta_{e_{j-1}}(x_{j-1}, x_n), \underbrace{\eta_{e_j=1}(0, x_n)}_{\frac{1}{2-x_n}}, \eta_{e_{j+1}}(x_j, x_n), \ldots, \eta_{e_n}(x_{n-1}, x_n)), && (2.24)
\end{aligned}
$$

$$
\begin{aligned}
&(B_j G_{e'} T)(x_1, \ldots, x_n) && (2.25)\\
&= (G_{e'} T)(x_1, \ldots, x_{j-1}, 1, x_j, \ldots, x_n) && (2.26)\\
&= T(\eta_{e'_1}(x_1, x_n), \ldots, \eta_{e'_{j-1}}(x_{j-1}, x_n), \underbrace{\eta_{e'_j=0}(1, x_n)}_{\frac{1}{2-x_n}}, \eta_{e'_{j+1}}(x_j, x_n), \ldots, \eta_{e'_n}(x_{n-1}, x_n)), && (2.27)
\end{aligned}
$$

.

- For $T \in Q_1(X)$, the cubes $A_1 G_0 T$ and $B_1 G_1 T$ are degenerate.

Checking directly,

$$
\begin{aligned}
A_1 G_0 T(x_1) &= G_0 T(0, x_1) = T(\eta_0(0, x_1)) = T(0), && (2.28)\\
B_1 G_1 T(x_1) &= G_1 T(1, x_1) = T(\eta_1(1, x_1)) = T(1), && (2.29)
\end{aligned}
$$

.

- For $n \geq 2$ and $e \in \mathcal{E}_n$, define $e^j \in \mathcal{E}_{n-1}$ by

$$e^j = (e_1, \ldots, e_{j-1}, e_{j+1}, \ldots, e_n), \quad j \leq n, \tag{2.30}$$

  . Then

$$
\begin{aligned}
e_j = 0 \quad &\Rightarrow \quad A_j G_e(T) = G_{e^j} A_j(T), && (2.31)\\
e_j = 1 \quad &\Rightarrow \quad B_j G_e(T) = G_{e^j} B_j(T), && (2.32)\\
& && (2.33)
\end{aligned}
$$

This is also shown directly:

$$
\begin{aligned}
&A_j G_e T(x, \ldots, x_n) && (2.34)\\
&= G_e T(x_1, \ldots, x_{j-1}, 0, x_j, \ldots, x_n) && (2.35)\\
&= T(\eta_{e_1}(x_1, x_n), \ldots, \eta_{e_{j-1}}(x_{j-1}, x_n), \underbrace{\eta_{e_j=0}(0, x_n)}_{0}, \eta_{e_{j+1}}(x_j, x_n), \ldots, \eta_{e_n}(x_{n-1}, x_n)) && (2.36)\\
&= A_j T(\eta_{e_1}(x_1, x_n), \ldots, \eta_{e_{j-1}}(x_{j-1}, x_n), \eta_{e_{j+1}}(x_j, x_n), \ldots, \eta_{e_n}(x_{n-1}, x_n)) && (2.37)\\
&= G_{e^j} A_j T(x_1, \ldots, x_n). && (2.38)
\end{aligned}
$$

$$
\begin{aligned}
&B_j G_e T(x, \ldots, x_n) && (2.39)\\
&= G_e T(x_1, \ldots, x_{j-1}, 1, x_j, \ldots, x_n) && (2.40)\\
&= T(\eta_{e_1}(x_1, x_n), \ldots, \eta_{e_{j-1}}(x_{j-1}, x_n), \underbrace{\eta_{e_j=1}(1, x_n)}_{1}, \eta_{e_{j+1}}(x_j, x_n), \ldots, \eta_{e_n}(x_{n-1}, x_n)) && (2.41)\\
&= B_j T(\eta_{e_1}(x_1, x_n), \ldots, \eta_{e_{j-1}}(x_{j-1}, x_n), \eta_{e_{j+1}}(x_j, x_n), \ldots, \eta_{e_n}(x_{n-1}, x_n)) && (2.42)\\
&= G_{e^j} B_j T(x_1, \ldots, x_n). && (2.43)
\end{aligned}
$$

Now introduce

$$\Phi_n : Q_n(X) \to Q_{n+1}(X), \tag{2.44}$$
$$\Phi_{n\geq 1}(T) := (-1)^{n+1} \sum_{e\in\mathcal{E}_n} G_e(T), \tag{2.45}$$
$$\Phi_0 = 0, \tag{2.46}$$

.

- For a singular n-cube $T : I^n \to X$, the following holds:

$$\partial_{n+1}\Phi_n(T) + \Phi_{n-1}\partial_n(T) = Sd_n(T) - T + (\text{degenerate cubes}). \tag{2.47}$$

We prove this directly.

$$\partial_{n+1}\Phi_n = (-1)^{n+1} \sum_{e\in\mathcal{E}_n} \partial_{n+1}G_e = (-1)^{n+1} \sum_{i=1}^{n+1}(-1)^i \sum_{e\in\mathcal{E}_n}(A_iG_e - B_iG_e). \tag{2.48}$$

In particular,

$$(-1)^{n+1}(-1)^{n+1} \sum_{e\in\mathcal{E}_n} A_{n+1}G_e = \sum_{e\in\mathcal{E}_n} F_e = Sd_n, \tag{2.49}$$
$$(-1)^{n+1}(-1)^{n+1} \sum_{e\in\mathcal{E}_n} B_{n+1}G_e = 1 + (\text{degenerate cubes}). \tag{2.50}$$

When $n \geq 2$,

$$\Phi_{n-1}\partial_n = \Phi_{n-1} \sum_{i=1}^{n}(-1)^i(A_i - B_i) = (-1)^n \sum_{i=1}^{n} \sum_{e\in\mathcal{E}_{n-1}} (-1)^i(G_eA_i - G_eB_i). \tag{2.51}$$

Therefore it is enough to show that

$$\sum_{i=1}^{n}(-1)^i \sum_{e\in\mathcal{E}_n}(A_iG_e - B_iG_e) - \sum_{i=1}^{n} \sum_{e\in\mathcal{E}_{n-1}} (-1)^i(G_eA_i - G_eB_i) \tag{2.52}$$

is degenerate. For any $i \in \{1, \ldots, n\}$, we can decompose

$$\sum_{e\in\mathcal{E}_n} = \sum_{e\in\mathcal{E}_n, e_i=0} + \sum_{e\in\mathcal{E}_n, e_i=1} \tag{2.53}$$

. Then

$$\sum_{e\in\mathcal{E}_n}(A_iG_e - B_iG_e) - \sum_{e\in\mathcal{E}_{n-1}} (G_eA_i - G_eB_i) \tag{2.54}$$
$$= \sum_{e\in\mathcal{E}_n, e_i=0} (A_iG_e - B_iG_e) + \sum_{e\in\mathcal{E}_n, e_i=1} (A_iG_e - B_iG_e) - \sum_{e\in\mathcal{E}_{n-1}} (G_eA_i - G_eB_i). \tag{2.55}$$

Here,

$$(2.21) \quad \Rightarrow \quad \sum_{e\in\mathcal{E}_n, e_i=1} A_iG_e = \sum_{e\in\mathcal{E}_n, e_i=0} B_iG_e, \tag{2.56}$$

$$(2.31) \quad \Rightarrow \quad \sum_{e\in\mathcal{E}_n, e_i=0} A_iG_e = \sum_{e\in\mathcal{E}_{n-1}} G_eA_j, \tag{2.57}$$

$$(2.32) \quad \Rightarrow \quad \sum_{e\in\mathcal{E}_n, e_i=1} B_i G_e = \sum_{e\in\mathcal{E}_{n-1}} G_e B_j \tag{2.58}$$

. When $n=1$, noting that $\Phi_0 = 0$, it is enough to show that

$$(-1)\sum_{e\in\mathcal{E}_1}(A_1G_e - B_1G_e) = -(A_1G_0 + A_1G_1 - B_1G_0 - B_1G_1) \tag{2.59}$$

is degenerate. This follows because A_1G_0 and B_1G_1 are degenerate, and (2.21) gives $A_1G_1 = B_1G_0$.

Thus Φ_n is, modulo $D_n(X)$, a chain homotopy connecting Sd_n and 1. Since G_e preserves degeneracy, Φ_n also preserves degeneracy. That is, $\Phi_n(D_n(X)) \subset D_{n+1}(X)$. Hence homomorphisms

$$\phi_n : C_n(X) \to C_{n+1}(X) \tag{2.60}$$

are induced, and (2.12) holds. □

- In [3] it is stated that the chain homotopy $\phi = \{\phi_n\}_n$ is natural; hence $\phi_n(T) \in C_{n+1}(X)$ is supported on $T(\Delta^n)$, and therefore its $(n+1)$-dimensional volume is zero. I have not fully understood why naturality of ϕ_n guarantees that $\phi_n T(\Delta^n)$ is supported on $T(\Delta^n)$, but from the explicit form of ϕ_n, namely the expression (2.20) for G_e, it is clear that $\phi_n T(\Delta^n)$ is supported on $T(\Delta^n)$. The question is whether the same property holds for an arbitrary subdivision operator and an arbitrary chain homotopy ϕ_n for it. To prove the same statement for an arbitrary chain homotopy, one presumably uses naturality or related properties.
- In [3] it is remarked that a differential character $f \in \hat{H}(X, \mathbb{R}/\Lambda) = \{f : Z_k(X) \to \mathbb{R}/\Lambda \mid f(\partial c) = \int_c \omega, \omega \in \Lambda^{k+1}, c \in C_{k+1}(X)\}$ is invariant under the subdivision operator. Indeed, let Δ be the subdivision operator and let ψ be a chain homotopy to 1, so that $1 - \Delta = \partial\psi + \psi\partial$. For $x \in Z_k$, we have $f(\Delta x) = f(x) - f(\partial\psi x)$. But $f(\partial\psi x) = \int_{\psi x}\omega$, and since ψx has zero $(k+1)$-dimensional volume in X, $f(\partial\psi x) = 0$. Thus $f(\Delta x) = f(x)$, so the differential character f is invariant under subdivision.

2.1 Memo: Naturality of the Chain Homotopy ψ and the Volume of $\psi(\sigma)$

As a general fact, suppose $\psi : C_k \to C_{k+p}$, $p \geq 1$, is a natural homomorphism. Then for $\sigma : \Delta^k \to X$, we show that $\psi_X(\sigma)$ is supported on $\sigma(\Delta^k)$. Consequently, the $(k+p)$-dimensional volume of $\psi_X(\sigma)$ is zero. The naturality assumption is the commutativity of

$$\begin{array}{ccc} C_k(X) & \xrightarrow{f_*} & C_k(Y) \\ {\scriptstyle\psi_X}\downarrow & & {\scriptstyle\psi_Y}\downarrow \\ C_{k+p}(X) & \xrightarrow{f_*} & C_{k+p}(Y) \end{array} . \tag{2.61}$$

In particular, for

$$\sigma : \Delta^k \to X \tag{2.62}$$

we apply naturality of ψ to the diagram

$$\begin{array}{ccc} C_k(\Delta^k) & \xrightarrow{\sigma_*} & C_k(X) \\ {\scriptstyle\psi_{\Delta^k}}\downarrow & & {\scriptstyle\psi_X}\downarrow \\ C_{k+p}(\Delta^k) & \xrightarrow{\sigma_*} & C_{k+p}(X) \end{array} . \tag{2.63}$$

$$\psi_X(\sigma) = \psi_X \circ \sigma_* \circ \mathrm{id}_{\Delta^k}, \quad \mathrm{id}_{\Delta^k} : \Delta^k \to \Delta^k \tag{2.64}$$

. By commutativity,

$$\psi_X(\sigma) = \sigma_* \circ \psi_{\Delta^k} \circ \mathrm{id}_{\Delta^k} : \Delta^{k+p} \to X \tag{2.65}$$

but since the right-hand side factors through σ, $\psi_X(\sigma)$ is contained in $\sigma(\Delta^k)$. More explicitly,

$$\psi_{\Delta^k}(\mathrm{id}_{\Delta^k}) : \Delta^{k+p} \to \Delta^k = \sum_j a_j \tau_j, \quad \tau_j \tag{2.66}$$

and each $\tau_j(\Delta^{k+p})$ is a subset of Δ^k. Therefore the singular simplex obtained by further applying σ,

$$\sigma_*(\tau_j) : \Delta^{k+p} \to X, \quad \sigma_*(\tau_j) : x \mapsto \sigma(\tau_j(x)) \tag{2.67}$$

is contained in the singular simplex $\sigma(\Delta^k)$.

3 The Eilenberg–Zilber Theorem

In this section we work with ordinary singular chain complexes. As is clear from the proof below, the only property used is that $\tilde{H}_n(\Delta^p) = 0$ for all n (this property is called acyclicity, and this method is called the method of acyclic models). Thus the same argument also applies to normalized cubical singular chains. Let $S_*(X)$ and $S^*(X, R)$ be the singular chain complex and singular cochain complex.

At the cochain level, the cup product is defined for $f, g \in S^*(X)$ by

$$(f \cup g)(c \in C_*(X)) = (f \otimes g)(\rho \circ d_*(c)) \tag{3.1}$$

. Here

$$d : X \to X \times X \tag{3.2}$$

is the diagonal map, and

$$\rho : S_*(X \times Y) \to S_*(X) \otimes S_*(Y) \tag{3.3}$$

is a natural chain map satisfying

$$\rho|_{S_0(X \times Y)}(x, y) = x \otimes y, \tag{3.4}$$

. A map satisfying this condition is called a diagonal approximation. The theorem below says that there is a natural chain homotopy between $S_*(X \times Y)$ and $S_*(X) \otimes S_*(Y)$. We sketch the proof following [2].

Theorem 3.1 (Eilenberg–Zilber theorem). *(i) There exists a natural chain map*

$$\rho : S_*(X \times Y) \to S_*(X) \otimes S_*(Y) \tag{3.5}$$

which in degree 0 *satisfies*

$$\rho(x, y) = x \otimes y \tag{3.6}$$

. Moreover, any two such maps satisfying (3.6) are chain homotopic.

(ii) There exists a natural chain map

$$\kappa : S_*(X) \otimes S_*(Y) \to S_*(X \times Y) \tag{3.7}$$

which in degree 0 *satisfies*

$$\kappa(x \otimes y) = (x, y) \tag{3.8}$$

. Moreover, any two such maps satisfying (3.8) are chain homotopic.

(iii) κ is a chain-homotopy inverse of ρ.

Remark. Naturality means commutativity with the following induced chain maps for

$$f : X \to X', \quad g : Y \to Y' \tag{3.9}$$

:

$$f_* \otimes g_* : S_*(X) \otimes S_*(Y) \to S_*(X') \otimes S_*(Y'), \tag{3.10}$$
$$f \times g : S_*(X \times Y) \to S_*(X' \times Y'). \tag{3.11}$$

(Sketch of proof.) (Existence of ρ.) We construct ρ_n by induction. For $n = 0$, define ρ_0 by (3.6). Assume that natural chain maps $\rho_{i<n}$ have been given:

$$\partial \rho_{r<n} = \rho_{r-1}\partial, \quad (f_* \otimes g_*)\rho_{r<n} = \rho_r(f \times g)_*. \tag{3.12}$$

Introduce the diagonal map

$$d : \Delta^n \to \Delta^n \times \Delta^n, \quad d(x) = (x, x) \tag{3.13}$$

. Since $d \in S_n(\Delta^n \times \Delta^n)$, the commutativity of ρ_{n-1} with ∂ applied to $\partial(d)$ gives

$$\partial \rho_{n-1}\partial(d) = \rho_{n-2}\partial\partial(d) = 0, \quad (n > 1), \tag{3.14}$$
$$\epsilon \rho_0 \partial(d) = 0. \tag{3.15}$$

Since all groups $\tilde{H}_*(\Delta^n) \otimes \tilde{H}_*(\Delta^n)$ are trivial, for $X = \Delta^n$ there exists ρ_n such that

$$\rho_{n-1}\partial(d) = \partial \rho_n(d), \quad \rho_n(d) \in (S_*(\Delta^n) \otimes S_*(\Delta^n))_n = \sum_{p+q=n} S_p(\Delta^n) \otimes S_q(\Delta^n) \tag{3.16}$$

.

Now a singular simplex σ' of $X \times Y$ has the form

$$\sigma' : \Delta^n \to X \times Y, \quad \sigma'(t_0, \dots, t_n) = (\sigma'_X(t_0, \dots, t_n), \sigma'_Y(t_0, \dots, t_n)) \tag{3.17}$$

and can be regarded as a pair (σ, τ) with $\sigma : \Delta^n \to X$ and $\tau : \Delta^n \to Y$. Define $\rho_n(\sigma, \tau)$ by

$$\rho_n(\sigma, \tau) = (\sigma_* \otimes \tau_*)(\rho_n(d)) \in \sum_{p+q=n} S_p(X) \otimes S_q(Y) \tag{3.18}$$

. Then ρ_n is a natural chain map. (To be checked.)

(Existence of κ.) Similarly. For $n = 0$, define it by (3.8). Assume that, for $r < n$, natural chain maps

$$\partial \kappa_{r<n} = \kappa_{r-1}\partial, \quad (f_* \times g_*)\kappa_r = \kappa_r(f_* \otimes g_*) \tag{3.19}$$

have been constructed.

First construct

$$\kappa_{n=p+q} : S_p(\Delta^p) \otimes S_q(\Delta^q) \to S_n(\Delta^n) \tag{3.20}$$

. For $\iota_p = id \in S_p(\Delta^p)$ and $\iota_q = id \in S_q(\Delta^q)$, since all reduced homology groups $\tilde{H}_n(\Delta^p \times \Delta^q)$ are zero,

$$\partial \kappa_{n-1}\partial(\iota_p \otimes \iota_q) = \kappa_{n-1}\partial\partial(\iota_p \otimes \iota_q) = 0 \tag{3.21}$$

we have

$$\kappa_{n-1}\partial(\iota_p \otimes \iota_q) = \partial \kappa_n(\Delta^p \times \Delta^q), \quad \kappa_n(\Delta^p \times \Delta^q) \in S_n(\Delta^q \times \Delta^q) \tag{3.22}$$

for some such element. Thus for a general $\sigma \otimes \tau \in S_p(X) \otimes S_q(Y)$, define

$$\kappa_n(\sigma \otimes \tau) = (\sigma \times \tau)_* \kappa_n(\iota_p \otimes \iota_q) \tag{3.23}$$

. Then κ_n is a natural chain map. (To be checked.)

(Uniqueness of natural chain maps $S_*(X \times Y) \to S_*(X \times Y)$.) For a natural chain map

$$\phi : S(X \times Y) \to S(X \times Y), \quad \phi|_{n=0} = 1, \tag{3.24}$$

we show that there exists a chain homotopy

$$\Phi : S(X \times Y) \to S(X \times Y) \tag{3.25}$$

connecting 1 to ϕ. We argue by induction. Set $\Phi_0 = 0$. Assume that for $1 \leq r \leq n-1$,

$$\partial \Phi_r + \Phi_{r-1}\partial = \phi_r - 1, \quad (f \times g)_* \Phi_r = \Phi_r (f \times g)_* \tag{3.26}$$

have been constructed. Then

$$\begin{aligned}
\partial(-\Phi_{n-1}\partial + \phi_n - 1) &= (-\partial\Phi_{n-1}\partial + \partial\phi_n - \partial) && (3.27)\\
&= ((1 - \phi_{n-1} + \Phi_{n-2}\partial)\partial + \partial\phi_n - \partial) && (3.28)\\
&= (\phi_{n-1}\partial + \partial\phi_n) = 0. && (3.29)
\end{aligned}$$

In particular, for the diagonal map $d \in S_n(\Delta^n \times \Delta^n)$,

$$\partial(-\Phi_{n-1}\partial + \phi_n - 1)(d) = 0 \tag{3.30}$$

and since $\tilde{H}_n(\Delta^n \times \Delta^n) = 0$ for all n, there exists

$$\partial\Phi_n(d) = (-\Phi_{n-1}\partial + \phi_n - 1)(d), \quad \Phi_n(d) \in S_{n+1}(\Delta^n \times \Delta^n) \tag{3.31}$$

. For a general $(\sigma, \tau) \in S_*(X \times Y)$, set

$$\Phi_n(\sigma, \tau) = (\sigma \times \tau)_* \Phi_n(d) \tag{3.32}$$

. Then $\{\Phi_n\}$ is the desired chain homotopy.

(Uniqueness of natural chain maps $S_*(X) \otimes S_*(Y) \to S_*(X) \otimes S_*(Y)$.) Omitted.

(Uniqueness of ρ, κ.) The composites $\rho\kappa$ and $\kappa\rho$ are natural chain maps satisfying $(\rho\kappa)_0 = 1$ and $(\kappa\rho)_0 = 1$, respectively.

$$\kappa\rho \cong 1, \quad \rho\kappa \cong 1. \tag{3.33}$$

Now suppose that another ρ' is given.

3.1 ρ, κ for Cubical Singular Chain Complexes

In general, for maps

$$f : A \to C, \quad g : B \to D \tag{3.34}$$

define

$$f \times g : A \times B \to C \times D, \quad (f \times g)(a, b) := (f(a), g(b)) \tag{3.35}$$

. We now return to the notation of [1]. For singular cubes

$$S : I^p \to X, \quad T : I^q \to Y \tag{3.36}$$

,

$$S \times T : I^{p+q} \to X \times Y, \quad (S \times T)(x_1, \ldots, x_{p+q}) = (S(x_1, \ldots, x_p), T(x_{p+1}, \ldots, x_{p+q})) \tag{3.37}$$

.

For the chain map

$$\kappa : Q_*(X) \otimes Q_*(Y) \to Q_*(X \times Y) \tag{3.38}$$

we may choose, independently of degree,

$$\kappa_{p,q}(S \otimes T) = S \times T, \quad S : I^p \to X, \quad T : I^q \to Y, \tag{3.39}$$

. Since κ sends degenerate cubes to degenerate cubes, $\kappa : C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$ is well-defined. See [1] for further properties of κ.

On the other hand, the chain map ρ has a more complicated expression.

$$\rho_n : Q_n(X \times Y) \to \sum_{p+q=n} Q_p(X) \otimes Q_q(Y) \tag{3.40}$$

We want to define this map. In degree 0 set

$$\rho_0(x, y) = x \otimes y, \quad x \in X, y \in Y. \tag{3.41}$$

We now define the higher-degree part. Let us derive it as an exercise in acyclic models. Note that $\sigma : I^n \to X \times Y$ can be written as a pair

$$(S, T), \quad S : I^n \to X, \quad T : I^n \to Y \tag{3.42}$$

. Assuming the construction up to $r < n$, construct

$$\rho_n : Q_n(I^n \times I^n) \to \sum_{p+q=n} Q_p(I^n) \otimes Q_q(I^n) \tag{3.43}$$

and induce it to

$$Q_*(I^n) \otimes Q_*(I^n) \xrightarrow{S_* \otimes T_*} Q_*(X) \otimes Q_*(Y) \tag{3.44}$$

.

First construct

$$\rho_1 : Q_1(I^1 \times I^1) \to Q_1(I^1) \otimes Q_0(I^1) + Q_0(I^1) \otimes Q_1(I^1) \tag{3.45}$$

.

$$d : I^1 \to I^1 \times I^1, \quad x \mapsto (x, x) \tag{3.46}$$

be the diagonal map. Then

$$\partial \rho_0 \partial(d) = 0. \tag{3.47}$$

Since $\tilde{H}_0(I^1) \otimes \tilde{H}(I^1) = 0$, there exists $\rho_1(d) \in (Q_*(I^1) \otimes Q_*(I^1))_1$ such that

$$\rho_0 \partial(d) = \partial \rho_1(d). \tag{3.48}$$

The left-hand side is

$$\rho_0((1, 1) - (0, 0)) = 1 \otimes 1 - 0 \otimes 0 \tag{3.49}$$

so indeed, choosing

$$\rho_1(d)(x) = x \otimes 1 + 0 \otimes x \tag{3.50}$$

satisfies the condition. For a general map

$$\rho_1 : Q_1(X \times Y) \to Q_1(X) \otimes Q_0(Y) + Q_0(X) \otimes Q_1(Y) \tag{3.51}$$

define

$$\rho_1(S,T) = (S_* \otimes T_*)(\rho_1(d)) \tag{3.52}$$

. That is,

$$\rho_1(S,T)(x) = S(x) \otimes T(1) + S(0) \otimes T(x) \tag{3.53}$$

. Equivalently,

$$\rho_1(S,T) = S \otimes B_1 T + A_1 S \otimes T \tag{3.54}$$

. (One needs to check that this ρ_1 satisfies the desired properties.)

Next construct

$$\rho_2 : Q_2(I^2 \times I^2) \to Q_2(I^2) \otimes Q_0(I^2) + Q_1(I^2) \otimes Q_1(I^2) + Q_0(I^2) \otimes Q_2(I^2) \tag{3.55}$$

.

$$d : I^2 \to I^2 \times I^2, \quad x \mapsto (x,x) \tag{3.56}$$

be the diagonal map. By naturality,

$$\partial\rho_1\partial(d) = \rho_0\partial\partial(d) = 0. \tag{3.57}$$

Since $(\tilde{H}_*(I^2) \otimes \tilde{H}(I^2))_1 = 0$, there exists $\sigma \in (Q_*(I^2) \otimes Q_*(I^2))_2$ such that

$$\rho_1\partial(d) = \partial\sigma. \tag{3.58}$$

The left-hand side is

$$\begin{aligned} \partial d(x) &= d(1,x) - d(0,x) - d(x,1) + d(x,0) && (3.59)\\ &= ((1,x),(1,x)) - ((0,x),(0,x)) - ((x,1),(x,1)) + ((x,0),(x,0)) && (3.60) \end{aligned}$$

and hence

$$\begin{aligned} \rho_1\partial d(x) &= (1,x) \otimes (1,1) - (0,x) \otimes (0,1) - (x,1) \otimes (1,1) + (x,0) \otimes (1,0) && (3.61)\\ &+ (1,0) \otimes (1,x) - (0,0) \otimes (0,x) - (0,1) \otimes (x,1) + (0,0) \otimes (x,0). && (3.62) \end{aligned}$$

$$\partial\sigma(x) = \sigma(1,x) - \sigma(0,x) - \sigma(x,1) + \sigma(x,0) \tag{3.63}$$

. Now

$$\sigma(x_1,x_2) = (x_1,x_2) \otimes (1,1) + (0,0) \otimes (x_1,x_2) - (0,x_1) \otimes (x_2,1) + (x_1,0) \otimes (1,x_2) \tag{3.64}$$

indeed satisfies the condition. For a general map

$$\rho_2 : Q_2(X \times Y) \to (Q_*(X) \otimes Q_*(Y))_2 \tag{3.65}$$

define

$$\rho_2(S,T) = (S_* \otimes T_*)\sigma \tag{3.66}$$

. That is,

$$\rho_2(S,T)(x) = S(x_1,x_2)\otimes T(1,1) + S(0,0)\otimes T(x_1,x_2) - S(0,x_1)\otimes T(x_2,1) + S(x_1,0)\otimes T(1,x_2) \tag{3.67}$$

. Equivalently,

$$\rho_2(S,T) = S\otimes B_1B_2T + A_1A_2S\otimes T - A_1S\otimes B_2T + A_2S\otimes B_1T \tag{3.68}$$

.

In general, the formula is

$$\rho_n : Q_n(X\times Y)\to (Q_*(X)\otimes Q_*(Y))_n, \tag{3.69}$$

$$\rho_n(S,T) = \sum_{J\subset\{1,\dots,n\}} (-1)^J A_{J'}S\otimes B_JT. \tag{3.70}$$

Here the sum over J runs over all subsets of $\{1,\dots,n\}$, and $(-1)^J$ denotes the sign of the permutation $\{1,\dots,n\}\to JJ'$. We write J as the ordered set $J=\{j_1<\dots<j_p\}$, put $J'=\{1,\dots,n\}\backslash J$, and write $A_J = A_{j_1}\cdots A_{j_q}$, $B_J = B_{j_1}\cdots B_{j_p}$, and so on. For example, the sign of $\{1,2\}\to\{2\}\{1\}$ is -1, while the sign of $\{1,2,3,4\}\to\{1,4\}\{2,3\}$ is 1. Let us check that this definition indeed satisfies $\partial\rho_n = \rho_{n-1}\partial$. [3]

$$\partial\rho_n(S,T) = \partial[\sum_{J\subset\{1,\dots,n\}} (-1)^J A_{J'}S\otimes B_JT] \tag{3.87}$$

$$= \sum_{J\subset\{1,\dots,n\}} (-1)^J(\partial A_{J'}S\otimes B_JT + (-1)^{n-|J'|}A_{J'}S\otimes\partial B_JT) \tag{3.88}$$

$$= \sum_{J\subset\{1,\dots,n\}} (-1)^J(\partial A_{J'}S\otimes B_JT + (-1)^{|J|}A_{J'}S\otimes\partial B_JT) \tag{3.89}$$

$$= \sum_{J\subset\{1,\dots,n\}} (-1)^J\left[\sum_{i=1}^{|J|}(-1)^i(A_i-B_i)A_{J'}S\otimes B_JT + (-1)^{|J|}\sum_{i=1}^{n-|J|}(-1)^iA_{J'}S\otimes(A_i-B_i)B_JT\right]. \tag{3.90}$$

[3] $n=2$:

$$\partial\rho_2(S,T) = \partial(S\otimes B_1B_2T + A_1A_2S\otimes T - A_1S\otimes B_2T + A_2S\otimes B_1T) \tag{3.71}$$

$$= \partial S\otimes B_1B_2T + A_1A_2S\otimes\partial T - \partial A_1S\otimes B_2T + A_1S\otimes\partial B_2T + \partial A_2S\otimes B_1T - A_2S\otimes\partial B_1T \tag{3.72}$$

$$= (-A_1S + B_1S + A_2S - B_2S)\otimes B_1B_2T \tag{3.73}$$

$$+ A_1A_2S\otimes(-A_1T + B_1T + A_2T - B_2T) \tag{3.74}$$

$$- (-A_1A_1S + B_1A_1S)\otimes B_2T + A_1S\otimes(-A_1B_2T + B_1B_2T) \tag{3.75}$$

$$+ (-A_1A_2S + B_1A_2S)\otimes B_1T - A_2S\otimes(-A_1B_1T + B_1B_1T) \tag{3.76}$$

$$= (B_1S - B_2S)\otimes B_1B_2T + A_1A_2S\otimes(-A_1T + A_2T) \tag{3.77}$$

$$- (B_1A_1S)\otimes B_2T + A_1S\otimes(-A_1B_2T) + (B_1A_2S)\otimes B_1T - A_2S\otimes(-A_1B_1T) \tag{3.78}$$

$$= (B_1S - B_2S)\otimes B_1B_2T + A_1A_2S\otimes(-A_1T + A_2T) \tag{3.79}$$

$$- (A_1B_2S)\otimes B_2T + A_1S\otimes(-A_1B_2T) + (B_1A_2S)\otimes B_1T - A_2S\otimes(-B_1A_2T), \tag{3.80}$$

$$\rho_1\partial(S,T) = \rho_1(-A_1 + B_1 + A_2 - B_2)(S,T) \tag{3.81}$$

$$= \rho_1(-(A_1S,A_1T) + (B_1S,B_1T) + (A_2S,A_2T) - (B_2S,B_2T)) \tag{3.82}$$

$$= -(A_1S\otimes B_1A_1T + A_1A_1S\otimes A_1T) + (B_1S\otimes B_1B_1T + A_1B_1S\otimes B_1T) \tag{3.83}$$

$$+ (A_2S\otimes B_1A_2T + A_1A_2S\otimes A_2T) - (B_2S\otimes B_1B_2T + A_1B_2S\otimes B_2T) \tag{3.84}$$

$$= -(A_1S\otimes A_1B_2T + A_1A_2S\otimes A_1T) + (B_1S\otimes B_1B_1T + B_1A_2S\otimes B_1T) \tag{3.85}$$

$$+ (A_2S\otimes B_1A_2T + A_1A_2S\otimes A_2T) - (B_2S\otimes B_1B_2T + A_1B_2S\otimes B_2T). \tag{3.86}$$

They indeed agree.

On the other hand,

$$\rho_{n-1}\partial(S,T) = \rho_{n-1}\sum_{i=1}^{n}(-1)^i(A_i - B_i)(S,T) \tag{3.91}$$

$$= \rho_{n-1}\sum_{i=1}^{n}(-1)^i((A_iS, A_iT) - (B_iS, B_iT)) \tag{3.92}$$

$$= \sum_{i=1}^{n}(-1)^i \sum_{J\subset\{1,\ldots,n-1\}} (-1)^J \left[A_{J'}A_iS \otimes B_JA_iT - A_{J'}B_iS \otimes B_JB_iT\right]. \tag{3.93}$$

Recall that for $1 \le i < j \le n$,

$$A_iA_j = A_{j-1}A_i, \quad B_iB_j = B_{j-1}B_i, \quad A_iB_j = B_{j-1}A_i, \quad B_iA_j = A_{j-1}B_i \tag{3.94}$$

. Equivalently, for $n \ge i \ge j \ge 1$,

$$A_iA_j = A_jA_{i+1}, \quad B_iB_j = B_jB_{i+1}, \quad A_iB_j = B_jA_{i+1}, \quad B_iA_j = A_jB_{i+1} \tag{3.95}$$

. Comparing the two sides, there are terms appearing on one side but not the other.

$$\partial\rho_n(S,T) \ni \sum_{J\subset\{1,\ldots,n\}} (-1)^J(-1)^{|J|} \sum_{i=1}^{n-|J|} (-1)^i A_{J'}S \otimes (-B_i)B_JT \tag{3.96}$$

does not appear in $\rho_{n-1}\partial(S,T)$. Looking more carefully, this term cancels with

$$\partial\rho_n(S,T) \ni \sum_{J\subset\{1,\ldots,n\}} (-1)^J \sum_{i=1}^{|J|}(-1)^i A_iA_{J'}S \otimes B_JT \tag{3.97}$$

. We show this explicitly. The sign $(-1)^J$ can be written explicitly using the set $J = \{j_1 < \cdots < j_{|J|}\}$. The sign associated with the permutation

$$12\cdots j_1(j_1+1)\cdots(j_2-1)j_2\cdots \to j_1 12\cdots(j_1-1)(j_1+1)\cdots(j_2-1)j_2\cdots \tag{3.98}$$

is $(-1)^{j_1-1}$. The sign associated with the permutation

$$j_1 12\cdots(j_1-1)(j_1+1)\cdots(j_2-1)j_2\cdots \to j_1j_2 12\cdots(j_1-1)(j_1+1)\cdots(j_2-1)(j_2+2)\cdots \tag{3.99}$$

is $(-1)^{j_2-2}$. Thus

$$(-1)^J = (-1)^{\sum_{a=1}^{|J|}(j_a-a)}. \tag{3.100}$$

Moreover, note that

$$B_iB_J = B_iB_{j_1}\cdots B_{j_{|J|}} = B_{j_1}\cdots B_{j_r}B_{i+r}B_{j_{r+1}}\cdots B_{j_{|J|}}, \quad j_r < i+r < j_{r+1}, \tag{3.101}$$

. Here r is the integer satisfying $j_r < i+r < j_{r+1}$, namely the largest integer satisfying $j_a - a < i$:

$$r = \max\{a \in \{1,\ldots,|J|\} | j_a - a < i\} \tag{3.102}$$

. Write the ordered set corresponding to B_iB_J as

$$J(i) = \{j_1 < \cdots j_r < i+r < j_{r+1} < \cdots < j_{|J|}\} \tag{3.103}$$

. Thus

$$B_iB_J = B_{J(i)}. \tag{3.104}$$

For given J and i, it is enough to show that the signs cancel for $\tilde{J}, \tilde{i}$ satisfying

$$B_i B_J = B_{\tilde{J}}, \quad A_{J'} = A_{\tilde{i}} A_{\tilde{J}'} \tag{3.105}$$

:

$$(-1)^J (-1)^{|J|} (-1)^i = (-1)^{\tilde{J}} (-1)^{\tilde{i}} \tag{3.106}$$

.

$$i + r = \tilde{i} + \tilde{r}, \quad \tilde{i} = r + 1 \tag{3.107}$$

We show that

For example, when

$$J = 1, 2, 5, 6, 8, 10, 12, \cdots \tag{3.108}$$
$$i = 4, \tag{3.109}$$

we have

$$4, 1, 2, 5, 6, 8, 10, 12, \cdots \tag{3.110}$$
$$\to 1, 5, 2, 5, 6, 8, 10, 12, \cdots \tag{3.111}$$
$$\to 1, 2, 6, 5, 6, 8, 10, 12, \cdots \tag{3.112}$$
$$\to 1, 2, 5, 7, 6, 8, 10, 12, \cdots \tag{3.113}$$
$$\to 1, 2, 5, 6, 8, 8, 10, 12, \cdots \tag{3.114}$$
$$\to 1, 2, 5, 6, 8, 9, 10, 12, \cdots = J(i) = \tilde{J}. \tag{3.115}$$

and the process stops there, so $r = 5$. Also

$$i + r = 9 \tag{3.116}$$

. In this case

$$J' = 3, 4, 7, 9, 11, \cdots \tag{3.117}$$

and correspondingly

$$\tilde{J}' = 3, 4, 7, \hat{9}, 11, \cdots \tag{3.118}$$

where $\hat{9}$ means that 9 is skipped. Therefore, to satisfy $\tilde{J}'(\tilde{i}) = J'$, if $\tilde{i}$ moves from left to right $\tilde{r}$ times, it is enough that

$$\tilde{i} + \tilde{r} = i + r \tag{3.119}$$

. Also, the numbers passed by i and $\tilde{i}$ as they move to the right are

$$1, 2, 3, 4, 5, 6, 7, 8 \tag{3.120}$$

and the number of these is $(i + r) - 1$; hence

$$r + \tilde{r} = i + r - 1 \tag{3.121}$$

. Thus (3.107) is shown. The signs are

$$(-1)^{\tilde{J}} (-1)^{\tilde{i}} = (-1)^{J(i)} (-1)^{r+1} = (-1)^{\sum_{a=1}^{r} (j_a - a) + ((i+r) - (r+1)) + \sum_{a=r+1}^{|J|} (j_a - a - 1)} (-1)^{r+1} \tag{3.122}$$
$$= (-1)^J (-1)^{i-1} (-1)^{|J| - r} (-1)^{r+1} = (-1)^J (-1)^{|J|} (-1)^i \tag{3.123}$$

and hence (3.106) holds. [4] Therefore

$$\partial\rho_n(S,T) = \sum_{J\subset\{1,\dots,n\}} (-1)^J \left[\sum_{i=1}^{|J|}(-1)^i(-B_i)A_{J'}S\otimes B_JT + (-1)^{|J|}\sum_{i=1}^{n-|J|}(-1)^i A_{J'}S\otimes A_iB_JT\right] \tag{3.124}$$

$$= \sum_{J\subset\{1,\dots,n\}} (-1)^J \left[(-1)^{|J|}\sum_{i=1}^{n-|J|}(-1)^i A_{\{1,\dots,n\}\setminus J}S\otimes A_iB_JT - \sum_{i=1}^{|J|}(-1)^i B_iA_{\{1,\dots,n\}\setminus J}S\otimes B_JT\right]. \tag{3.125}$$

It remains to show that this agrees with

$$\rho_{n-1}\partial(S,T) = \sum_{i=1}^{n}(-1)^i \sum_{J\subset\{1,\dots,n-1\}} (-1)^J \left[A_{\{1,\dots,n-1\}\setminus J}A_iS\otimes B_JA_iT - A_{\{1,\dots,n-1\}\setminus J}B_iS\otimes B_JB_iT\right] \tag{3.126}$$

. Here, to avoid confusion, we have explicitly written that for $J\subset\{1,\dots,n-1\}$, J' means $J' = \{1,\dots,n-1\}\setminus J$. Again, we do not check the complete term-by-term agreement including existence or nonexistence of terms; we only check the agreement of signs.

Let us check the agreement of the first terms. First find $\tilde{J},\tilde{i}$ such that

$$A_iB_J = B_{\tilde{J}}A_{\tilde{i}} \tag{3.127}$$

. For example, when

$$J = 1,2,5,6,8,10,12 = n, \tag{3.128}$$
$$i = 4, \tag{3.129}$$

we have

$$4,1,2,5,6,8,10,12 \tag{3.130}$$
$$\to 1,5,2,5,6,8,10,12 \tag{3.131}$$
$$\to 1,2,6,5,6,8,10,12 \tag{3.132}$$
$$\to 1,2,5,7,6,8,10,12 \tag{3.133}$$
$$\to 1,2,5,6,8,8,10,12 \tag{3.134}$$
$$\to 1,2,5,6,8,\underbrace{9}_{i+r},10,12 \tag{3.135}$$
$$\to 1,2,5,6,8,9,9,12 \tag{3.136}$$
$$\to 1,2,5,6,8,9,11,9 \tag{3.137}$$

. After $i+r$, each time the term moves to the right, j_a decreases by one, $j_a \mapsto j_a - 1$. Thus, for given J,i, using the same definition (3.102) of r as before,

$$j_r < i+r < j_{r+1} \tag{3.138}$$

,

$$ij_1\cdots j_{|J|} \to j_1\cdots j_r(i+r)j_{r+1}\cdots j_{|J|} \to j_1\cdots j_r(j_{r+1}-1)\cdots(j_{|J|}-1)(i+r). \tag{3.139}$$

(By the definition of r, note that $j_r < j_{r+1}-1$.) Hence

$$\tilde{i} = i+r, \quad \tilde{J} = \{j_1,\dots,j_r,(j_{r+1}-1),\dots,(j_{|J|}-1)\}. \tag{3.140}$$

At this point,

$$A_{\{1,\dots,n\}\setminus J} = A_{\{1,\dots,n-1\}\setminus\tilde{J}}A_{\tilde{i}} \tag{3.141}$$

[4] Strictly speaking, one also needs to handle cases such as $i+r > |J|$, but we omit this.

is automatically satisfied. In the example above,

$$\{1,\ldots,n-1\}\backslash\tilde{J}=\{3,4,7,10\} \tag{3.142}$$

so

$$3,4,7,10,{\color{red}9}\to 3,4,7,{\color{red}9},11. \tag{3.143}$$

On the other hand,

$$\{1,\ldots,n\}\backslash J=\{3,4,7,9,11\}. \tag{3.144}$$

Thus it remains only to check the agreement of signs.

$$(-1)^{\tilde{i}}(-1)^{\tilde{J}}=(-1)^{i+r}(-1)^{\sum_{a=1}^{r}(j_a-a)+\sum_{a=r+1}^{|J|}(j_a-a-1)}=(-1)^{i+r}(-1)^{J}(-1)^{-(|J|-r)}=(-1)^{i}(-1)^{J}(-1)^{|J|} \tag{3.145}$$

and hence they agree. The second terms are handled similarly.

4 Cup Products

At the cochain level, the cup product is defined by

$$(f\cup g)(c\in C_*(X))=(f\otimes g)(\rho\circ d_*(c)) \tag{4.1}$$

where

$$d:X\to X\times X \tag{4.2}$$

is the diagonal map. There is some freedom in the choice of $\rho:C_*(X\times Y)\to C_*(X)\otimes C_*(Y)$, but if we take the map from the preceding section,

$$\rho_n(S,T)=\sum_{J\subset\{1,\ldots,n\}}(-1)^J A_{J'}S\otimes B_J T \tag{4.3}$$

then for a singular cube $T:I^n\to X$,

$$d_*(T)=(T,T):I^n\to X\times X, \tag{4.4}$$

$$\rho\circ d_*(T)=\rho(T,T)=\sum_{J\subset\{1,\ldots,n\}}(-1)^J A_{J'}T\otimes B_J T. \tag{4.5}$$

Therefore

$$(f\cup g)(T)=(f\otimes g)(\rho\circ d_*(T))=\sum_{J\subset\{1,\ldots,n\}}(-1)^J f(A_{J'}T)g(B_J T). \tag{4.6}$$

If we look only at fixed degrees, for $f^p\in C^p(X,R)$, $g^q\in C^q(X,R)$, and $T_{p+q}:I^{p+q}\to X$,

$$(f^p\cup g^q)(T_{p+q})=\sum_{J\subset\{1,\ldots,p+q\},|J|=p}(-1)^J f^p(A_{J'}T_{p+q})g^q(B_J T_{p+q}). \tag{4.7}$$

In particular,

$$(f^0\cup g^n)(T_n)=\sum_{J\subset\{1,\ldots,n\},|J|=0}(-1)^J f^0(A_{J'}T_n)g^n(B_J T_n) \tag{4.8}$$

$$=f^0(T_n(0,\ldots,0))g^n(T_n), \tag{4.9}$$

$$(f^n \cup g^0)(T_n) = \sum_{J\subset\{1,\ldots,n\},|J|=n} (-1)^J f^n(A_{J'}T_n)g^0(B_JT_n) \tag{4.10}$$
$$= f^n(T_n)g^0(T_n(1,\ldots,1)). \tag{4.11}$$

For $p = q = 1$, we get

$$(f^1 \cup g^1)(T_2) = \sum_{J\subset\{1,2\},|J|=1} (-1)^J f^1(A_{J'}T_2)g^1(B_JT_2) \tag{4.12}$$
$$= f^1(A_2T_2)g^1(B_1T_2) - f^1(A_1T_2)g^1(B_2T_2) \tag{4.13}$$
$$= f^1(T_2(x,0))g^1(T_2(1,x)) - f^1(T_2(0,x))g^1(T_2(x,1)) \tag{4.14}$$

. [5]

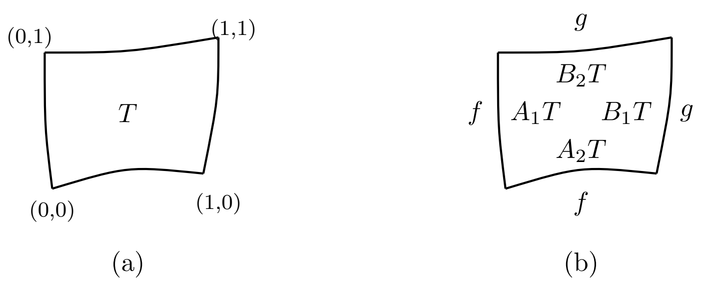

Figure 1: The cup product in the case $p = q = 1$.

5 The de Rham Map and Cup Products

For a differential form

$$\omega \in \Omega^n(X) \tag{5.1}$$

,

$$R : \Omega^n(X) \to C(X,\mathbb{R}), \tag{5.2}$$
$$\omega \mapsto R\omega : \left(T \to \int_T \omega\right) \in C^n(X,\mathbb{R}), \quad T : I^n \to X \in C_n(X), \tag{5.3}$$

is called the de Rham map.

- By the properties of differential forms, the cochain $R\omega^n \in C^n(X,\mathbb{R})$ obtained from the de Rham map, unlike a general $f \in C^n(X,\mathbb{R})$, is invariant under changes of the "parametrization" of $T : I^n \to X$, namely replacement by an orientation-preserving self-diffeomorphism $\phi : I^n \to I^n$:

$$R\omega^n(T) = R\omega^n(T \circ \phi). \tag{5.4}$$

  Thus the value $R\omega^n(T)$ is determined only by the "shape" of $T : I^n \to X$.
- R is a natural chain map.
- The de Rham map is invariant under the subdivision operator $\Delta : C_*(X) \to C_*(X)$. [6]

$$\Delta^* R\omega = R\omega. \tag{5.5}$$

[5]Compare this with the formula using the Alexander–Whitney map for ordinary singular chain complexes,

$$(f^1 \cup g^1)(012) = f^1(01)g(12) \tag{4.15}$$

.

[6]The subdivision operator is the one defined in Section 2; here we write $sd = \{sd_n\}_n$ as Δ.

This is because subdivision corresponds to subdividing the domain of integration:

$$(\Delta^* R\omega)(T) = R\omega(\Delta T) = \int_{\Delta T} \omega = \int_T \omega = R\omega(T) \tag{5.6}$$

. Here we used the equality of sets $\Delta T(I^n) = T(I^n)$.

5.1 Cup Products and the Subdivision Operator

The cup product and the de Rham map do not commute, but one expects an expression resembling a discrete approximation to an integral.

$$\omega^p \in \Omega^p(X), \theta^q \in \Omega^q(X) \tag{5.7}$$

and estimate

$$R(\omega^p \wedge \theta^q) - R\omega^p \cup R\theta^q \tag{5.8}$$

.

$$R(\omega^p \wedge \theta^q)(T) = \int_T \omega^p \wedge \theta^q = \int_{I^{p+q}} T^*(\omega^p \wedge \theta^q) = \int_{I^{p+q}} T^*\omega^p \wedge T^*\theta^q. \tag{5.9}$$

$$\begin{aligned}
(R\omega^p \cup R\theta^q)(T) &= \sum_{J\subset\{1,\ldots,n\},|J|=p} (-1)^J R\omega^p(A_{J'}T)R\theta^q(B_J T) && (5.10)\\
&= \sum_{J\subset\{1,\ldots,n\},|J|=p} (-1)^J \int_{A_{J'}T} \omega^p \int_{B_J T} \theta^q && (5.11)\\
&= \sum_{J\subset\{1,\ldots,n\},|J|=p} (-1)^J \int_{A_{J'}I^{p+q}} T^*\omega^p \int_{B_J I^{p+q}} T^*\theta^q. && (5.12)
\end{aligned}$$

Thus the comparison reduces to a comparison for differential forms on the unit cube I^{p+q}. That is, for

$$\omega, \theta \in \Omega^*(I^{p+q}) \tag{5.13}$$

we need to estimate

$$\int_{I^{p+q}} \omega^p \wedge \theta^q - \sum_{J\subset\{1,\ldots,n\},|J|=p} (-1)^J \int_{A_{J'}I^{p+q}} \omega^p \int_{B_J I^{p+q}} \theta^q \tag{5.14}$$

.

For example, in the case $p = q = 1$, write

$$\begin{aligned}
\omega &= \omega_x(x, y)dx + \omega_y(x, y)dy, && (5.15)\\
\theta &= \theta_x(x, y)dx + \theta_y(x, y)dy && (5.16)
\end{aligned}$$

. Then

$$R(\omega \wedge \theta)(\mathrm{id}) = \int_{[0,1]^{\times 2}} (\omega_x(x, y)\theta_y(x, y) - \omega_y(x, y)\theta_x(x, y))dxdy. \tag{5.17}$$

On the other hand,

$$(R\omega^1 \cup R\theta^1)(\mathrm{id}) = \int_{[0,1]^{\times 2}} (\omega_x(x, 0)\theta_y(1, y) - \omega_y(0, y)\theta_x(x, 1))dxdy. \tag{5.18}$$

Thus the cup product of differential forms may be viewed as an expression approximating the integral by endpoint values. Naively, the subdivision operator should improve the accuracy of this approximation. Writing this out in the case $p = q = 1$,

$$\begin{aligned}
&\Delta^*(R\omega^1 \cup R\theta^1)(\mathrm{id}) && (5.19)\\
&= \int_{[0,1/2]\times[0,1/2]} (\omega_x(x,0)\theta_y(1/2,y) - \omega_y(0,y)\theta_x(x,1/2))dxdy && (5.20)\\
&+ \int_{[1/2,1]\times[0,1/2]} (\omega_x(x,0)\theta_y(1,y) - \omega_y(1/2,y)\theta_x(x,1/2))dxdy && (5.21)\\
&+ \int_{[0,1/2]\times[1/2,1]} (\omega_x(x,1/2)\theta_y(1/2,y) - \omega_y(0,y)\theta_x(x,1))dxdy && (5.22)\\
&+ \int_{[1/2,1]\times[1/2,1]} (\omega_x(x,1/2)\theta_y(1,y) - \omega_y(1/2,y)\theta_x(x,1))dxdy. && (5.23)
\end{aligned}$$

Therefore one expects, at least naively, that

$$\lim_{n\to\infty} (\Delta^*)^n (R\omega \cup R\theta) = R(\omega \wedge \theta) \tag{5.24}$$

holds [3]. [7]

Using the limiting formula (5.24) and the chain homotopy for the subdivision operator,

$$1 - \Delta = \partial\psi + \psi\partial \tag{5.25}$$

define

$$E : \Omega^p(X) \otimes \Omega^q(X) \to C^{p+q-1}(X) \tag{5.26}$$

by

$$E(\omega,\theta) := -\sum_{n=0}^{\infty} (\Delta^*)^n \psi^* (R\omega \cup R\theta), \tag{5.27}$$

that is,

$$E(\omega,\theta)(c \in C_*(X)) := -\sum_{n=0}^{\infty} (R\omega \cup R\theta)(\psi\Delta^n c) \tag{5.28}$$

as in [3].

- One needs to check whether the infinite sum in (5.28) converges. [3] states: "A straightforward estimate shows that the right hand side of (1.8) is dominated by a geometric series and hence convergences."

[7] In [3], it is stated that cubical singular chain complexes are used for the limiting formula (5.24), and (5.24) itself is attributed to [4]. However, I could not determine whether [4] depends on some property of cubical singular chain complexes rather than the usual simplicial singular chain complex. At first glance, [4] itself also appears to use a construction based on the simplicial singular chain complex.

Now, noting that R and Δ are natural chain maps and that ψ is natural,

$$\begin{aligned}
\delta E(\omega,\theta)(c) &= E(\omega,\theta)(\partial c) && (5.29)\\
&= -\sum_{n=0}^{\infty}(\Delta^*)^n\psi^*(R\omega\cup R\theta)(\partial c) && (5.30)\\
&= -\sum_{n=0}^{\infty}(R\omega\cup R\theta)(\psi\Delta^n\partial c) && (5.31)\\
&= -\sum_{n=0}^{\infty}(R\omega\cup R\theta)(\psi\partial\Delta^n c) && (5.32)\\
&= -\sum_{n=0}^{\infty}(R\omega\cup R\theta)((1-\Delta-\partial\psi)\Delta^n c) && (5.33)\\
&= -\sum_{n=0}^{\infty}(R\omega\cup R\theta)((\Delta^n-\Delta^{n+1})c)+\sum_{n=0}^{\infty}(R\omega\cup R\theta)(\partial\psi\Delta^n c) && (5.34)\\
&= -\lim_{N\to\infty}(R\omega\cup R\theta)((1-\Delta^{N+1})c)+\sum_{n=0}^{\infty}\delta(R\omega\cup R\theta)(\psi\Delta^n c) && (5.35)\\
&= -(R\omega\cup R\theta)(c)+\lim_{N\to\infty}(\Delta^*)^{N+1}(R\omega\cup R\theta)(c)+\sum_{n=0}^{\infty}(\delta R\omega\cup R\theta+(-1)^{|\omega|}R\omega\cup\delta R\theta)(\partial\psi\Delta^n c) && (5.36)\\
&= -(R\omega\cup R\theta)(c)+R(\omega\wedge\theta)(c)+\sum_{n=0}^{\infty}(Rd\omega\cup R\theta+(-1)^{|\omega|}R\omega\cup Rd\theta)(\partial\psi\Delta^n c) && (5.37)\\
&= -(R\omega\cup R\theta)(c)+R(\omega\wedge\theta)(c)-E(d\omega,\theta)(c)-(-1)^{|\omega|}E(\omega,d\theta)(c). && (5.38)
\end{aligned}$$

Thus

$$R(\omega\wedge\theta)-R\omega\cup R\theta=\delta E(\omega,\theta)+E(d\omega,\theta)+(-1)^{|\omega|}E(\omega,d\theta), \tag{5.39}$$

or equivalently,

$$R(\omega\wedge\theta)-R\omega\cup R\theta=\delta E(\omega\otimes\theta)+Ed(\omega\otimes\theta) \tag{5.40}$$

.

5.1.1 On the Convergence of the Infinite Sum (5.24)

To examine convergence of (5.24), consider the case $\omega,\theta\in\Omega^1(X)$.

$$\begin{aligned}
E(\omega,\theta)(T:I\to X) &= -\sum_{n=0}^{\infty}(R\omega\cup R\theta)(\psi\Delta^n T) && (5.41)\\
&= -\sum_{n=0}^{\infty}\left[R\omega(A_2\psi\Delta^n T)R\theta(B_1\psi\Delta^n T)-R\omega(A_1\psi\Delta^n T)R\theta(B_2\psi\Delta^n T)\right]. && (5.42)
\end{aligned}$$

The explicit form of ψ from (2.45) is

$$\begin{aligned}
\psi(T) &= \sum_{e\in\mathcal{E}_1}G_e(T)=G_0(T)+G_1(T), && (5.43)\\
\psi(T)(x_1,x_2) &= T(\eta_0(x_1,x_2))+T(\eta_1(x_1,x_2)). && (5.44)
\end{aligned}$$

Thus for a general $\sigma:I\to X$,

$$\begin{aligned}
(R\omega\cup R\theta)(\psi\sigma) &= (R\omega\cup R\theta)(G_0\sigma+G_1\sigma) && (5.45)\\
&= R\omega(A_2G_0\sigma)R\theta(B_1G_0\sigma)-R\omega(A_1G_0\sigma)R\theta(B_2G_0\sigma) && (5.46)\\
&\quad+R\omega(A_2G_1\sigma)R\theta(B_1G_1\sigma)-R\omega(A_1G_1\sigma)R\theta(B_2G_1\sigma). && (5.47)
\end{aligned}$$

Here note that

$$A_1G_0\sigma : x \mapsto \sigma(\eta_0(0,x)) = \sigma(0), \tag{5.48}$$
$$B_1G_1\sigma : x \mapsto \sigma(\eta_1(1,x)) = \sigma(1), \tag{5.49}$$
$$B_2G_1\sigma : x \mapsto \sigma(\eta_1(x,1)) = \sigma(1), \tag{5.50}$$

so only the first term remains:

$$(R\omega \cup R\theta)(\psi\sigma) = R\omega(A_2G_0\sigma)R\theta(B_1G_0\sigma) \tag{5.51}$$
$$= R\omega(x \mapsto \sigma(\eta_0(x,0)))R\theta(x \mapsto \sigma(\eta_0(1,x))) \tag{5.52}$$
$$= R\omega(x \mapsto \sigma(\frac{x}{2}))R\theta(x \mapsto \sigma(\frac{1}{2-x})). \tag{5.53}$$

Here we use that a cochain obtained from the de Rham map is independent of the parametrization of $T : I^1 \to X$. Therefore $(R\omega \cup R\theta)(\psi\sigma)$ is the product of the line integral of ω along the first half of the path $\sigma(x \in [0,1/2])$ and the line integral of θ along the second half $\sigma(x \in [1/2,1])$. We estimate this from above. Write the maximum of a one-form and the length of a path as

$$\mu_{\omega,\sigma} = \max_{v \in T_yX, |v|=1, y \in \sigma([0,1])} |\omega(v)|, \tag{5.54}$$
$$\ell_{\sigma,[a,b]} = \int_a^b \|\frac{d\sigma(t)}{dt}\| dt \tag{5.55}$$

and so on. Then

$$|(R\omega \cup R\theta)(\psi\sigma)| \le \mu_{\omega,\sigma}\mu_{\theta,\sigma}\ell_{\sigma,[0,1/2]}\ell_{\sigma,[1/2,1]} \le \mu_{\omega,\sigma}\mu_{\theta,\sigma}(\frac{\ell_\sigma}{2})^2. \tag{5.56}$$

Here

$$\ell_\sigma = \ell_{\sigma,[0,1]} = \ell_{\sigma,[0,1/2]} + \ell_{\sigma,[1/2,1]} \tag{5.57}$$

and we used the arithmetic–geometric mean inequality

$$\ell_{\sigma,[0,1/2]}\ell_{\sigma,[1/2,1]} \le (\frac{\ell_\sigma}{2})^2 \tag{5.58}$$

. Since the subdivision operator Δ only subdivides the domain of integration, similarly

$$|(R\omega \cup R\theta)(\psi\Delta^n\sigma)| \le \mu_{\omega,\sigma}\mu_{\theta,\sigma}\sum_{p=0}^{2^n-1}\ell_{\sigma,[\frac{2p}{2^{n+1}},\frac{2p+1}{2^{n+1}}]}\ell_{\sigma,[\frac{2p+1}{2^{n+1}},\frac{2p+2}{2^{n+1}}]} \tag{5.59}$$
$$\le \mu_{\omega,\sigma}\mu_{\theta,\sigma}\frac{1}{4}\sum_{p=0}^{2^n-1}\left(\ell_{\sigma,[\frac{2p}{2^{n+1}},\frac{2p+1}{2^{n+1}}]} + \ell_{\sigma,[\frac{2p+1}{2^{n+1}},\frac{2p+2}{2^{n+1}}]}\right)^2 \tag{5.60}$$
$$= \mu_{\omega,\sigma}\mu_{\theta,\sigma}\frac{1}{4}2^n\sum_{p=0}^{2^n-1}\frac{1}{2^n}\left(\ell_{\sigma,[\frac{2p}{2^{n+1}},\frac{2p+1}{2^{n+1}}]} + \ell_{\sigma,[\frac{2p+1}{2^{n+1}},\frac{2p+2}{2^{n+1}}]}\right)^2 \tag{5.61}$$
$$\le \mu_{\omega,\sigma}\mu_{\theta,\sigma}\frac{1}{4}2^n\left(\sum_{p=0}^{2^n-1}\frac{1}{2^n}(\ell_{\sigma,[\frac{2p}{2^{n+1}},\frac{2p+1}{2^{n+1}}]} + \ell_{\sigma,[\frac{2p+1}{2^{n+1}},\frac{2p+2}{2^{n+1}}]})\right)^2 \tag{5.62}$$
$$= \mu_{\omega,\sigma}\mu_{\theta,\sigma}\frac{1}{4}2^n\left(\frac{1}{2^n}\ell_\sigma\right)^2 \tag{5.63}$$
$$= \mu_{\omega,\sigma}\mu_{\theta,\sigma}\frac{1}{4}2^{-n}(\ell_\sigma)^2. \tag{5.64}$$

Here we used the inequality for the convex function $x \mapsto x^2$,

$$\sum_p a_p x^2 \le (\sum_p a_p x)^2, \quad \sum_p a_p = 1 \tag{5.65}$$

. Therefore the infinite sum (5.41) is absolutely convergent for fixed T.

The convergence for general $\omega \in \Omega^p$ and $\theta \in \Omega^q$ should also be discussed, but we leave it for later.

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