

Chain Complexes, Cochain Complexes, and Descent Equations

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I follow the computation in Sec. IV of [2]. I also referred to [1, 3].

1 Chain complexes and cochain complexes

Let Λ be a discrete subset of \mathbb{R}^D with no accumulation points. (Would a finite set also be sufficient?) For $n \geq 0$, we say that $A = \{A_{p_0 \dots p_n}\}_{p_0, \dots, p_n \in \Lambda}$ is an n -chain if $A_{p_0 \dots p_n}$ is completely antisymmetric in p_0, \dots, p_n and decays exponentially when one moves away from the diagonal $p_0 = p_1 = \dots = p_n$. We write the set of n -chains as $C_n(\Lambda)$. Define the boundary operator by

$$\partial : C_n(\Lambda) \rightarrow C_{n-1}(\Lambda), \quad (\partial A)_{p_1 \dots p_n} := \sum_{p_0 \in \Lambda} A_{p_0 \dots p_n}. \quad (1)$$

Because the sum over $p_0 \in \Lambda$ gives at most an $O(1)$ factor, ∂A also preserves the property of exponential decay. By the complete antisymmetry of $A_{p_0 \dots p_n}$,

$$(\partial^2 A)_{p_2 \dots p_n} = \sum_{p_1 \in \Lambda} (\partial A)_{p_1 \dots p_n} = \sum_{p_0, p_1 \in \Lambda} A_{p_0 p_1 \dots p_n} = 0. \quad (2)$$

Thus $\partial^2 = 0$, so $\bigoplus_{n \geq 0} C_n(\Lambda)$ is a chain complex.

We say that $\alpha : \Lambda^{n+1} \rightarrow \mathbb{R}$ is an n -cochain if the function $\alpha(p_0, \dots, p_n)$ is bounded and completely antisymmetric in p_0, \dots, p_n , and moreover we impose a condition on α so that the pairing between $C_n(\Lambda)$ and $C^n(\Lambda)$,

$$\langle A, \alpha \rangle = \frac{1}{(n+1)!} \sum_{p_0, \dots, p_n} A_{p_0 \dots p_n} \alpha(p_0, \dots, p_n), \quad (3)$$

converges. For example, when $n = 1$, if one writes formally

$$\langle A, \alpha \rangle = \frac{1}{2} \sum_{p, q} A_{pq} \alpha(p, q), \quad (4)$$

then the neighborhood of the diagonal contributes. Writing

$$\langle A, \alpha \rangle \sim \sum_p \sum_\delta A_{p, p+\delta} \alpha(p, p+\delta), \quad (5)$$

we see that, if $A_{p, p+\delta} \alpha(p, p+\delta)$ were $O(1)$ for arbitrary p , the sum would diverge. Here we impose the following condition:

- the intersection between the support of α and a neighborhood of the diagonal subset of Λ^{n+1} is a finite set.

This seems to be the property called cocontrolledness [1].

The coboundary operator

$$\delta : C^n(\Lambda) \rightarrow C^{n+1}(\Lambda) \quad (6)$$

is defined by

$$\langle A, \delta\alpha \rangle := \langle \partial A, \alpha \rangle. \quad (7)$$

The right-hand side is

$$\frac{1}{(n+1)!} \sum_{p_1, \dots, p_{n+1}} (\partial A)_{p_1 \dots p_{n+1}} \alpha(p_1, \dots, p_{n+1}) = \frac{1}{(n+1)!} \sum_{p_0, p_1, \dots, p_{n+1}} A_{p_0 \dots p_n} \alpha(p_1, \dots, p_{n+1}) \quad (8)$$

$$= (n+2) \frac{1}{(n+2)!} \sum_{p_0, p_1, \dots, p_{n+1}} A_{p_0 \dots p_n} \alpha(p_1, \dots, p_{n+1}), \quad (9)$$

and hence

$$(\delta\alpha)(p_0, \dots, p_{n+1}) \sim (n+2)\alpha(p_1, \dots, p_{n+1}). \quad (10)$$

Antisymmetrizing this expression, one obtains

$$(\delta\alpha)(p_0, \dots, p_{n+1}) = \sum_{j=0}^{n+1} (-1)^j \alpha(p_0, \dots, p_{j-1}, p_{j+1}, \dots, p_{n+1}). \quad (11)$$

For example, the coboundaries of 0- and 1-cochains are

$$(\delta\alpha)(p_0, p_1) = \alpha(p_1) - \alpha(p_0), \quad (12)$$

$$(\delta\alpha)(p_0, p_1, p_2) = \alpha(p_1, p_2) - \alpha(p_0, p_2) + \alpha(p_0, p_1). \quad (13)$$

The cup product between cochains,

$$\cup : C^n(\Lambda) \times C^m(\Lambda) \rightarrow C^{n+m}(\Lambda), \quad (14)$$

is defined by antisymmetrizing the expression

$$\alpha(p_0, \dots, p_n) \gamma(p_{n+1}, \dots, p_{n+m}). \quad (15)$$

Namely,

$$(\alpha \cup \gamma)(p_0, \dots, p_{n+m}) = \frac{1}{(n+m+1)!} \sum_{\sigma \in S_{n+m+1}} (-1)^{\text{sgn}\sigma} \alpha(p_{\sigma(0)}, \dots, p_{\sigma(n)}) \gamma(p_{\sigma(n+1)}, \dots, p_{\sigma(n+m)}). \quad (16)$$

Since $\text{sgn}(\sigma\sigma') = \text{sgn}\sigma \text{sgn}\sigma'$, it is clear that antisymmetry is preserved. For example, the cup product of a 0-cochain and a 1-cochain is

$$(\alpha \cup \gamma)(p_0, p_1) = \frac{1}{2} (\alpha(p_0) \gamma(p_0, p_1) - \alpha(p_1) \gamma(p_1, p_0)). \quad (17)$$

The cup product of a 1-cochain and a 1-cochain is

$$(\alpha \cup \gamma)(p_0, p_1, p_2) = \frac{1}{3!} (\alpha(p_0, p_1) \gamma(p_1, p_2) + \alpha(p_1, p_2) \gamma(p_2, p_0) + \alpha(p_2, p_0) \gamma(p_0, p_1) \quad (18)$$

$$- \alpha(p_1, p_0) \gamma(p_0, p_2) - \alpha(p_2, p_1) \gamma(p_1, p_0) - \alpha(p_0, p_2) \gamma(p_2, p_1)). \quad (19)$$

Also, the sum defining $\alpha \cup \gamma$ contains the term

$$(-1)^{\frac{(n+m)(n+m+1)}{2}} \alpha(p_{n+m}, \dots, p_m) \gamma(p_m, \dots, p_0). \quad (20)$$

Since

$$(-1)^{\frac{(n+m)(n+m+1)}{2}} \alpha(p_{n+m}, \dots, p_m) \gamma(p_m, \dots, p_0) \quad (21)$$

$$= (-1)^{\frac{(n+m)(n+m+1)}{2}} (-1)^{\frac{n(n+1)}{2}} (-1)^{\frac{m(m+1)}{2}} \gamma(p_0, \dots, p_m) \alpha(p_m, \dots, p_{n+m}) \quad (22)$$

$$= (-1)^{nm} \gamma(p_0, \dots, p_m) \alpha(p_m, \dots, p_{n+m}), \quad (23)$$

we have

$$\alpha \cup \gamma = (-1)^{nm} \gamma \cup \alpha. \quad (24)$$

(One also has to show that $\alpha \cup \gamma \propto \gamma \cup \alpha$.) The relation

$$\delta(\alpha \cup \gamma) = \delta\alpha \cup \gamma + (-1)^n \alpha \cup \delta\gamma \quad (25)$$

holds. Let us show this directly. We have

$$(\delta(\alpha \cup \gamma))(p_0, \dots, p_{n+m+1}) \quad (26)$$

$$= \sum_{j=0}^{n+m+1} (-1)^j (\alpha \cup \gamma)(p_0, \dots, p_{j-1}, p_{j+1}, \dots, p_{n+m+1}), \quad (27)$$

and this is

$$\sum_{j=0}^n (-1)^j \alpha(p_0, \dots, p_{j-1}, p_{j+1}, \dots, p_{n+1}) \gamma(p_{n+1}, \dots, p_{n+m+1}) \quad (28)$$

$$+ \alpha(p_0, \dots, p_n) \sum_{j=n+1}^{n+m+1} (-1)^j \gamma(p_n, \dots, p_{j-1}, p_{j+1}, \dots, p_{n+m+1}) \quad (29)$$

$$= \sum_{j=0}^{n+1} (-1)^j \alpha(p_0, \dots, p_{j-1}, p_{j+1}, \dots, p_{n+1}) \gamma(p_{n+1}, \dots, p_{n+m+1}) - (-1)^{n+1} \alpha(p_0, \dots, p_n) \gamma(p_{n+1}, \dots, p_{n+m+1}) \quad (30)$$

$$+ \alpha(p_0, \dots, p_n) \sum_{j=n}^{n+m+1} (-1)^j \gamma(p_n, \dots, p_{j-1}, p_{j+1}, \dots, p_{n+m+1}) - (-1)^n \alpha(p_0, \dots, p_n) \gamma(p_{n+1}, \dots, p_{n+m+1}) \quad (31)$$

$$= \sum_{j=0}^{n+1} (-1)^j \alpha(p_0, \dots, p_{j-1}, p_{j+1}, \dots, p_{n+1}) \gamma(p_{n+1}, \dots, p_{n+m+1}) \quad (32)$$

$$+ (-1)^n \alpha(p_0, \dots, p_n) \sum_{j=n}^{n+m+1} (-1)^{j-n} \gamma(p_n, \dots, p_{j-1}, p_{j+1}, \dots, p_{n+m+1}) \quad (33)$$

$$= (\delta\alpha)(p_0, \dots, p_{n+1}) \gamma(p_{n+1}, \dots, p_{n+m+1}) + (-1)^n \alpha(p_0, \dots, p_n) (\delta\gamma)(p_n, \dots, p_{n+m+1}). \quad (34)$$

Antisymmetrizing gives the desired formula.

As usual, define

$$B_n(\Lambda) = \text{Im} [\partial : C_{n+1}(\Lambda) \rightarrow C_n(\Lambda)], \quad (35)$$

$$Z_n(\Lambda) = \text{Ker} [\partial : C_n(\Lambda) \rightarrow C_{n-1}(\Lambda)], \quad (36)$$

$$H_n(\Lambda) = Z_n(\Lambda)/B_n(\Lambda), \quad (37)$$

$$B^n(\Lambda) = \text{Im} [\delta : C^{n-1}(\Lambda) \rightarrow C^n(\Lambda)], \quad (38)$$

$$Z^n(\Lambda) = \text{Ker} [\delta : C^n(\Lambda) \rightarrow C^{n+1}(\Lambda)], \quad (39)$$

$$H^n(\Lambda) = Z^n(\Lambda)/B^n(\Lambda). \quad (40)$$

The pairing defines

$$\langle \cdot, \cdot \rangle : H_n(\Lambda) \times H^n(\Lambda) \rightarrow \mathbb{R}. \quad (41)$$

Also,

$$\delta((\alpha + \delta\beta) \cup (\gamma + \delta\epsilon)) = \delta(\alpha + \delta\beta) \cup (\gamma + \delta\epsilon) + (-1)^n (\alpha + \delta\beta) \cup \delta(\gamma + \delta\epsilon) \quad (42)$$

$$= \delta\alpha \cup (\gamma + \delta\epsilon) + (-1)^n (\alpha + \delta\beta) \cup \delta\gamma \quad (43)$$

$$= \delta(\alpha \cup \gamma) + \delta(\alpha \cup \delta\epsilon + (-1)^n \beta \cup \delta\gamma), \quad (44)$$

so the cup product is well-defined as

$$\cup : H^n(\Lambda) \times H^m(\Lambda) \rightarrow H^{n+m}(\Lambda). \quad (45)$$

Suppose that, in addition to having no accumulation points, Λ has the property that for every point of \mathbb{R}^D there is a point of Λ within distance δ . Then it seems that

$$H^n(\Lambda) \cong H^n(\mathbb{R}^D) = \begin{cases} \mathbb{R} & (n = D), \\ 0 & (\text{else}), \end{cases} \quad (46)$$

holds. $H^D(\Lambda)$ is generated by

$$\delta f_1 \cup \cdots \cup \delta f_D. \quad (47)$$

Here f_j is a function satisfying

$$f_j(x_1, \dots, x_D) = 1 \quad \text{for} \quad x_j \gg 0, \quad (48)$$

$$f_j(x_1, \dots, x_D) = 0 \quad \text{for} \quad x_j \ll 0. \quad (49)$$

Note that

$$(\delta f_j)(p, q) = f_j(q) - f_j(p). \quad (50)$$

2 Descent equations

Above, the cochains were \mathbb{R} -valued. Now consider an n -cochain with values in $(n+2)$ -forms on a manifold X ,

$$F^{(n+2)} = \{F_{p_0 \cdots p_n}^{(n+2)}\}. \quad (51)$$

Starting from a 0-cochain $F_p^{(2)}$ valued in 2-forms, define the $(n+2)$ -form-valued n -cochain $F^{(n+2)}$ by the descent equation

$$dF^{(n+1)} = \partial F^{(n+2)}, \quad n \geq 1. \quad (52)$$

Writing the Λ indices explicitly, the descent equation is

$$dF_{p_1 \cdots p_n}^{(n+1)} = \sum_{p_0 \in \Lambda} F_{p_0 \cdots p_n}^{(n+2)}. \quad (53)$$

When $F^{(n+1)}$ is given, a solution of the descent equation has the ambiguity of adding an n -cycle:

$$F^{(n+2)} \mapsto F^{(n+2)} + B^{(n+2)}, \quad \partial B^{(n+2)} = 0. \quad (54)$$

The pairing defines an $(n+2)$ -form,

$$\langle F^{(n+2)}, \alpha \rangle \in \Omega^{(n+2)}(X). \quad (55)$$

By the descent equation, the exterior derivative is

$$d \langle F^{(n+2)}, \alpha \rangle = \langle dF^{(n+2)}, \alpha \rangle = \langle \partial F^{(n+3)}, \alpha \rangle = \langle F^{(n+3)}, \delta \alpha \rangle. \quad (56)$$

A sufficient condition for the integral $\int_{C_n} \langle F^{(n+2)}, \alpha \rangle$ to be independent of the choice of cycle $C_n \in Z_n(X)$ is $\delta \alpha = 0$, namely that α is an n -cocycle, $\alpha \in Z^n(\Lambda)$. On the other hand, when α is a coboundary, $\alpha = \delta \beta$, we have

$$\langle F^{(n+2)}, \delta \beta \rangle = \langle \partial F^{(n+2)}, \beta \rangle = \langle dF^{(n+1)}, \beta \rangle = d \langle F^{(n+1)}, \beta \rangle, \quad (57)$$

and hence the integral over a cycle $C_n \in Z_n(X)$ vanishes. Therefore only $[\alpha] \in H^n(\Lambda)$ can give a nontrivial value upon integration over a cycle $C_n \in Z_n(X)$. $H^n(\Lambda)$ is nontrivial only in degree $n = D$, and its generator was commented on above. Thus the only closed form whose integral has meaning is

$$\Omega^{(D+2)}(f_1, \dots, f_D) = \langle F^{(D+2)}, \delta f_1 \cup \dots \cup \delta f_D \rangle. \quad (58)$$

Let g be a function with finite support. When we change $f_1 \mapsto f_1 + g$, the change in $\Omega^{(D+2)}(f_1, \dots, f_D)$ is

$$\Omega^{(D+2)}(g, f_2, \dots, f_D) = \langle F^{(D+2)}, \delta g \cup \delta f_2 \cup \dots \cup \delta f_D \rangle. \quad (59)$$

Since g has finite support, $g \cup \delta f_2 \cup \dots \cup \delta f_D$ is a D -cochain, and therefore

$$\Omega^{(D+2)}(g, f_2, \dots, f_D) = \langle F^{(D+2)}, \delta(g \cup \delta f_2 \cup \dots \cup \delta f_D) \rangle \quad (60)$$

$$= \langle \partial F^{(D+2)}, g \cup \delta f_2 \cup \dots \cup \delta f_D \rangle \quad (61)$$

$$= \langle dF^{(D+1)}, g \cup \delta f_2 \cup \dots \cup \delta f_D \rangle \quad (62)$$

$$= d \langle F^{(D+1)}, g \cup \delta f_2 \cup \dots \cup \delta f_D \rangle. \quad (63)$$

This is exact, so the integral is invariant.

The issue is the effect of the ambiguity $F^{(D+2)} \mapsto F^{(D+2)} + B^{(D+2)}$, $\partial B^{(D+2)} = 0$, on the integral. It gives

$$\langle B^{(D+2)}, \delta f_1 \cup \dots \cup \delta f_D \rangle. \quad (64)$$

This is nothing but the pairing, with values in $(D+2)$ -forms, $H^D(\Lambda) \times H_D(\Lambda) \rightarrow \Omega^{(D+2)}(X)$, so in general it does not vanish and is not even a closed form. For example, when $D = 1$,

$$\langle B^{(3)}, \delta f \rangle = \frac{1}{2} \sum_{pq} B_{pq}^{(3)} (f(q) - f(p)). \quad (65)$$

The difference $f(q) - f(p)$ is cocontrolled, but $f(p)$ and $f(q)$ are not individually cocontrolled. One would like to compute, for instance,

$$\sum_{pq} B_{pq}^{(3)} f(q) = \sum_q f(q) \sum_p B_{pq}^{(3)} = 0, \quad (66)$$

but if the order of summation is exchanged, then

$$\sum_q B_{pq}^{(3)} f(q) \quad (67)$$

diverges.

According to [3], when one starts from

$$F_p^{(2)} = \frac{i}{2} \oint \frac{dz}{2\pi i} \text{Tr}(GdHG^2 dH_p), \quad (68)$$

it is written that Kapustin and Spodyneiko argued that

$$\partial B^{(n+2)} = 0 \quad (69)$$

has only trivial solutions

$$B^{(n+2)} = \partial C^{(n+1)} \quad (70)$$

(citation [48] in [3]). Accepting this, the ambiguity of the solution of the descent equation becomes

$$F^{(n+2)} + \partial C^{(n+2)}. \quad (71)$$

Correspondingly, the descent equation determining $F^{(n+3)}$ is

$$d(F^{(n+2)} + \partial C^{(n+2)}) = \partial F^{(n+3)}, \quad (72)$$

or equivalently

$$dF^{(n+2)} = \partial(F^{(n+3)} - dC^{(n+2)}). \quad (73)$$

Since $d^2 = 0$, the ambiguity $\partial C^{(n+2)}$ does not affect the descent equation that determines $F^{(n+4)}$. Formally, if one thinks of $F^{(2)}$ as also being determined from $F^{(1)}$ by the descent equation

$$dF^{(1)} = \partial F^{(2)}, \quad (74)$$

then it is natural to assume the ambiguity

$$F_p^{(2)} \rightarrow F^{(2)} + \partial C^{(2)}. \quad (75)$$

The ambiguity of $F^{(n+2)}$ for $n > 0$ is given by

$$F^{(n+2)} \rightarrow F^{(n+2)} + \partial C^{(n+2)} + dC^{(n-1)} \quad (76)$$

according to [3].

References

- [1] Anton Kapustin, Lev Spodyneiko, *Thermal Hall conductance and a relative topological invariant of gapped two-dimensional systems*, arXiv:1905.06488.
- [2] Anton Kapustin and Lev Spodyneiko, *Higher-dimensional generalizations of Berry curvature*, PHYSICAL REVIEW B 101, 235130 (2020).
- [3] Xueda Wen, Marvin Qi, Agnes Beaudry, Juan Moreno, Markus J. Pflaum, Daniel Spiegel, Ashvin Vishwanath, Michael Hermele, *Flow of (higher) Berry curvature and bulk-boundary correspondence in parametrized quantum systems*, arXiv:2112.07748.