Memo: A Direct Proof of Quantization of $\mathbb{Z}$ Topological Invariants

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May 31, 2026

- Add the relation between d and δ.

We directly show that the Chern number

$$ch = \left(\frac{i}{2\pi}\right)^n \int_{M_{2n}} \mathrm{tr}[F^n] \tag{1}$$

is quantized. It is enough to show that ch does not change under an infinitesimal deformation. In other words, we show that the variation of the integrand is an exact form.

There are several equivalent expressions, up to overall constants:

$$ch \sim \int_{M_{2n}\times\mathbb{R}} \mathrm{tr}[GdG^{-1}]^{2n+1}, \tag{2}$$

$$ch \sim \int_{M_{2n}} \mathrm{tr}[P(dP)^{2n}]. \tag{3}$$

For $n=1$ and a one-dimensional projector P, this reduces to

$$ch \sim \int_{M_2} \langle d\psi | d\psi \rangle \,. \tag{4}$$

1 $\mathrm{tr}[F^n]$

$$\delta \mathrm{tr}[F^n] = d\, n\, \mathrm{tr}[\delta A\, F^{n-1}]. \tag{5}$$

Here δA may be regarded as a global 1-form on M_{2n}. Indeed, the gauge field A is defined patchwise, and the patching is given by a gauge transformation

$$A^U = A^V + d\chi \quad \text{on} \quad U \cap V. \tag{6}$$

Since an infinitesimal deformation $\delta A = A' - A$ does not change the topological sector, the patching function χ may be kept fixed:

$$\delta\chi = 0. \tag{7}$$

Then on overlaps,

$$\delta A^U = \delta A^V \quad \text{on} \quad U \cap V, \tag{8}$$

so the infinitesimal variation of the gauge field is a global 1-form,

$$\delta A \in \Omega^{(1)}(M_{2n}). \tag{9}$$

Therefore

$$n\, \mathrm{tr}[\delta A\, F^{n-1}] \in \Omega^{(2n-1)}(M_{2n}) \tag{10}$$

is a global $(2n-1)$-form, and hence

$$\delta \mathrm{tr}[F^n] = d\, n\, \mathrm{tr}[\delta A\, F^{n-1}] \tag{11}$$

is exact.

2 $\mathrm{tr}[GdG^{-1}]^{2n+1}$

Let

$$G : M_{2n} \times \mathbb{R} \to GL_m(\mathbb{C}). \tag{12}$$

Then

$$\begin{aligned}
&\frac{1}{2n+1}\delta\mathrm{tr}[GdG^{-1}]^{2n+1} &(13)\\
&= \mathrm{tr}[\delta(GdG^{-1})(GdG^{-1})^{2n}] &(14)\\
&= \mathrm{tr}[(\delta G\, dG^{-1} + Gd\delta G^{-1})(GdG^{-1})^{2n}] &(15)\\
&= \mathrm{tr}[(\delta G\, dG^{-1} - dG\,\delta G^{-1})(GdG^{-1})^{2n} - G\delta G^{-1} d(GdG^{-1})^{2n}] &(16)\\
&\quad + d\,\mathrm{tr}[G\delta G^{-1}(GdG^{-1})^{2n}]. &(17)
\end{aligned}$$

Using

$$d(GdG^{-1}) = dG\, dG^{-1} = -GdG^{-1}GdG^{-1}, \tag{18}$$

we have

$$d(GdG^{-1})^{2n} = 0. \tag{19}$$

Moreover,

$$\begin{aligned}
&\mathrm{tr}[(\delta G\, dG^{-1} - dG\,\delta G^{-1})(GdG^{-1})^{2n}] &(20)\\
&= \mathrm{tr}[(-G\delta G^{-1}GdG^{-1} + GdG^{-1}G\delta G^{-1})(GdG^{-1})^{2n}] = 0. &(21)
\end{aligned}$$

Thus

$$\delta\mathrm{tr}[GdG^{-1}]^{2n+1} = d(2n+1)\mathrm{tr}[G\delta G^{-1}(GdG^{-1})^{2n}]. \tag{22}$$

3 $\mathrm{tr}[P(dP)^{2n}]$

First, it is closed:

$$d\mathrm{tr}[P(dP)^{2n}] = \mathrm{tr}[dP(dP)^{2n}]. \tag{23}$$

Using $P = P^2$, hence $dP = dPP + PdP$ and $PdPP = 0$, this becomes

$$\mathrm{tr}[(dPP + PdP)(dP)^{2n}] = 2\mathrm{tr}[P(dP)^{2n+1}] = 2\mathrm{tr}[P^2(dP)^{2n+1}] = 2\mathrm{tr}[P(dP)^{2n+1}P] = 2\mathrm{tr}[P(dP)P(dP)^{2n}] = 0. \tag{24}$$

4 $\langle d\psi | d\psi \rangle$

Let the patch transformation be

$$|\psi^U\rangle = |\psi^V\rangle\, e^{i\theta} \quad \text{on} \quad U \cap V. \tag{25}$$

Then the Berry connection transforms as

$$\langle \psi^U | d\psi^U \rangle = \langle \psi^V | d\psi^V \rangle + id\theta. \tag{26}$$

Under an infinitesimal deformation of the frame $|\psi\rangle$, the topological sector is not changed, so the patch transformation may be kept fixed:

$$\delta\theta = 0. \tag{27}$$

Then

$$\delta \langle \psi^U | d\psi^U \rangle = \delta \langle \psi^V | d\psi^V \rangle \quad \text{on} \quad U \cap V. \tag{28}$$

Hence the infinitesimal variation of the Berry connection is a global 1-form,

$$\delta \langle \psi | d\psi \rangle \in \Omega^{(1)}(M_2). \tag{29}$$

Therefore

$$\delta \langle d\psi | d\psi \rangle = d\, \delta \langle \psi | d\psi \rangle \tag{30}$$

is exact.