Note: Bound-State Solutions of a One-Dimensional Dirac Hamiltonian

Ken Shiozaki

May 31, 2026

Consider the Hamiltonian

$$\mathcal{H} = -i\partial_x \tau_z + m\big(\cos\theta(x)\tau_x + \sin\theta(x)\tau_y\big), \quad m > 0. \tag{1}$$

We find the localized solution when $\theta(x)$ has a kink,

$$\theta(x) = \begin{cases} 0 & (x < 0), \\ \Delta\theta & (x > 0), \end{cases} \quad -\pi < \Delta\theta < \pi. \tag{2}$$

Let E be the energy eigenvalue. For $x < 0$, assume that the wave function behaves as

$$\phi(x) \propto e^{\kappa x}, \quad \kappa > 0. \tag{3}$$

The characteristic equation is

$$\begin{vmatrix} -i\kappa - E & m \\ m & i\kappa - E \end{vmatrix} = \kappa^2 + E^2 - m^2 = 0. \tag{4}$$

Thus

$$\kappa = \sqrt{m^2 - E^2}. \tag{5}$$

The eigenfunction is

$$\begin{pmatrix} m \\ E + i\kappa \end{pmatrix} e^{\kappa x} \sim \begin{pmatrix} 1 \\ e^{i\varphi_E} \end{pmatrix} e^{\kappa x}, \tag{6}$$

where we set

$$e^{i\varphi_E} = \frac{E}{m} + i\frac{\kappa}{m}, \quad 0 < \varphi_E < \pi. \tag{7}$$

For $x > 0$, first note that

$$\mathcal{H} = e^{-i\tau_z \Delta\theta/2} \left[-i\partial_x \tau_z + m\tau_x\right] e^{i\tau_z \Delta\theta/2}. \tag{8}$$

For a localized solution $\phi(x) \propto e^{-\kappa' x}$, the characteristic equation is

$$\begin{vmatrix} i\kappa' - E & m \\ m & -i\kappa' - E \end{vmatrix} = (\kappa')^2 + E^2 - m^2 = 0. \tag{9}$$

Hence

$$\kappa' = \kappa = \sqrt{m^2 - E^2}. \tag{10}$$

The eigenfunction is

$$e^{-i\tau_z \Delta\theta/2} \begin{pmatrix} m \\ E - i\kappa \end{pmatrix} e^{-\kappa x} \sim \begin{pmatrix} 1 \\ e^{i\Delta\theta - i\varphi_E} \end{pmatrix} e^{-\kappa x}. \tag{11}$$

The continuity of the eigenfunction at $x = 0$ gives the condition

$$2\varphi_E = \Delta\theta. \tag{12}$$

Solving it, one obtains

$$\varphi_E = \begin{cases} \frac{\Delta\theta}{2} & (0 < \Delta\theta < \pi), \\ \pi + \frac{\Delta\theta}{2} & (-\pi < \Delta\theta < 0). \end{cases} \tag{13}$$

Therefore the bound-state eigenfunction and energy are

$$\phi_{\text{bound}}(x) \sim \begin{cases} \begin{pmatrix} 1 \\ e^{i\Delta\theta/2} \end{pmatrix} e^{-\kappa|x|}, \quad \kappa = m\left|\sin\frac{\Delta\theta}{2}\right|, \quad E = m\cos\frac{\Delta\theta}{2} & (0 < \Delta\theta < \pi), \\ \begin{pmatrix} 1 \\ -e^{i\Delta\theta/2} \end{pmatrix} e^{-\kappa|x|}, \quad \kappa = m\sin\left(-\frac{\Delta\theta}{2}\right), \quad E = -m\cos\frac{\Delta\theta}{2} & (-\pi < \Delta\theta < 0). \end{cases} \tag{14}$$