Note: Winding Numbers (Draft)

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For a map

$$q : M_{2n+1} \to U(n), \tag{1}$$

the $(2n+1)$-dimensional winding number is defined by

$$W_{2n+1}[q] = \frac{(-1)^n n!}{(2n+1)!} \left(\frac{i}{2\pi}\right)^{n+1} \int_{M_{2n+1}} \mathrm{tr}\,[q^\dagger dq]^{2n+1} \tag{2}$$

$$= \frac{(-1)^n n!}{(2n+1)!} \left(\frac{i}{2\pi}\right)^{n+1} \int_{M_{2n+1}} \mathrm{tr}\,[-d(q^\dagger dq)(q^\dagger dq)^{2n-1}]. \tag{3}$$

Since $q^\dagger dq = -dq^\dagger q$, note that

$$W_{2n+1}[q^\dagger] = -W_{2n+1}[q]. \tag{4}$$

1 Additivity

Let

$$U, V : M_{2n+1} \to U(n) \tag{5}$$

and set

$$q = UV^\dagger. \tag{6}$$

Since

$$q^\dagger dq = V(U^\dagger dU - V^\dagger dV)V^\dagger, \tag{7}$$

we have

$$\mathrm{tr}\,[q^\dagger dq]^{2n+1} = \mathrm{tr}\,[X - Y]^{2n+1}, \tag{8}$$

where

$$X = U^\dagger dU, \quad Y = V^\dagger dV. \tag{9}$$

We use identities such as

$$X^2 = -dX, \quad Y^2 = -dY, \tag{10}$$

$$X dX = dX X, \quad Y dY = dY Y, \tag{11}$$

$$dX^2 = 0, \quad dY^2 = 0. \tag{12}$$

For $2n+1 = 1$, we immediately obtain

$$W_1[UV^\dagger] = W_1[U] - W_1[V]. \tag{13}$$

For $2n+1=3$, since

$$\mathrm{tr}\,[(X-Y)^3] = \mathrm{tr}\,[X^3 - Y^3 + 3d(XY)], \tag{14}$$

we also get

$$W_3[UV^\dagger] = W_3[U] - W_3[V]. \tag{15}$$

Let M_2 be a two-dimensional space cobordant to a point, and let

$$\widetilde{q} : X_3 \to U(n) \tag{16}$$

be an extension of $q : M_2 \to U(n)$. Writing the WZW term as

$$WZW_2[q] = \frac{1}{24\pi^2} \int_{X_3} \mathrm{tr}\,[\widetilde{q}^\dagger d\widetilde{q}]^3 \in \mathbb{R}/2\pi\mathbb{Z}, \tag{17}$$

one obtains the Polyakov–Wiegmann formula

$$WZW_2[UV^\dagger] = WZW_2[U] - WZW_2[V] + \frac{1}{8\pi^2} \int_{M_2} \mathrm{tr}\,[U^\dagger dU\, V^\dagger dV]. \tag{18}$$

The generalization to the case where M_2 is not cobordant to a point must be treated carefully. For example, related discussions can be found in [1].

For $2n+1=5$,

$$\begin{aligned} \mathrm{tr}\,[(X-Y)^5] = \mathrm{tr}\,[&X^5 - Y^5 - 5X^4Y + 5X^3Y^2 + 5X^2YXY + 5XY^4 && (19)\\ &- 5X^2Y^3 - 5XYXY^2]. && (20)\end{aligned}$$

Using

$$\begin{aligned} d(XY^3) &= dXY^3 - XdYY^2 = -X^2Y^3 + XY^4, && (21)\\ d(X^3Y) &= dXX^2Y - X^3dY = -X^4Y + X^3Y^2, && (22)\\ d(XYXY) &= dXYXY - XdYXY + XYdXY - XYXdY && (23)\\ &= -X^2YXY + XY^2XY - XYX^2Y + XYXY^2 \sim_{\mathrm{tr}} -2X^2YXY + 2XYXY^2, && (24)\end{aligned}$$

we obtain

$$\mathrm{tr}\,[(X-Y)^5] = \mathrm{tr}\,[X^5] - \mathrm{tr}\,[Y^5] + 5d\,\mathrm{tr}\,\left[X^3Y + XY^3 - \frac{1}{2}XYXY\right]. \tag{25}$$

This gives

$$W_5[UV^\dagger] = W_5[U] - W_5[V], \tag{26}$$

and

$$\begin{aligned} WZW_4[UV^\dagger] = WZW_4[U] &- WZW_4[V] \\ &+ \frac{(-1)^2 2!}{(2n)!}\left(\frac{i}{2\pi}\right)^3 \int_{M_4} \mathrm{tr}\,\big[(U^\dagger dU)^3 V^\dagger dV + U^\dagger dU (V^\dagger dV)^3 && (27)\\ &-\frac{1}{2} U^\dagger dU V^\dagger dV U^\dagger dU V^\dagger dV\big]. && (28)\end{aligned}$$

2 When $q \in O(n)$

Let $q : M_3 \to O(n)$. We would like to show

$$W_3[q] \in 2\mathbb{Z}. \tag{29}$$

References

[1] Domenico Monaco and Clément Tauber, "Gauge-theoretic invariants for topological insulators: A bridge between Berry, Wess-Zumino, and Fu-Kane-Mele", arXiv:1611.05691.