Notes on *Topological terms of (2+1)d flag-manifold sigma models*

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- Calculation of the second Chern number.
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1.1 Effective action

We have

$$\tilde{\Omega}_4^{\mathrm{Spin}^c}(S^2) = \mathbb{Z}. \tag{1}$$

If A is a Spin^c field and $a = -iz^\dagger dz$ is the $U(1)$ field defined by $\boldsymbol{n} = z^\dagger \boldsymbol{\sigma} z$, $[z] \in \mathbb{C}P^1$, then the theta term is given by [4]

$$\left(\frac{1}{8\pi^2}\int (F+f)^2 + \frac{\sigma}{8}\right) - \left(\frac{1}{8\pi^2}\int F^2 + \frac{\sigma}{8}\right) = \frac{1}{8\pi^2}\int f(f+2F). \tag{2}$$

The effective action of the corresponding $(2+1)$-dimensional invertible phase is

$$\frac{1}{4\pi}\int ada + \frac{1}{2\pi}\int Ada. \tag{3}$$

The first term is known as the Hopf term, and it means that the wave function acquires the phase (-1) under a 2π rotation of a skyrmion [1]. The second term means that the skyrmion carries Spin^c charge one. As a comment, compared with $\Omega_*^{\mathrm{Spin}^c}(S^2)$, the first and second terms are not separately bordism invariants.

- As a fact, for the $\mathbb{C}P^1$ model one has identically $f^2 = 0$.
- Unlike the case of $BU(1)$, when a comes from $\mathbb{C}P^1$, the integral $\int ada$ is a homotopy invariant.
- Since $\Omega_3^{SO}(S^2) = 0$, $\int ada$ is not a bordism invariant. This is not in contradiction with the fact that the Hopf map $S^3 \to S^2$ is null-bordant, because it is realized as the boundary of $\mathbb{C}P_2^2$.
- Abanov argues that one way to derive the Hopf term $\int ada$ is to first embed $[z] \in \mathbb{C}P^1$ into $\mathbb{C}P^M$ with $M > 1$. Then $\int ada$ is no longer a homotopy invariant, and the term $\int ada$ can be obtained by varying the action [5].

1.2 Lattice model and numerical calculation

A naive lattice model is as follows. I write only the single-particle Hamiltonian:

$$\mathcal{H}(k_x, k_y, \boldsymbol{n}) \tag{4}$$

$$= \sin k_x \tau_x + \sin k_y \tau_y + n_1 \sigma_x \tau_z + n_2 \sigma_y \tau_z + (n_3 - 2 + \cos k_x + \cos k_y)\sigma_z \tau_z \tag{5}$$

$$= e^{ik_x}\frac{\sigma_z\tau_z - i\tau_x}{2} + e^{-ik_x}\frac{\sigma_z\tau_z + i\tau_x}{2} + e^{ik_y}\frac{\sigma_z\tau_z - i\tau_y}{2} + e^{-ik_y}\frac{\sigma_z\tau_z + i\tau_y}{2} + n_1\sigma_x\tau_z + n_2\sigma_y\tau_z + (n_3 - 2)\sigma_z\tau_z. \tag{6}$$

The real-space representation is

$$\mathcal{H}(x-1, y; x, y) = \frac{\sigma_z\tau_z - i\tau_x}{2}, \tag{7}$$

$$\mathcal{H}(x+1, y; x, y) = \frac{\sigma_z\tau_z + i\tau_x}{2}, \tag{8}$$

$$\mathcal{H}(x, y-1; x, y) = \frac{\sigma_z\tau_z - i\tau_y}{2}, \tag{9}$$

$$\mathcal{H}(x, y+1; x, y) = \frac{\sigma_z\tau_z + i\tau_y}{2}, \tag{10}$$

$$\mathcal{H}(x, y; x, y) = n_1\sigma_x\tau_z + n_2\sigma_y\tau_z + (n_3 - 2)\sigma_z\tau_z. \tag{11}$$

Matrix elements that are not defined above are zero. A skyrmion configuration can be introduced as follows[1]. For sites $x \in \{1, \dots, L_x\}$ and $y \in \{1, \dots, L_y\}$, take

$$n_1(x, y) = \sin\left(\frac{2\pi x}{L_x} - \pi\right) \cos\left(\frac{\pi y}{L_y} - \frac{\pi}{2}\right), \tag{12}$$

$$n_2(x, y) = \sin\left(\frac{2\pi y}{L_y} - \pi\right) \cos^2\left(\frac{\pi x}{L_x} - \frac{\pi}{2}\right), \tag{13}$$

$$n_3(x, y) = 1 - 2\cos^2\left(\frac{\pi x}{L_x} - \frac{\pi}{2}\right) \cos^2\left(\frac{\pi y}{L_y} - \frac{\pi}{2}\right). \tag{14}$$

We compute whether the number of electrons in the ground state changes depending on whether a skyrmion is present. Numerical calculations for the above model show that, with $L_x L_y$ sites,

- without a skyrmion: $2L_x L_y$;
- with a skyrmion: $2L_x L_y - 1$.

1.3 Berry phase

Consider a parameter-dependent Hamiltonian $\hat{H}(t)$. Assume that after one period it is unitarily equivalent to the original Hamiltonian[2]. Here we take

$$\hat{H}(T) = \hat{H}(0). \tag{15}$$

Assume further that, for any value of the parameter, there is a finite energy gap between the ground state and the first excited state of $\hat{H}(t)$. Assume also that the ground state is nondegenerate, and write the instantaneous ground state of $\hat{H}(t)$ as $|\Psi(t)\rangle$:

$$\hat{H}(t) |\Psi(t)\rangle = E(t) |\Psi(t)\rangle . \tag{16}$$

Then, in the adiabatic limit, the time evolution of the ground state over one period is

$$Te^{-i\int_0^T \hat{H}(t)dt} |\Psi(t=0)\rangle \sim e^{-i\int_0^T E(t)dt} e^{i\gamma} |\Psi(t=0)\rangle . \tag{17}$$

Here $e^{i\gamma}$ is the Berry phase,

$$e^{i\gamma} = \prod_t \langle \Psi(t+\delta t)|\Psi(t)\rangle . \tag{18}$$

We compute the Berry phase of the ground state when the skyrmion is rotated by 2π. The Hamiltonian is

$$\hat{H}(t) = \sum_{x,y} c_x^\dagger \mathcal{H}_{x,y}(\boldsymbol{n}(t)) c_y, \tag{19}$$

$$\boldsymbol{n}(t) = \left(\cos\frac{2\pi t}{T} n_1 - \sin\frac{2\pi t}{T} n_2,\ \sin\frac{2\pi t}{T} n_1 + \cos\frac{2\pi t}{T} n_2,\ n_3\right). \tag{20}$$

Here n_1, n_2, n_3 are the skyrmion configuration given above. The ground state is the state in which the negative-energy eigenstates are occupied. Write the eigenvectors of the single-particle Hamiltonian $H(t)$ as $u_j(t)$:

$$H(t) u_j = \epsilon_j u_j, \quad \epsilon_1 \le \epsilon_2 \le \cdots . \tag{21}$$

[1] For example, [3] gives a short explanation.

[2] In general, the Berry phase can be defined as long as the Hamiltonian is unitarily equivalent after one period.

Then

$$|\Psi(t)\rangle = \prod_{\epsilon_j<0} (c^\dagger u_j(t)) |0\rangle . \tag{22}$$

As is well known, the overlap of single-particle Slater determinants is given by a determinant,

$$\langle \Psi(t+\delta t)|\Psi(t)\rangle = \det[\Psi(t+\delta t)^\dagger \Psi(t)]. \tag{23}$$

Here

$$\Psi(t) = (u_1(t), \dots, u_M(t)), \quad \epsilon_{j\le M}(t) < 0, \quad \epsilon_{j>M}(t) > 0. \tag{24}$$

Since we are now interested only in the $U(1)$ phase, the eigenvectors $u_j(t)$ need not be normalized.

Numerically, one obtains the Berry phase $e^{i\gamma} = -1$.

- Can one show that the trivial contribution coming from the ground-state energy does not contribute to the effective action $\frac{1}{4\pi} ada$?

1.4 Second Chern number

Regard the Hamiltonian as a Hamiltonian on $T^2 \times S^2$ and compute the second Chern number ch_2 of the model. Parametrize $\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Then

$$\mathcal{H}(k_x, k_y, \theta, \phi) = \sin k_x \tau_x + \sin k_y \tau_y + \sin\theta\cos\phi\, \sigma_x\tau_z + \sin\theta\sin\phi\, \sigma_y\tau_z + (\cos\theta - 2 + \cos k_x + \cos k_y)\sigma_z\tau_z . \tag{25}$$

Consider how to numerically compute the second Chern number for a Dirac model $\mathcal{H} = \sum_{\mu=1}^{5} h_\mu \gamma_\mu$.

$$ch_2 = \left(-\frac{1}{2}\right)^5 \frac{1}{2} \left(\frac{i}{2\pi}\right)^2 \int \mathrm{tr}\,[\hat{h}\, d\hat{h}\, d\hat{h}\, d\hat{h}\, d\hat{h}], \quad \hat{h} = \mathrm{sgn}\, h = \frac{h}{|h|}. \tag{26}$$

Since

$$d\hat{h} = d\frac{h}{|h|} = \frac{dh}{|h|} - \frac{h\, d|h|}{|h|^2}, \tag{27}$$

we get

$$\begin{aligned}
&\mathrm{tr}\left[\frac{h}{|h|}\left(\frac{dh}{|h|} - \frac{h\,d|h|}{|h|^2}\right)\left(\frac{dh}{|h|} - \frac{h\,d|h|}{|h|^2}\right)\left(\frac{dh}{|h|} - \frac{h\,d|h|}{|h|^2}\right)\left(\frac{dh}{|h|} - \frac{h\,d|h|}{|h|^2}\right)\right] &(28)\\
&= \frac{1}{|h|^5}\mathrm{tr}\,[h(dh)^4] - \frac{1}{|h|^6}\mathrm{tr}\,[h(dh)^3 h\, d|h| + h(dh)^2 h\, d|h|\, dh + h\, dh\, h\, d|h|(dh)^2 + hh\, d|h|(dh)^3] &(29)\\
&= \frac{1}{|h|^5}\mathrm{tr}\,[h(dh)^4] - \frac{1}{|h|^6}\mathrm{tr}\,[h(dh)^3 h - h(dh)^2 h\, dh + h\, dh\, h(dh)^2 - hh(dh)^3]\, d|h| &(30)\\
&= \frac{1}{|h|^5}\mathrm{tr}\,[h(dh)^4]. &(31)
\end{aligned}$$

Here $d|h|$ is a c-number. Therefore, for a Dirac model, the second Chern number can be computed as

$$ch_2 = \frac{1}{2^8\pi^2} \int \frac{1}{|h|^5} \mathrm{tr}\,[h(dh)^4]. \tag{32}$$

For the Hamiltonian $H(k_x, k_y, \theta, \phi)$ above, the numerical calculation gives $ch_2 = -1$.

- Qi, Hughes, and Zhang [6] discuss the relation between the second Chern number and the charge of localized states, so this should be checked.

2 2d class D

- In $(2+1)$ dimensions, for the Hamiltonian
$$H = k_x \sigma_x \tau_x + k_y \sigma_y \tau_x + m_1 \sigma_z \tau_x + m_2 \tau_y + m_3 \tau_z, \tag{33}$$
it is known that integrating out the fermion field gives a sigma model on S^2 for the mass term $\boldsymbol{m} = (m_1, m_2, m_3) \in S^2$ [1]. In particular, [1] says that a Hopf term is obtained. This needs to be checked by calculation.

- Combining this fact with the fact that the Hopf term is a spin Chern-Simons term, one should be able to confirm that the skyrmion is a fermion by introducing a skyrmion of the mass term in a suitable fermionic lattice model and measuring the fermion parity. In this note I record a concrete lattice model and the numerical results.

- A similar object in $(3+1)$ dimensions has also been calculated. In [2], a texture based on the Hopf map $\pi_3(S^2) = \mathbb{Z}$ is introduced in a three-dimensional fermionic lattice model, and it is argued that this texture is again a fermion. On the other hand, I do not know whether the topological term of the sigma model describing the fermionic nature of this three-dimensional texture is known.

2.1 Ground-state fermion parity of a free-fermion model

For a free-fermion model
$$\hat{H} = \frac{i}{4} \sum_{ij} A_{ij} a_i a_j, \quad \{a_i, a_j\} = 2\delta_{ij}, \quad a_i^\dagger = a_i, \quad A^T = A, \quad A^\dagger = -A, \tag{34}$$
the fermion parity of the ground state is given by the sign of the Pfaffian of the antisymmetric matrix A:
$$\langle GS|(-1)^F|GS\rangle = \operatorname{sign} \operatorname{Pf}[A]. \tag{35}$$

Let us derive a formula for obtaining the antisymmetric matrix A from a given BdG Hamiltonian. For the BdG Hamiltonian
$$\hat{H} = \frac{1}{2} \sum_{ij} (c_i^\dagger, c_i) \begin{pmatrix} h_{ij} & \Delta_{ij} \\ \Delta_{ij}^\dagger & -h_{ij}^T \end{pmatrix} \begin{pmatrix} c_j \\ c_j^\dagger \end{pmatrix} \tag{36}$$
$$= \frac{1}{2} \sum_{ij} (h_{ij} c_i^\dagger c_j - h_{ij}^T c_i c_j^\dagger) + \frac{1}{2} \sum_{ij} (\Delta_{ij} c_i^\dagger c_j^\dagger + \Delta_{ij}^\dagger c_i c_j), \tag{37}$$
introduce Majorana fermions a_{2i-1}, a_{2i} by
$$a_{2i-1} = c_i + c_i^\dagger, \quad a_{2i} = -i(c_i - c_i^\dagger). \tag{38}$$
Then
$$c_i = \frac{a_{2i-1} + i a_{2i}}{2}, \quad c_i^\dagger = \frac{a_{2i-1} - i a_{2i}}{2}. \tag{39}$$
We compute
$$h_{ij} c_i^\dagger c_j - h_{ij}^T c_i c_j^\dagger \tag{40}$$
$$= h_{ij} \frac{a_{2i-1} - i a_{2i}}{2} \frac{a_{2j-1} + i a_{2j}}{2} - h_{ij}^T \frac{a_{2i-1} + i a_{2i}}{2} \frac{a_{2j-1} - i a_{2j}}{2} \tag{41}$$
$$= \frac{1}{4} \left[(h_{ij} - h_{ij}^T) a_{2i-1} a_{2j-1} + (i h_{ij} + i h_{ij}^T) a_{2i-1} a_{2j} + (-i h_{ij} - i h_{ij}^T) a_{2i} a_{2j-1} + (h_{ij} - h_{ij}^T) a_{2i} a_{2j} \right] \tag{42}$$
$$= \frac{i}{4} \left[(-i h_{ij} + i h_{ij}^T) a_{2i-1} a_{2j-1} + (h_{ij} + h_{ij}^T) a_{2i-1} a_{2j} + (-h_{ij} - h_{ij}^T) a_{2i} a_{2j-1} + (-i h_{ij} + i h_{ij}^T) a_{2i} a_{2j} \right]. \tag{43}$$

Similarly,

$$\Delta_{ij}c_i^\dagger c_j^\dagger + \Delta_{ij}^\dagger c_i c_j \tag{44}$$

$$= \Delta_{ij}\frac{a_{2i-1}-ia_{2i}}{2}\frac{a_{2j-1}-ia_{2j}}{2} + \Delta_{ij}^\dagger\frac{a_{2i-1}+ia_{2i}}{2}\frac{a_{2j-1}+ia_{2j}}{2} \tag{45}$$

$$= \frac{1}{4}\Big[(\Delta_{ij}+\Delta_{ij}^\dagger)a_{2i-1}a_{2j-1} + (-i\Delta_{ij}+i\Delta_{ij}^\dagger)a_{2i-1}a_{2j} + (-i\Delta_{ij}+i\Delta_{ij}^\dagger)a_{2i}a_{2j-1} + (-\Delta_{ij}-\Delta_{ij}^\dagger)a_{2i}a_{2j}\Big] \tag{46}$$

$$= \frac{i}{4}\Big[(-i\Delta_{ij}-i\Delta_{ij}^\dagger)a_{2i-1}a_{2j-1} + (-\Delta_{ij}+\Delta_{ij}^\dagger)a_{2i-1}a_{2j} + (-\Delta_{ij}+\Delta_{ij}^\dagger)a_{2i}a_{2j-1} + (i\Delta_{ij}+i\Delta_{ij}^\dagger)a_{2i}a_{2j}\Big]. \tag{47}$$

Therefore

$$\hat{H} = \frac{1}{2}\frac{i}{4}\sum_{ij}\Big[(-ih_{ij}+ih_{ij}^T-i\Delta_{ij}-i\Delta_{ij}^\dagger)a_{2i-1}a_{2j-1} \tag{48}$$

$$+ (h_{ij}+h_{ij}^T-\Delta_{ij}+\Delta_{ij}^\dagger)a_{2i-1}a_{2j} + (-h_{ij}-h_{ij}^T-\Delta_{ij}+\Delta_{ij}^\dagger)a_{2i}a_{2j-1} \tag{49}$$

$$+ (-ih_{ij}+ih_{ij}^T+i\Delta_{ij}+i\Delta_{ij}^\dagger)a_{2i}a_{2j}\Big] \tag{50}$$

$$= \frac{1}{2}\frac{i}{4}\sum_{ij}(a_{2i-1},a_{2i})\begin{pmatrix} -ih_{ij}+ih_{ij}^T-i\Delta_{ij}-i\Delta_{ij}^\dagger & h_{ij}+h_{ij}^T-\Delta_{ij}+\Delta_{ij}^\dagger \\ -h_{ij}-h_{ij}^T-\Delta_{ij}+\Delta_{ij}^\dagger & -ih_{ij}+ih_{ij}^T+i\Delta_{ij}+i\Delta_{ij}^\dagger \end{pmatrix}\begin{pmatrix} a_{2j-1} \\ a_{2j}\end{pmatrix}. \tag{51}$$

Thus for given matrices h,Δ, one may take

$$A = \frac{1}{2}\begin{pmatrix} -ih+ih^T-i\Delta-i\Delta^\dagger & h+h^T-\Delta+\Delta^\dagger \\ -h-h^T-\Delta+\Delta^\dagger & -ih+ih^T+i\Delta+i\Delta^\dagger \end{pmatrix}. \tag{52}$$

2.2 Model

In the Dirac model

$$H = k_x\tau_x + k_y\tau_y + M\tau_z, \quad C = \tau_x K, \tag{53}$$

particle-hole symmetry implies $M^* = M$. Also, since $M^\dagger = M$ and $M^2 = 1$, M can be regarded as a class-AI Hamiltonian. There are three possible mass terms:

$$M = m_1\sigma_x + m_2\sigma_y\mu_y + m_3\sigma_z. \tag{54}$$

The full Hamiltonian is an 8×8 model.

Write $(m_1,m_2,m_3) = m\boldsymbol{n}$, with $m>0$, $\boldsymbol{n}\in S^2$, $|\boldsymbol{n}|=1$. According to [1], integrating out the fermions produces an action corresponding to the Hopf term,

$$\mathrm{Hopf}[\boldsymbol{n}] = \frac{ik}{4\pi}\int ada. \tag{55}$$

The precise meaning of the right-hand side should be understood as a spin Chern-Simons term. The coefficient k should be derived carefully, and I do not derive it here. The skyrmion should give $(-1)^k$ both under a 2π rotation and under measurement of fermion parity.

A naive lattice model is

$$H_{\mathrm{BdG}}(k_x,k_y,\boldsymbol{n}) \tag{56}$$

$$= \sin k_x\tau_x + \sin k_y\tau_y + n_1\sigma_x\tau_z + n_2\sigma_y\mu_y\tau_z + (n_3-2+\cos k_x+\cos k_y)\sigma_z\tau_z \tag{57}$$

$$= e^{ik_x}\frac{\sigma_z\tau_z-i\tau_x}{2} + e^{-ik_x}\frac{\sigma_z\tau_z+i\tau_x}{2} + e^{ik_y}\frac{\sigma_z\tau_z-i\tau_y}{2} + e^{-ik_y}\frac{\sigma_z\tau_z+i\tau_y}{2} + n_1\sigma_x\tau_z + n_2\sigma_y\mu_y\tau_z + (n_3-2)\sigma_z\tau_z. \tag{58}$$

The real-space representation is

$$H_{\rm BdG}(x-1,y;x,y) = \frac{\sigma_z\tau_z - i\tau_x}{2}, \tag{59}$$
$$H_{\rm BdG}(x+1,y;x,y) = \frac{\sigma_z\tau_z + i\tau_x}{2}, \tag{60}$$
$$H_{\rm BdG}(x,y-1;x,y) = \frac{\sigma_z\tau_z - i\tau_y}{2}, \tag{61}$$
$$H_{\rm BdG}(x,y+1;x,y) = \frac{\sigma_z\tau_z + i\tau_y}{2}, \tag{62}$$
$$H_{\rm BdG}(x,y;x,y) = n_1\sigma_x\tau_z + n_2\sigma_y\mu_y\tau_z + (n_3-2)\sigma_z\tau_z. \tag{63}$$

Matrix elements not defined above are zero. If, in τ space, we define the normal Hamiltonian h and the gap function Δ by

$$H_{\rm BdG} = \begin{pmatrix} h & \Delta \\ \Delta^\dagger & -h^T \end{pmatrix}, \tag{64}$$

then

$$h(x-1,y;x,y) = \frac{\sigma_z}{2}, \tag{65}$$
$$h(x+1,y;x,y) = \frac{\sigma_z}{2}, \tag{66}$$
$$h(x,y-1;x,y) = \frac{\sigma_z}{2}, \tag{67}$$
$$h(x,y+1;x,y) = \frac{\sigma_z}{2}, \tag{68}$$
$$h(x,y;x,y) = n_1\sigma_x + n_2\sigma_y\mu_y + (n_3-2)\sigma_z, \tag{69}$$
$$\Delta(x-1,y;x,y) = \frac{-i}{2}, \tag{70}$$
$$\Delta(x+1,y;x,y) = \frac{i}{2}, \tag{71}$$
$$\Delta(x,y-1;x,y) = \frac{-1}{2}, \tag{72}$$
$$\Delta(x,y+1;x,y) = \frac{1}{2}. \tag{73}$$

Introduce a skyrmion configuration suitably. Take real space to be a torus T^2. To construct a degree-one continuous map $T^2 \to S^2$, it is useful to refer to the smash product $S^1 \wedge S^1 \cong S^2$. For example, [3] gives a short explanation. For sites $x \in \{1,\dots,L_x\}$ and $y \in \{1,\dots,L_y\}$, one may take

$$n_1(x,y) = \sin\left(\frac{2\pi x}{L_x} - \pi\right)\cos\left(\frac{\pi y}{L_y} - \frac{\pi}{2}\right), \tag{74}$$
$$n_2(x,y) = \sin\left(\frac{2\pi y}{L_y} - \pi\right)\cos^2\left(\frac{\pi x}{L_x} - \frac{\pi}{2}\right), \tag{75}$$
$$n_3(x,y) = 1 - 2\cos^2\left(\frac{\pi x}{L_x} - \frac{\pi}{2}\right)\cos^2\left(\frac{\pi y}{L_y} - \frac{\pi}{2}\right). \tag{76}$$

- Numerically, when a skyrmion is introduced, the ground-state fermion parity indeed differs from that of the ground state without a skyrmion. Thus one concludes that the skyrmion is a fermion.

2.3 Another model

A model with two S^1 textures can produce a fermion localized at the intersection of the two textures as follows: (i) introduce an x-direction texture in a two-dimensional superconducting system and create a one-dimensional superconductor extended in the y direction, and (ii) introduce a y-direction texture in the one-dimensional superconducting system to produce a fermion. This is not a skyrmion, but I record the calculation as a check of the numerical method.

2.3.1 Model 1

The model is a $(p_x + ip_y) \oplus (p_x - ip_y)$ superconductor with a texture in the mass term and a texture in the phase of the gap function:

$$H = \begin{pmatrix} (\cos\phi - 2 + \cos k_x + \cos k_y)\sigma_x + \sin\phi\,\sigma_z & e^{i\theta}(\sin k_x - i\sin k_y) \\ e^{i\theta}(\sin k_x + i\sin k_y) & -(\cos\phi - 2 + \cos k_x + \cos k_y)\sigma_x - \sin\phi\,\sigma_z \end{pmatrix}. \tag{77}$$

Thus

$$h = (\cos\phi - 2 + \cos k_x + \cos k_y)\sigma_x + \sin\phi\,\sigma_z, \tag{78}$$

$$\Delta = e^{i\theta}(\sin k_x - i\sin k_y), \tag{79}$$

and the real-space representation is

$$h(x-1, y; x, y) = h(x+1, y; x, y) = \frac{1}{2}\sigma_x, \tag{80}$$

$$h(x, y-1; x, y) = h(x, y+1; x, y) = \frac{1}{2}\sigma_x, \tag{81}$$

$$h(x, y; x, y) = (\cos\phi - 2)\sigma_x + \sin\phi\,\sigma_z, \tag{82}$$

$$\Delta(x, y; x+1, y) = -\Delta(x+1, y; x, y) = \frac{-ie^{i\theta}}{2}, \tag{83}$$

$$\Delta(x, y; x, y+1) = -\Delta(x, y+1; x, y) = \frac{-e^{i\theta}}{2}. \tag{84}$$

Taking

$$\phi = \frac{2\pi x}{L_x}, \quad \theta = \frac{2\pi y}{L_y}, \tag{85}$$

one expects to obtain a ground state whose fermion parity changes.

- Numerically, however, the result is not very clean. The ratio of fermion parities depends on the system size and on the presence or absence of the θ/ϕ textures.

2.3.2 1D Kitaev chain

$$H = \begin{pmatrix} \cos k_x & e^{i\theta}\sin k_x \\ e^{-i\theta}\sin k_x & -\cos k_x \end{pmatrix}, \quad \theta = \frac{2\pi x}{L_x}. \tag{86}$$

- This gives a clean result in which the fermion parity changes depending on whether the texture is present.

3 On $ch(V \oplus W) = ch(V) + ch(W)$

The second-order part in the curvature of the Chern-character identity

$$ch(V \oplus W) = ch(V) + ch(W) \tag{87}$$

is

$$\mathrm{tr}\,F_{V\oplus W}^2 = \mathrm{tr}\,F_V^2 + \mathrm{tr}\,F_W^2, \tag{88}$$

but one should note that the direct-sum decomposition $V \oplus W$ is global.

Let us see what happens when the direct-sum decomposition is not global. As an example, consider the occupied states of the following 4×4 Hamiltonian:

$$H = \sum_{i=1}^{5} h_i\gamma_i, \quad \{\gamma_i,\gamma_j\} = 2\delta_{ij}, \quad h_ih_i = 1. \tag{89}$$

Since $H^2 = 1$, the projection onto the occupied states with energy eigenvalue -1 is

$$P = \frac{1-H}{2}. \tag{90}$$

Using the formula $\mathrm{tr}\,F^n = \mathrm{tr}\,P(dP)^{2n}$ for the Berry curvature of the occupied states, we get

$$\mathrm{tr}\,F^2 = -\frac{1}{2^5}\mathrm{tr}\,[H(dH)^4] = \frac{1}{2^3}\epsilon_{ijklm}h_i\,dh_j\,dh_k\,dh_l\,dh_m. \tag{91}$$

Here I used $\mathrm{tr}\,[\gamma_i\gamma_j\gamma_k\gamma_l\gamma_m] = -4\epsilon_{ijklm}$. For example, if the base space is S^4 and we take $h = \mathrm{id} : S^4 \to S^4$, then $\frac{1}{2}\int_{S^4}\mathrm{tr}\,(iF/2\pi)^2 = \pm 1$.

There are two occupied states. Let us write them explicitly. Put $(\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5) = (\boldsymbol{\sigma}\tau_x,\tau_y,\tau_z)$. Using spherical coordinates on S^4, write $h = (\sin\theta\cos\phi\,\boldsymbol{n}, \sin\theta\sin\phi, \cos\theta)$, $\boldsymbol{n}^2 = 1$[3]. A direct calculation gives

$$H = e^{-i\boldsymbol{n}\cdot\boldsymbol{\sigma}\tau_z\phi/2}e^{-i\boldsymbol{n}\cdot\boldsymbol{\sigma}\tau_y\theta/2}\tau_z e^{i\boldsymbol{n}\cdot\boldsymbol{\sigma}\tau_y\theta/2}e^{i\boldsymbol{n}\cdot\boldsymbol{\sigma}\tau_z\phi/2}. \tag{92}$$

Locally, the occupied states are

$$\Psi = e^{-i\boldsymbol{n}\cdot\boldsymbol{\sigma}\tau_z\phi/2}e^{-i\boldsymbol{n}\cdot\boldsymbol{\sigma}\tau_y\theta/2}\sigma_0 \otimes \begin{pmatrix}0\\1\end{pmatrix}_\tau \tag{93}$$

$$\sim \begin{pmatrix}-\boldsymbol{n}\cdot\boldsymbol{\sigma}\sin\frac{\theta}{2}\\ e^{i\boldsymbol{n}\cdot\boldsymbol{\sigma}\phi}\cos\frac{\theta}{2}\end{pmatrix}_\tau. \tag{94}$$

Choose a basis of the $\boldsymbol{\sigma}$ degrees of freedom satisfying $\boldsymbol{n}\cdot\boldsymbol{\sigma}\,|\boldsymbol{n},\pm\rangle = \pm\,|\boldsymbol{n},\pm\rangle$. The following discussion does not require a concrete expression, but one may take, for example,

$$(|\boldsymbol{n},+\rangle_\sigma, |\boldsymbol{n},-\rangle_\sigma) = \begin{pmatrix} e^{-i\phi_n}\cos\frac{\theta_n}{2} & -\sin\frac{\theta_n}{2}\\ \sin\frac{\theta_n}{2} & e^{i\phi_n}\cos\frac{\theta_n}{2}\end{pmatrix}_\sigma, \tag{95}$$

where (θ_n,ϕ_n) are the spherical coordinates of $\boldsymbol{n}$. The occupied states of H are

$$|\psi_1\rangle = |\boldsymbol{n},+\rangle_\sigma \otimes \begin{pmatrix}-\sin\frac{\theta}{2}\\ e^{i\phi}\cos\frac{\theta}{2}\end{pmatrix}_\tau, \quad |\psi_2\rangle = |\boldsymbol{n},-\rangle_\sigma \otimes \begin{pmatrix}\sin\frac{\theta}{2}\\ e^{-i\phi}\cos\frac{\theta}{2}\end{pmatrix}_\tau. \tag{96}$$

A direct calculation gives

$$F_1 = \langle d\psi_1|d\psi_1\rangle = -f_\sigma + f_\tau, \quad F_2 = \langle d\psi_2|d\psi_2\rangle = f_\sigma - f_\tau, \tag{97}$$

$$F_1^2 = 2f_\sigma f_\tau, \quad F_2^2 = -2f_\sigma f_\tau. \tag{98}$$

Here $f_\sigma = i\,d(\cos^2\frac{\theta_n}{2})\,d\phi_n$ and $f_\tau = i\,d(\cos^2\frac{\theta}{2})\,d\phi$. Thus

$$F_1^2 + F_2^2 = 0, \tag{99}$$

and $\mathrm{tr}\,F^2 = F_1^2 + F_2^2$ does not hold. In this example, the cross terms between $|\psi_1\rangle$ and $|\psi_2\rangle$ should contribute to $\mathrm{tr}\,F^2$.

[3] $S^2 \wedge S^2$.

4 Note: computing the integral of the CS3 form on S^3 as the boundary of $\mathbb{C}P^2_2$

4.1 Construction of $\mathbb{C}P^2_2 \to \mathbb{C}P^1$

We want to construct a map $f : \mathbb{C}P^2_2 \to \mathbb{C}P^1$. For the handle decomposition of $\mathbb{C}P^2$, see for example App. C of [7]. A p-handle in dimension d means the pair $(D^p \times D^q, \partial D^p \times D^q)$. The submanifold $\partial D^p \times D^q \cong S^{p-1} \times D^q$ is called the attaching region. The manifold $\mathbb{C}P^2$ decomposes as

$$D^4 \cong \mathbb{C}P^2_0 \subset \mathbb{C}P^2_2 \subset \mathbb{C}P^2. \tag{100}$$

A characteristic feature of $\mathbb{C}P^2$ is that when the 2-handle is attached to the 0-handle, the circle S^1 has framing $+1$.

The space $\mathbb{C}P^2_2$ and the 4-handle D^4 are described using homogeneous coordinates $[z_1, z_2, z_3]$ on $\mathbb{C}P^2$, with $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$, as follows. Choose a real number r with $0 < r < 1$. For $r < |z_3| \leq 1$, $(z_1/z_3, z_2/z_3)$ can be chosen as local coordinates on $\mathbb{C}P^2$. Since

$$|z_1/z_3|^2 + |z_2/z_3|^2 = \frac{1 - |z_3|^2}{|z_3|^2}, \quad 0 \leq \frac{1 - |z_3|^2}{|z_3|^2} < \frac{1 - r^2}{r^2}, \tag{101}$$

the set

$$\{[z_1, z_2, z_3] \in \mathbb{C}P^2 \mid r < |z_3| \leq 1\} \tag{102}$$

is a 4-disk D^4, namely the 4-handle. On the other hand,

$$\mathbb{C}P^2_2 = \{[z_1, z_2, z_3] \in \mathbb{C}P^2 \mid 0 \leq |z_3| < r\} \tag{103}$$

gives $\mathbb{C}P^2_2$. The attaching region is

$$\{[z_1, z_2, z_3] \in \mathbb{C}P^2 \mid |z_3| = r\} = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1 - r^2\} \cong S^3. \tag{104}$$

We want to construct a continuous map $\tilde{f} : \mathbb{C}P^2_2 \to \mathbb{C}P^1$. It suffices to define $\tilde{f}$ so that, when restricted to the boundary $|z_3| = r$, it becomes the Hopf fibration

$$S^1 \to S^3 \xrightarrow{f} \mathbb{C}P^1. \tag{105}$$

Using homogeneous coordinates on $\mathbb{C}P^2_2$, define

$$\tilde{f} : \mathbb{C}P^2_2 \to \mathbb{C}P^1, \quad [z_1, z_2, z_3] \mapsto \left[\frac{z_1}{\sqrt{1 - |z_3|^2}}, \frac{z_2}{\sqrt{1 - |z_3|^2}} \right]. \tag{106}$$

4.2 Calculation of $\int_{S^3} \omega d\omega$

Since $H^2(S^3, \mathbb{Z}) = 0$, a $U(1)$ field on S^3 is simply given as a 1-form. For any 1-form ω on S^3, one can compute $\int_{S^3} \omega d\omega$. Here, as an example, we compute it for a 1-form ω used in the definition of the Hopf invariant, for which the nontrivial value $\int_{S^3} \omega d\omega = 1$ is known.

First define ω following Bott and Tu. Let $\alpha \in \Omega^2_{DR}(S^2)$ be the volume form on the unit-radius S^2, normalized by $\int_{S^2} \alpha = 1$. The pullback $f^*\alpha$ under the Hopf map $f : S^3 \to S^2$ satisfies $f^*\alpha = d\omega$ for some 1-form $\omega \in \Omega^1_{DR}(S^3)$, because $H^2_{DR}(S^3) = 0$. Equation (17.23.3) of Bott-Tu gives the following expression for α in terms of homogeneous coordinates $[w_1, w_2]$ on $\mathbb{C}P^1$:

$$\alpha = \frac{i}{2\pi}\frac{(w_2 dw_1 - w_1 dw_2)(\bar{w}_2 d\bar{w}_1 - \bar{w}_1 d\bar{w}_2)}{(|w_1|^2 + |w_2|^2)^2}. \tag{107}$$

Using $|w_1|^2 + |w_2|^2 = 1$ and

$$dw_1\bar{w}_1 + w_1 d\bar{w}_1 + dw_2\bar{w}_2 + w_2 d\bar{w}_2 = 0, \tag{108}$$

we find

$$(w_2 dw_1 - w_1 dw_2)(\bar{w}_2 d\bar{w}_1 - \bar{w}_1 d\bar{w}_2) \tag{109}$$
$$= |w_2|^2 dw_1 d\bar{w}_1 - w_2\bar{w}_1 dw_1 d\bar{w}_2 - w_1\bar{w}_2 dw_2 d\bar{w}_1 + |w_1|^2 dw_2 d\bar{w}_2 \tag{110}$$
$$= |w_2|^2 dw_1 d\bar{w}_1 + \bar{w}_1 dw_1(dw_1\bar{w}_1 + w_1 d\bar{w}_1 + dw_2\bar{w}_2) \tag{111}$$
$$+ \bar{w}_2 dw_2(dw_1\bar{w}_1 + dw_2\bar{w}_2 + w_2 d\bar{w}_2) + |w_1|^2 dw_2 d\bar{w}_2 \tag{112}$$
$$= dw_1 d\bar{w}_1 + dw_2 d\bar{w}_2. \tag{113}$$

Thus

$$\alpha = \frac{i}{2\pi}(dw_1 d\bar{w}_1 + dw_2 d\bar{w}_2) = -\frac{i}{2\pi}d(\bar{w}_1 dw_1 + \bar{w}_2 dw_2) = -\frac{i}{2\pi}d(w^\dagger dw), \tag{114}$$

where $w = (w_1, w_2)^T$. The volume form on $\mathbb{C}P^1$ is therefore the exterior derivative of the 1-form $-\frac{i}{2\pi}w^\dagger dw$. This 1-form is only locally defined on $\mathbb{C}P^1$.

Now compute the pullback $\tilde{f}^*\alpha$ to $\mathbb{C}P^2_2$ by setting

$$w_1 = \frac{z_1}{\sqrt{1 - |z_3|^2}}, \quad w_2 = \frac{z_2}{\sqrt{1 - |z_3|^2}}. \tag{115}$$

Let $z = (z_1, z_2)^T$ and $\rho = \sqrt{1 - |z_3|^2}$. Note that $z^\dagger z = |z_1|^2 + |z_2|^2 = \rho^2$. Then

$$dw^\dagger dw = d(\rho^{-1}z^\dagger)d(\rho^{-1}z) \tag{116}$$
$$= (d\rho^{-1}z^\dagger + \rho^{-1}dz^\dagger)(d\rho^{-1}z + \rho^{-1}dz) \tag{117}$$
$$= \rho^{-1}d\rho^{-1}z^\dagger dz - \rho^{-1}d\rho^{-1}dz^\dagger z + \rho^{-2}dz^\dagger dz \tag{118}$$
$$= \rho^{-1}d\rho^{-1}z^\dagger dz - \rho^{-1}d\rho^{-1}(2\rho d\rho - z^\dagger dz) + \rho^{-2}dz^\dagger dz \tag{119}$$
$$= 2\rho^{-1}d\rho^{-1}z^\dagger dz + \rho^{-2}dz^\dagger dz \tag{120}$$
$$= d(\rho^{-2}z^\dagger dz). \tag{121}$$

Therefore

$$\omega = -\frac{i}{2\pi}\rho^{-2}z^\dagger dz. \tag{122}$$

First compute the integral $\int_{S^3}\omega d\omega$ over the sphere of radius ρ using this expression. Put $z_1 = x_1 + ix_2$ and $z_2 = x_3 + ix_4$. Then

$$\omega = \rho^{-2}\frac{1}{\pi}(x_1 dx_2 + x_3 dx_4), \quad \int_{S^3}\omega d\omega = \rho^{-4}\frac{2}{\pi^2}\int_{S^3} x_1\, dx_2\, dx_3\, dx_4 = 1. \tag{123}$$

This gives the usual Hopf invariant.

Next compute the integral $\int_{\mathbb{C}P^2_2} d\omega d\omega$. By Stokes' theorem, the result must agree with the previous one. Write $\omega = \rho^{-2}\omega_0$, where $\omega_0 = \frac{1}{\pi}(x_1 dx_2 + x_3 dx_4)$. Then

$$d\omega = d\rho^{-2}\omega_0 + \rho^{-2}d\omega_0, \tag{124}$$

and

$$
\begin{aligned}
d\omega d\omega &= (d\rho^{-2}\omega_0 + \rho^{-2}d\omega_0)(d\rho^{-2}\omega_0 + \rho^{-2}d\omega_0) && (125)\\
&= 2\rho^{-2}d\rho^{-2}\omega_0 d\omega_0 + \rho^{-4}d\omega_0 d\omega_0 && (126)\\
&= \frac{2}{\pi^2}\left(2\rho^{-2}d\rho^{-2}x_1dx_2dx_3dx_4 + \rho^{-4}dx_1dx_2dx_3dx_4\right). && (127)
\end{aligned}
$$

Set

$$
\begin{aligned}
x_1 &= \rho\sin\xi\sin\phi\cos\theta, && (128)\\
x_2 &= \rho\sin\xi\sin\phi\sin\theta, && (129)\\
x_3 &= \rho\sin\xi\cos\phi, && (130)\\
x_4 &= \rho\cos\xi. && (131)
\end{aligned}
$$

Then

$$
x_1dx_2dx_3dx_4 = \rho^4\sin^4\xi\sin^3\phi\cos^2\theta\, d\theta\, d\phi\, d\xi + (\cdots)d\rho, \tag{132}
$$

and

$$
dx_1dx_2dx_3dx_4 = d(x_1dx_2dx_3dx_4) = 4\rho^3\sin^4\xi\sin^3\phi\cos^2\theta\, d\rho\, d\theta\, d\phi\, d\xi. \tag{133}
$$

This seems to give

$$
d\omega d\omega = 0 \tag{134}
$$

which is puzzling.

5 Relations among curvatures in complex flag-manifold models

Motivation 1. The relation for the Chern character

$$
ch(V \oplus W) = ch(V) + ch(W) \tag{135}
$$

naively suggests

$$
\mathrm{tr}\, F^n_{V\oplus W} = \mathrm{tr}\, F^n_V + \mathrm{tr}\, F^n_W, \tag{136}
$$

but I cannot give a direct proof. Perhaps an analogous relation can be proven directly for complex flag manifolds, rather than for general complex vector bundles with connection.

Motivation 2. From the fibration

$$
U(N+M+K) \to \frac{U(N+M+K)}{U(N)\times U(M)\times U(K)} \to BU(N)\times BU(M)\times BU(K), \tag{137}
$$

the E_2 page of the Leray-Serre spectral sequence is

$$
\begin{array}{c|ccccc}
3 & \mathbb{Z} & & & & \\
2 & 0 & 0 & & & \\
1 & \mathbb{Z} & 0 & \mathbb{Z}^3 & & \\
0 & \mathbb{Z} & 0 & \mathbb{Z}^3 & 0 & \mathbb{Z}^L \\
\hline
E_2^{p,q} & 0 & 1 & 2 & 3 & 4
\end{array} \tag{138}
$$

Here L depends on whether each of N, M, K is 1 or at least 2; for example, for $(N, M, K) = (1, 1, 1)$, $L = 6$. The group $E_2^{0,1} = H^1(U(N+M+K), \mathbb{Z}) = \mathbb{Z}$ is generated by $\mathrm{tr}\,[U^\dagger dU]$, and $E_2^{0,3} = H^3(U(N+M+K), \mathbb{Z}) = \mathbb{Z}$ is generated by $\mathrm{tr}\,[U^\dagger dU]^3$. It is then natural to interpret the differentials d_2, d_4 as

- d_2: constraints on the curvature of the $U(N)\times U(M)\times U(K)$ gauge fields coming from $d\mathrm{tr}\,[U^\dagger dU]=0$;
- d_4: constraints on the curvature of the $U(N)\times U(M)\times U(K)$ gauge fields coming from $d\mathrm{tr}\,[U^\dagger dU]^3=0$.

Below I show a relation among the curvatures of the $U(N)\times U(M)\times U(K)$ gauge fields, which seems closely related to this expectation.

5.1 Preparation

Let $U: U_i \to U(N+M+K)$ be a local field configuration, written as

$$U = (|u_1\rangle, \dots, |u_N\rangle, |v_1\rangle, \dots, |v_M\rangle, |w_1\rangle, \dots, |w_K\rangle). \tag{139}$$

Define projection operators on $\mathbb{C}^{N+M+K}$ by

$$P_1 := \sum_{i=1}^{N} |u_i\rangle\langle u_i|, \tag{140}$$
$$P_2 := \sum_{i=1}^{M} |v_i\rangle\langle v_i|, \tag{141}$$
$$P_3 := \sum_{i=1}^{K} |w_i\rangle\langle w_i|. \tag{142}$$

In the basis U of $\mathbb{C}^{N+M+K}$, they are represented as

$$P_1U = U\begin{pmatrix}1 & & \\ & 0 & \\ & & 0\end{pmatrix}, \quad P_2U = U\begin{pmatrix}0 & & \\ & 1 & \\ & & 0\end{pmatrix}, \quad P_3U = U\begin{pmatrix}0 & & \\ & 0 & \\ & & 1\end{pmatrix}. \tag{143}$$

The $U(N)$, $U(M)$, and $U(K)$ gauge fields are

$$A_1 = P_1U^\dagger dUP_1, \tag{144}$$
$$A_2 = P_2U^\dagger dUP_2, \tag{145}$$
$$A_3 = P_3U^\dagger dUP_3. \tag{146}$$

Their curvatures are

$$F_j = dA_j + A_j^2 = P_j(-U^\dagger dU\,U^\dagger dU)P_j + P_jU^\dagger dUP_jU^\dagger dUP_j = -P_jU^\dagger dU(1-P_j)U^\dagger dUP_j. \tag{147}$$

Writing $X = U^\dagger dU$,

$$F_1 = -P_1X(P_2+P_3)XP_1, \tag{148}$$
$$F_2 = -P_2X(P_1+P_3)XP_2, \tag{149}$$
$$F_3 = -P_3X(P_1+P_2)XP_3. \tag{150}$$

Note that

$$dX = -X^2. \tag{151}$$

5.2 d_2

We have

$$0 = d\mathrm{tr}\,[U^\dagger dU] = -\mathrm{tr}\,[X^2], \tag{152}$$

which can be viewed as

$$\sum_{i,j\in\{1,2,3\}^2} \mathrm{tr}\,[P_i X P_j X] = 0. \tag{153}$$

Introduce the abbreviation

$$(ij) := \mathrm{tr}\,[P_i X P_j X]. \tag{154}$$

By the properties of the trace,

$$(ij) = -(ji). \tag{155}$$

In particular, $(ii) = 0$. The relation $d\mathrm{tr}\,[U^\dagger dU] = 0$ is

$$\sum_{i,j\in\{1,2,3\}^2} (ij) = (11) + (12) + (13) + (21) + (22) + (23) + (31) + (32) + (33) \tag{156}$$

$$= (12) + (13) + (21) + (23) + (31) + (32). \tag{157}$$

On the other hand, the curvatures of $U(N), U(M), U(K)$ are

$$-\,\mathrm{tr}\,F_1 = (12) + (13), \tag{158}$$

$$-\,\mathrm{tr}\,F_2 = (21) + (23), \tag{159}$$

$$-\,\mathrm{tr}\,F_3 = (31) + (32). \tag{160}$$

Thus $d\mathrm{tr}\,[U^\dagger dU] = 0$ is equivalent to

$$\mathrm{tr}\,F_1 + \mathrm{tr}\,F_2 + \mathrm{tr}\,F_3 = 0. \tag{161}$$

This can also be proven directly using only $(ij) = -(ji)$. Is it independent of $\mathrm{tr}\,[U^\dagger dU] = 0$?

5.3 d_4

Similarly,

$$0 = d\mathrm{tr}\,[U^\dagger dU]^3 = -\mathrm{tr}\,[X^4] \tag{162}$$

gives

$$\sum_{i,j,k,l\in\{1,2,3\}^4} (ijkl) = 0, \tag{163}$$

where

$$(ijkl) = \mathrm{tr}\,[P_i X P_j X P_k X P_l X]. \tag{164}$$

Note that

$$(ijkl) = -(jkli). \tag{165}$$

The curvature squares are

$$\mathrm{tr}\,F_1^2 = (1212) + (1312) + (1213) + (1313), \tag{166}$$

$$\mathrm{tr}\,F_2^2 = (2323) + (2123) + (2321) + (2121), \tag{167}$$

$$\mathrm{tr}\,F_3^2 = (3131) + (3231) + (3132) + (3232). \tag{168}$$

We show:

- $\mathrm{tr}\, F_1^2 + \mathrm{tr}\, F_2^2 + \mathrm{tr}\, F_3^2$ is a globally exact term.

Proof.

$$\mathrm{tr}\, F_1^2 + \mathrm{tr}\, F_2^2 + \mathrm{tr}\, F_3^2 = (1213) + (2123) + (3231) + (1 \leftrightarrow 2). \tag{169}$$

Use $P_3 = 1 - P_1 - P_2$ to eliminate P_3:

$$\begin{aligned}
(1213) &= \mathrm{tr}\,[P_1XP_2XP_1XP_3X] && (170)\\
&= \mathrm{tr}\,[P_1XP_2XP_1X(1-P_1-P_2)X] && (171)\\
&= \mathrm{tr}\,[P_1XP_2XP_1X^2] - (1211) - (1212), && (172)
\end{aligned}$$

$$\begin{aligned}
(2123) &= \mathrm{tr}\,[P_2XP_1XP_2XP_3X] && (173)\\
&= \mathrm{tr}\,[P_2XP_1XP_2X(1-P_1-P_2)X] && (174)\\
&= \mathrm{tr}\,[P_2XP_1XP_2X^2] - (2121) - (2122), && (175)
\end{aligned}$$

and

$$\begin{aligned}
(3231) &= \mathrm{tr}\,[(1-P_1-P_2)XP_2X(1-P_1-P_2)XP_1X] && (176)\\
&= \mathrm{tr}\,[XP_2X^2P_1X] - \mathrm{tr}\,[XP_2XP_1XP_1X] - \mathrm{tr}\,[XP_2XP_2XP_1X] - \mathrm{tr}\,[P_1XP_2X^2P_1X] && (177)\\
&\quad + (1211) + (1221) - \mathrm{tr}\,[P_2XP_2X^2P_1X] + (2211) + (2221). && (178)
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathrm{tr}\, F_1^2 + \mathrm{tr}\, F_2^2 + \mathrm{tr}\, F_3^2 &= 2\mathrm{tr}\,[P_1XP_2XP_1X^2 + P_2XP_1XP_2X^2 + XP_2X^2P_1X && (179)\\
&\qquad - XP_2XP_1XP_1X - XP_2XP_2XP_1X - P_1XP_2X^2P_1X - P_2XP_2X^2P_1X] && (180)\\
&= 2\mathrm{tr}\,[P_1XP_2XP_1X^2 - P_1XP_2X^2P_1X + P_1X^2P_2XP_1X && (181)\\
&\qquad + P_2XP_1XP_2X^2 - P_2XP_1X^2P_2X + P_2X^2P_1XP_2X && (182)\\
&\qquad - P_1X^2P_2X^2] && (183)\\
&= -2\mathrm{tr}\,[P_1XP_2XP_1dX - P_1XP_2dXP_1X + P_1dXP_2XP_1X && (184)\\
&\qquad + P_2XP_1XP_2dX - P_2XP_1dXP_2X + P_2dXP_1XP_2X && (185)\\
&\qquad + P_1dXP_2dX] && (186)\\
&= -2d\mathrm{tr}\,[P_1XP_2XP_1X + P_2XP_1XP_2X + P_1XP_2dX] && (187)\\
&= -2d\mathrm{tr}\,[P_1XP_2XP_1X + P_1XP_2XP_2X - P_1XP_2X^2] && (188)\\
&= -2d\mathrm{tr}\,[P_1XP_2XP_1X + P_1XP_2XP_2X - P_1XP_2X(P_1+P_2+P_3)X] && (189)\\
&= 2d\mathrm{tr}\,[P_1XP_2XP_3X]. && (190)
\end{aligned}$$

This proves the claim.

- The proof above seems unrelated to $d\mathrm{tr}\,[U^\dagger dU]^3 = 0$. Among the 81 terms $(ijkl)$ generated by $\mathrm{tr}\,[X^4]$, only 12 are used. If the relations among the remaining $81 - 12 = 69$ terms are some sort of trivial relations following only from cyclicity of the trace, then it may be fair to say that $\mathrm{tr}\, F_1^2 + \mathrm{tr}\, F_2^2 + \mathrm{tr}\, F_3^2 = d(\cdots)$ is equivalent to $d\mathrm{tr}\,[U^\dagger dU]^3 = 0$.
- The existence of the exact term naively requires a modification of the action of the three-dimensional theory. I consider this point in the next section.
- The expression resembles a cocycle in noncommutative geometry. Is there any relation?

In the relation

$$\operatorname{tr} F_1^2 + \operatorname{tr} F_2^2 + \operatorname{tr} F_3^2 = 2d\operatorname{tr}[P_1 X P_2 X P_3 X], \tag{191}$$

the left-hand side is real, while the right-hand side is real up to a total derivative:

$$\operatorname{tr}[P_1 X P_2 X P_3 X]^* = \operatorname{tr}[P_1 X P_2 X P_3 X]^\dagger = \operatorname{tr}[P_1 X P_3 X P_2 X]. \tag{192}$$

Here we used $X^\dagger = -X$.

5.4 $U(3)/U(1)^3$ model

Write a field configuration $U : \mathcal{M} \to U(3)$ as

$$U = (u_1, u_2, u_3), \quad u_i^\dagger u_j = \delta_{ij}. \tag{193}$$

The diagonal Berry connections and Berry curvatures are

$$a_i = u_i^\dagger du_i, \quad f_i = da_i = du_i^\dagger du_i. \tag{194}$$

Let us write the right-hand side of the relation derived in the previous section,

$$f_1^2 + f_2^2 + f_3^2 = 2d\operatorname{tr}[P_1 X P_2 X P_3 X], \tag{195}$$

explicitly. We have

$$X_{ij} = [U^\dagger dU]_{ij} = u_i^\dagger du_j = -du_i^\dagger u_j. \tag{196}$$

Therefore

$$2\operatorname{tr}[P_1 X P_2 X P_3 X] = 2X_{12}X_{23}X_{31} = 2u_1^\dagger du_2\, u_2^\dagger du_3\, u_3^\dagger du_1. \tag{197}$$

In terms of the $U(3)$ Berry connection $a_{ij} = u_i^\dagger du_j$,

$$2\operatorname{tr}[P_1 X P_2 X P_3 X] = 2a_{12}a_{23}a_{31}. \tag{198}$$

Thus the relation implies

$$a_1 da_1 + a_2 da_2 + a_3 da_3 = 2a_{12}a_{23}a_{31} + (\text{closed 3-form}). \tag{199}$$

5.5 General flag manifolds

We look for analogous relations. First rewrite the $k = 3$ case in a form that is easy to generalize:

$$\operatorname{tr} F_1^2 + \operatorname{tr} F_2^2 + \operatorname{tr} F_3^2 = d\operatorname{tr}[P_1 X P_2 X P_3 X + P_1 X P_3 X P_2 X] \tag{200}$$

$$= \frac{1}{3}\sum_{\sigma\in A_3} d\operatorname{tr}[P_{\sigma_1} X P_{\sigma_2} X P_{\sigma_3} X] \tag{201}$$

$$= \sum_{\sigma\in A_2} d\operatorname{tr}[P_3 X P_{\sigma_1} X P_{\sigma_2} X]. \tag{202}$$

A direct guess for the general case is

$$\sum_{i=1}^{k} \operatorname{tr} F_i^2 = \sum_{a,b,c\in\{1,\ldots,k\}} d\operatorname{tr}[P_a X P_b X P_c X]. \tag{203}$$

Start from

$$0 = d\mathrm{tr}\,[X^3] = \sum_{i,j,k\in\{1,\ldots,p\}} d\mathrm{tr}\,[P_i X P_j X P_k X]. \tag{204}$$

We look for combinations that vanish trivially. Use

$$\sum_i P_i = 1, \quad dX = -X^2. \tag{205}$$

Since

$$F_i = -P_i X(1-P_i)XP_i, \tag{206}$$

we have

$$\sum_i \mathrm{tr}\, F_i^2 = \sum_i \mathrm{tr}\,[P_i X(1-P_i)XP_i X(1-P_i)X]. \tag{207}$$

Let $I \in \{1,\ldots,N-1\}$. Then

$$F_i = -P_i X(1-P_i)XP_i, \tag{208}$$

and

$$\begin{aligned}
\sum_i \mathrm{tr}\, F_i^2 &= \sum_i \mathrm{tr}\,[P_i X(1-P_i)XP_i X(1-P_i)X] && (209)\\
&= \sum_I \mathrm{tr}\,[P_I X(1-P_I)XP_I X(1-P_I)X] + \mathrm{tr}\,[P_N X(1-P_N)XP_N X(1-P_N)X] && (210)\\
&= \sum_I \mathrm{tr}\,[P_I X(1-P_I)XP_I X(1-P_I)X] && (211)\\
&\quad + \sum_{J,L} \mathrm{tr}\,[(1-\sum_I P_I)XP_J X(1-\sum_K P_K)XP_L X] && (212)\\
&= \sum_I \mathrm{tr}\,[P_I X^2 P_I X^2 - P_I X^2 P_I X P_I X - P_I X P_I X P_I X^2 + P_I X P_I X P_I X P_I X] && (213)\\
&\quad + \sum_{J,L} \mathrm{tr}\,[XP_J X^2 P_L X] - \sum_{I,J,L} \mathrm{tr}\,[P_I X P_J X^2 P_L X] - \sum_{J,K,L} \mathrm{tr}\,[XP_J X P_K X P_L X] && (214)\\
&\quad + \sum_{I,J,K,L} \mathrm{tr}\,[P_I X P_J X P_K X P_L X] && (215)\\
&= \sum_I \mathrm{tr}\,[P_I X^2 P_I X^2 - P_I X^2 P_I X P_I X - P_I X P_I X P_I X^2] && (216)\\
&\quad - \sum_{I,J} \mathrm{tr}\,[P_I X^2 P_J X^2] - \sum_{I,J,K} \mathrm{tr}\,[P_I X P_J X^2 P_K X] + \sum_{I,J,K} \mathrm{tr}\,[P_I X P_J X P_K X^2] && (217)\\
&= \sum_I \mathrm{tr}\,[P_I dX P_I dX + P_I dX P_I X P_I X + P_I X P_I X P_I dX] && (218)\\
&\quad - \sum_{I,J} \mathrm{tr}\,[P_I dX P_J dX] + \sum_{I,J,K} \mathrm{tr}\,[P_I X P_J dX P_K X] - \sum_{I,J,K} \mathrm{tr}\,[P_I X P_J X P_K dX] && (219)\\
&= \sum_I d\mathrm{tr}\,[P_I X P_I dX + \frac{2}{3} P_I X P_I X P_I X] && (220)\\
&\quad - \sum_{I,J} d\mathrm{tr}\,[P_I X P_J dX] - \sum_{I,J,K} d\mathrm{tr}\,[\frac{2}{3} P_I X P_J X P_K X] && (221)\\
&= \sum_I d\mathrm{tr}\,[-P_I X P_I X^2 + \frac{2}{3} P_I X P_I X P_I X] + \sum_{I,J} d\mathrm{tr}\,[P_I X P_J X^2] && (222)\\
&\quad - \sum_{I,J,K} d\mathrm{tr}\,[\frac{2}{3} P_I X P_J X P_K X]. && (223)
\end{aligned}$$

Here I used

$$\mathrm{tr}\,[P_I dX P_I X P_I X] = -\mathrm{tr}\,[P_I X P_I dX P_I X] = \mathrm{tr}\,[P_I X P_I X P_I dX] = \frac{1}{3} d\mathrm{tr}\,[P_I X P_I X P_I X], \tag{224}$$

$$\sum_{I,J,K} \mathrm{tr}\,[P_I dX P_J X P_K X] = -\sum_{I,J,K} \mathrm{tr}\,[P_I X P_J dX P_K X] = \sum_{I,J,K} \mathrm{tr}\,[P_I X P_J X P_K dX] \tag{225}$$

$$= \frac{1}{3} \sum_{I,J,K} d\mathrm{tr}\,[P_I X P_J X P_K X]. \tag{226}$$

Simplifying further,

$$\sum_{I,J} \mathrm{tr}\,[P_I X P_J X^2] = \sum_{I} \mathrm{tr}\,[P_I X P_I X^2] + \sum_{I,J,\ I\neq J} \mathrm{tr}\,[P_I X P_J X^2], \tag{227}$$

and

$$\sum_{I,J,K} \mathrm{tr}\,[P_I X P_J X P_K X] = \sum_{I} \mathrm{tr}\,[P_I X P_I X P_I X] + 3 \sum_{I,J,\ I\neq J} \mathrm{tr}\,[P_I X P_I X P_J X] \tag{228}$$

$$+ \sum_{I,J,K,\ I\neq J,\ J\neq K,\ K\neq I} \mathrm{tr}\,[P_I X P_J X P_K X]. \tag{229}$$

Thus

$$\sum_i \mathrm{tr}\, F_i^2 = d\mathrm{tr}\,\Bigg[\sum_{I,J,\ I\neq J} P_I X P_J X^2 - 2 \sum_{I,J,\ I\neq J} P_I X P_I X P_J X \tag{230}$$

$$-\frac{2}{3} \sum_{I,J,K,\ I\neq J,\ J\neq K,\ K\neq I} P_I X P_J X P_K X \Bigg] \tag{231}$$

$$= d\mathrm{tr}\,\Bigg[\sum_{I,J,K,\ I\neq J} P_I X P_J X P_K X + \sum_{I,J,\ I\neq J} P_I X P_J X P_N X \tag{232}$$

$$-2 \sum_{I,J,\ I\neq J} P_I X P_I X P_J X - \frac{2}{3} \sum_{I,J,K,\ I\neq J,\ J\neq K,\ K\neq I} P_I X P_J X P_K X \Bigg]. \tag{233}$$

Using

$$\sum_{I,J,K,\ I\neq J} \mathrm{tr}\,[P_I X P_J X P_K X] = \sum_{I,J,\ I\neq J} \mathrm{tr}\,[P_I X P_J X (P_I + P_J + \sum_{K,\ K\neq I,\ K\neq J} P_K) X] \tag{234}$$

$$= 2 \sum_{I,J,\ I\neq J} \mathrm{tr}\,[P_I X P_J X P_I X] + \sum_{I,J,K,\ I\neq J,\ K\neq I,\ K\neq J} \mathrm{tr}\,[P_I X P_J X P_K X], \tag{235}$$

we obtain

$$\sum_i \mathrm{tr}\, F_i^2 = d\mathrm{tr}\,\Bigg[\sum_{I,J,K,\ I\neq J,\ K\neq I,\ K\neq J} P_I X P_J X P_K X + \sum_{I,J,\ I\neq J} P_I X P_J X P_N X \tag{236}$$

$$-\frac{2}{3} \sum_{I,J,K,\ I\neq J,\ J\neq K,\ K\neq I} P_I X P_J X P_K X \Bigg] \tag{237}$$

$$= d\mathrm{tr}\,\Bigg[\frac{1}{3} \sum_{I,J,K,\ I\neq J,\ J\neq K,\ K\neq I} P_I X P_J X P_K X + \sum_{I,J,\ I\neq J} P_I X P_J X P_N X \Bigg] \tag{238}$$

$$= d\mathrm{tr}\,\Bigg[\sum_{I<J<K<N} P_I X P_J X P_K X + \sum_{I<J<K<N} P_I X P_K X P_J X + \sum_{I,J,\ I\neq J} P_I X P_J X P_N X \Bigg]. \tag{239}$$

Thus we get a gauge-invariant exact form.

6 Topological action of the $\mathbb{C}P^1$ model

6.1 Effective action

Since $\Omega_3^{\text{spin}}(S^2) = \mathbb{Z}_2$, there is a spin-bordism invariant action in three spacetime dimensions. It is known to be given by the Hopf term:

$$\exp \frac{i}{4\pi} \int_M \text{“}ada\text{''}. \tag{240}$$

Here the $U(1)$ field a is defined from the configuration $M \to S^2$, $(t, x, y) \mapsto \boldsymbol{n}(t, x, y)$, by writing $z \in \mathbb{C}^2$, $|z| = 1$, and

$$a = -iz^\dagger dz, \quad z^\dagger \boldsymbol{\sigma} z = \boldsymbol{n}. \tag{241}$$

The field $\boldsymbol{n}$ is invariant under the gauge transformation $z \mapsto ze^{i\alpha}$. The gauge-invariant form da should be expressible directly in terms of $\boldsymbol{n}$. To see this, note that

$$\boldsymbol{n} \cdot \boldsymbol{\sigma} = z_a^\dagger \sigma_{ab}^i z_b \sigma_{cd}^i = z_a^\dagger z_b (2\delta_{ad}\delta_{bc} - \delta_{ab}\delta_{cd}) = 2z_c z_d^\dagger - \delta_{cd} = 2zz^\dagger - 1. \tag{242}$$

Taking $P = zz^\dagger = (1 + \boldsymbol{n} \cdot \boldsymbol{\sigma})/2$ as a projection matrix,

$$\operatorname{tr}[P(dP)^2] = \operatorname{tr}[zz^\dagger d(zz^\dagger) d(zz^\dagger)] = dz^\dagger dz = ida, \tag{243}$$

and

$$\operatorname{tr}[P(dP)^2] = \frac{1}{8} \operatorname{tr}[(1 + \boldsymbol{n} \cdot \boldsymbol{\sigma}) d\boldsymbol{n} \cdot \boldsymbol{\sigma}\, d\boldsymbol{n} \cdot \boldsymbol{\sigma}] = \frac{i}{4} \epsilon_{ijk} n_i\, dn_j\, dn_k. \tag{244}$$

Thus

$$\frac{1}{2\pi} da = \frac{1}{8\pi} \epsilon_{ijk} n_i\, dn_j\, dn_k. \tag{245}$$

This shows that a skyrmion line corresponds to a monopole line of a. Also note that z cannot be chosen globally near a skyrmion.

In general, a is not globally defined, for example when a single skyrmion exists. Thus the above effective action requires introducing a four-dimensional manifold X with $\partial X = M$. Then $\boldsymbol{n} : M \to S^2$ cannot in general be extended to the spin manifold X, because $\Omega_3^{\text{spin}}(S^2) = \mathbb{Z}_2$. However, since $\Omega_3^{\text{spin}}(BU(1)) = 0$, or equivalently since enlarging by one component gives $\Omega_3^{\text{spin}}(\mathbb{C}P^2) = 0$, if we regard it as an arbitrary $U(1)$ field it can be extended over the spin manifold X. That is, there exists

$$\tilde{a} : X \to BU(1), \quad \tilde{a}|_M = a. \tag{246}$$

Equivalently, there exists

$$\tilde{z} : X \to \mathbb{C}^\infty, \quad \tilde{z}^\dagger \tilde{z} = 1, \quad \tilde{z}|_M = \begin{pmatrix} z_1 \\ z_2 \\ 0 \\ 0 \\ \vdots \end{pmatrix}. \tag{247}$$

Choose such an extension and **define**

$$\exp \frac{i}{4\pi} \int_M ada := \exp \frac{i}{4\pi} \int_X \tilde{f}^2. \tag{248}$$

From the general theory of spin Chern-Simons theory, when X is a closed spin manifold and $\tilde{a}$ is a $U(1)$ field,

$$\frac{i}{4\pi}\int_X \tilde{f}^2 \in 2\pi i\mathbb{Z}. \tag{249}$$

Therefore the definition is independent of the extension to X.

With the action of the $\mathbb{C}P^1$ model defined as above, its spin-bordism invariance is shown as follows. Let $(M_{12}, a_{12}) : (M_1, a_1) \sim (M_2, a_2)$ be a spin bordism of $\mathbb{C}P^1$ fields. If X is a bounding manifold for M_1, then a bounding manifold for M_2 can be taken as $X \cup M_{12}$. Therefore, by definition,

$$\frac{i}{4\pi}\int_{M_2} \text{“}ada\text{''} = \frac{i}{4\pi}\int_{X\cup M_{12}} f^2. \tag{250}$$

Since, as a property of the $\mathbb{C}P^1$ field, $f^2 = (da_{12})^2 = 0$ on M_{12},

$$\frac{i}{4\pi}\int_{M_2} \text{“}ada\text{''} = \frac{i}{4\pi}\int_{X} f^2, \tag{251}$$

which is precisely the Chern-Simons effective action on M_1.

6.1.1 Example calculation

In the $\mathbb{C}P^1$ model on $S^2 \times S^1$, the configuration corresponding to a 2π rotation of a skyrmion is

$$z(\theta,\phi,\alpha) = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\alpha}e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}. \tag{252}$$

Here (θ, ϕ) are coordinates on S^2, and α is the coordinate on S^1. Since $\Omega_3^{SO}(S^2) = 0$, there should exist a four-dimensional manifold with this configuration as its boundary, but I cannot find it[4].

If we embed the configuration into the $\mathbb{C}P^2$ model as follows, there exists a four-dimensional manifold with $S^1 = \partial D^2$ [8]:

$$\tilde{z}(\theta,\phi,\alpha,\rho) = \begin{pmatrix} 1 & & \\ & \rho e^{i\alpha} & \sqrt{1-\rho^2} \\ & -\sqrt{1-\rho^2} & \rho e^{-i\alpha} \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \\ 0 \end{pmatrix}, \quad \rho\in[0,1]. \tag{253}$$

This means that a 2π rotation of a skyrmion in components 1 and 2 is bordant to a zero rotation of a skyrmion in components 1 and 3. Indeed,

$$\tilde{z}(\theta,\phi,\alpha,\rho=0) = \begin{pmatrix} \cos\frac{\theta}{2} \\ 0 \\ -e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}. \tag{254}$$

Compute $\frac{i}{4\pi}\int \text{“}ada\text{''}$. We have

$$\tilde{z} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \rho e^{i(\phi+\alpha)}\sin\frac{\theta}{2} \\ -\sqrt{1-\rho^2}e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}. \tag{255}$$

[4] Together with $\Omega_3^{\mathrm{spin}}(S^2) = \mathbb{Z}_2$ and the fact, seen below, that the effective action takes the nontrivial value (-1), this means that it is not null-bordant within spin manifolds, so a non-spin manifold must be introduced.

Note that $\tilde{a}^* = \tilde{a}$.

$$\tilde{a} = -i\tilde{z}^\dagger d\tilde{z} = \rho^2 d(\phi+\alpha)\sin^2\frac{\theta}{2} + (1-\rho^2)d\phi\sin^2\frac{\theta}{2} \tag{256}$$

$$= d\phi\sin^2\frac{\theta}{2} + \rho^2 d\alpha\sin^2\frac{\theta}{2}, \tag{257}$$

$$\tilde{f} = d\sin^2\frac{\theta}{2}\,d\phi + d\rho^2\,d\alpha\,\sin^2\frac{\theta}{2} + \rho^2 d\sin^2\frac{\theta}{2}\,d\alpha, \tag{258}$$

$$\tilde{f}^2 = 2\sin^2\frac{\theta}{2}\,d\rho^2\,d\alpha\,d\sin^2\frac{\theta}{2}\,d\phi. \tag{259}$$

Thus

$$\frac{i}{4\pi}\int_{S^2\times D^2}\tilde{f}^2 = \frac{i}{4\pi}\int_{S^2\times D^2} 2\sin^2\frac{\theta}{2}\,d\sin^2\frac{\theta}{2}\,d\phi\,d\rho^2\,d\alpha = i\pi. \tag{260}$$

6.2 Memo: calculation example for flag $U(3)/U(1)^3$

As an application of the calculation above, consider a three-dimensional configuration in the flag manifold $U(3)/U(1)^3$ in which a skyrmion is put only in the first $U(1)$ component and rotated by 2π. We compute integrals such as $a_1 da_1$, $a_2 da_2$, and $a_1 da_2$. Below we choose the configuration so that $\det(z_1, z_2, z_3) = 1$:

$$z_1 = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \\ 0 \end{pmatrix}, \quad z_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad z_3 = \begin{pmatrix} -e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \\ 0 \end{pmatrix}. \tag{261}$$

Skyrmions exist in z_1 and z_3, and each is rotated by 2π. Unlike the $\mathbb{C}P^1$ model, within the flag manifold this is bordant to zero rotation. Indeed, the following extension to $S^2 \times D^2$ can be taken:

$$\tilde{z}_i = \begin{pmatrix} 1 & & \\ & \rho e^{i\alpha} & \sqrt{1-\rho^2} \\ & -\sqrt{1-\rho^2} & \rho e^{-i\alpha} \end{pmatrix} z_i, \tag{262}$$

namely

$$\tilde{z}_1 = \begin{pmatrix} \cos\frac{\theta}{2} \\ \rho e^{i(\phi+\alpha)}\sin\frac{\theta}{2} \\ -\sqrt{1-\rho^2}e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}, \quad \tilde{z}_2 = \begin{pmatrix} 0 \\ \sqrt{1-\rho^2} \\ \rho e^{-i\alpha} \end{pmatrix}, \tag{263}$$

$$\tilde{z}_3 = \begin{pmatrix} -e^{-i\phi}\sin\frac{\theta}{2} \\ \rho e^{i\alpha}\cos\frac{\theta}{2} \\ -\sqrt{1-\rho^2}\cos\frac{\theta}{2} \end{pmatrix}. \tag{264}$$

Let

$$\tilde{a}_i = -i\tilde{z}_i^\dagger d\tilde{z}_i. \tag{265}$$

At $\rho = 0$, the α dependence disappears:

$$\tilde{z}_1|_{\rho=0} = \begin{pmatrix} \cos\frac{\theta}{2} \\ 0 \\ -e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}, \quad \tilde{z}_2|_{\rho=0} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \tag{266}$$

$$\tilde{z}_3|_{\rho=0} = \begin{pmatrix} -e^{-i\phi}\sin\frac{\theta}{2} \\ 0 \\ -\cos\frac{\theta}{2} \end{pmatrix}. \tag{267}$$

Now compute the $U(1)$ gauge fields $\tilde{a}_i = -i\tilde{z}_i^\dagger d\tilde{z}_i$.

$$\tilde{a}_1 = \sin^2\frac{\theta}{2}\,d\phi + \rho^2\sin^2\frac{\theta}{2}\,d\alpha, \tag{268}$$
$$\tilde{a}_2 = -\rho^2 d\alpha, \tag{269}$$
$$\tilde{a}_3 = -\sin^2\frac{\theta}{2}\,d\phi + \rho^2\cos^2\frac{\theta}{2}\,d\alpha. \tag{270}$$

Since $\det(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3) = 1$, note that $\tilde{a}_1 + \tilde{a}_2 + \tilde{a}_3 = 0$. The curvatures are

$$\tilde{f}_1 = d\sin^2\frac{\theta}{2}\,d\phi + \rho^2 d\sin^2\frac{\theta}{2}\,d\alpha + \sin^2\frac{\theta}{2}\,d\rho^2\,d\alpha, \tag{271}$$
$$\tilde{f}_2 = -d\rho^2\,d\alpha, \tag{272}$$
$$\tilde{f}_3 = -d\sin^2\frac{\theta}{2}\,d\phi + \rho^2 d\cos^2\frac{\theta}{2}\,d\alpha + \cos^2\frac{\theta}{2}\,d\rho^2\,d\alpha. \tag{273}$$

Therefore

$$\frac{i}{4\pi}\int_{S^2\times D^2}\tilde{f}_1^2 = i\pi, \tag{274}$$
$$\frac{i}{4\pi}\int_{S^2\times D^2}\tilde{f}_2^2 = 0, \tag{275}$$
$$\frac{i}{4\pi}\int_{S^2\times D^2}\tilde{f}_3^2 = \frac{i}{4\pi}\int_{S^2\times D^2} -2\cos^2\frac{\theta}{2}d\sin^2\frac{\theta}{2}\,d\phi\,d\rho^2\,d\alpha = -i\pi, \tag{276}$$
$$\frac{i}{4\pi}\int_{S^2\times D^2}\tilde{f}_1\tilde{f}_2 = \frac{i}{4\pi}\int_{S^2\times D^2} -d\sin^2\frac{\theta}{2}\,d\phi\,d\rho^2\,d\alpha = -i\pi. \tag{277}$$

Next compute $\mathrm{Re}\,[a_{12}a_{23}a_{31}]$. Since it is gauge invariant, it is enough to compute it on $S^2 \times S^1$. Because $a_{12} = -iz_1^\dagger dz_2 = 0$, we get $\mathrm{Re}\,[a_{12}a_{23}a_{31}] = 0$. In particular,

$$\frac{i}{4\pi}\int_{S^2\times D^2}(\tilde{f}_1^2 + \tilde{f}_2^2 + \tilde{f}_1\tilde{f}_2 - \mathrm{Re}\,[a_{12}a_{23}a_{31}]) = 0. \tag{278}$$

This calculation is not in contradiction with $\Omega_3^{\mathrm{spin}}(U(3)/U(1)^3) = 0$. By analogy with $\mathbb{C}P^1$, the relation in the flag manifold

$$f_1^2 + f_2^2 + f_1\tilde{f}_2 - \mathrm{Re}\,[a_{12}a_{23}a_{31}] = 0 \tag{279}$$

suggests the existence of an effective action quantized in $\mathbb{Z}_2$,

$$\exp\frac{i}{4\pi}\int_X(\tilde{f}_1^2 + \tilde{f}_2^2 + \tilde{f}_1\tilde{f}_2 - \mathrm{Re}\,[a_{12}a_{23}a_{31}]) \in \{\pm 1\}. \tag{280}$$

Here $\tilde{a}_1, \tilde{a}_2$ are extensions not as flag fields but as independent $U(1)$ fields. However, if $\tilde{a}_1, \tilde{a}_2$ are independent $U(1)$ fields, then even when X is spin,

$$\frac{i}{4\pi}\int_X \tilde{f}_1\tilde{f}_2 \in \pi i\mathbb{Z}, \tag{281}$$

so the expression is ill-defined.

7 Imaginary-time evolution and Berry phase

As in real-time evolution, a discrete eigenstate also acquires a Berry phase in the adiabatic limit under imaginary-time evolution.

Consider imaginary-time evolution with an imaginary-time-dependent Hamiltonian $H(\tau)$. The Schrodinger equation is

$$\frac{d}{d\tau}\left|\psi(\tau)\right\rangle = -H(\tau)\left|\psi(\tau)\right\rangle . \tag{282}$$

Assume that the eigenvalues of $H(\tau)$ are discrete, and write the instantaneous eigenstates as

$$H(\tau)\left|n(\tau)\right\rangle = E_n(\tau)\left|n(\tau)\right\rangle , \quad n = 1, 2, \dots . \tag{283}$$

Then

$$\langle m(\tau)|\frac{dH}{d\tau}(\tau)|n(\tau)\rangle = \frac{dE_n}{d\tau}(\tau)\delta_{nm} + (E_n(\tau) - E_m(\tau))\,\langle m(\tau)|\frac{dn}{d\tau}(\tau)\rangle . \tag{284}$$

In particular,

$$\langle n(\tau)|\frac{dH}{d\tau}(\tau)|n(\tau)\rangle = \frac{dE_n}{d\tau}(\tau) \tag{285}$$

is the Hellmann-Feynman formula. Also, when $E_n(\tau) \neq E_m(\tau)$,

$$\langle m(\tau)|\frac{dn}{d\tau}(\tau)\rangle = \frac{\langle m(\tau)|\frac{dH}{d\tau}(\tau)|n(\tau)\rangle}{E_n(\tau) - E_m(\tau)}. \tag{286}$$

Expand the wave function as

$$\left|\psi(\tau)\right\rangle = \sum_n c_n(\tau)\left|n(\tau)\right\rangle . \tag{287}$$

Substituting this into the Schrodinger equation gives

$$\frac{d}{d\tau}c_n(\tau) = -E_n(\tau)c_n(\tau) - \sum_m \langle n(\tau)|\frac{dm}{d\tau}(\tau)\rangle\, c_m(\tau). \tag{288}$$

Assume $E_n(\tau) \neq E_m(\tau)$ for $m \neq n$. Then

$$\frac{d}{d\tau}c_n(\tau) = -E_n(\tau)c_n(\tau) - \langle n(\tau)|\frac{dn}{d\tau}(\tau)\rangle\, c_n(\tau) - \sum_{m\neq n}\frac{\langle n(\tau)|\frac{dH}{d\tau}(\tau)|m(\tau)\rangle}{E_m(\tau) - E_n(\tau)}c_m(\tau). \tag{289}$$

If the imaginary-time evolution of the system is sufficiently slow compared with the energy difference $E_m(\tau) - E_n(\tau)$, the third term can be neglected:

$$\frac{d}{d\tau}c_n(\tau) = -E_n(\tau)c_n(\tau) - \langle n(\tau)|\frac{dn}{d\tau}(\tau)\rangle\, c_n(\tau). \tag{290}$$

Thus

$$c_n(\tau) = e^{-\int_0^\tau E_n(\tau')d\tau'}e^{-\int_0^\tau \langle n(\tau')|\frac{dn}{d\tau}(\tau')\rangle d\tau'}c_n(\tau = 0). \tag{291}$$

If $H(\tau)$ is Hermitian, the first factor does not affect the $U(1)$ phase of the wave function. Assuming $|n(\tau)\rangle$ is normalized, $\langle n|n\rangle = 1$, we have

$$\langle n|dn\rangle^* = \langle dn|n\rangle = -\langle n|dn\rangle , \tag{292}$$

so the second factor is a $U(1)$ phase factor. In particular, for a closed path $H(T) = H(0)$, one obtains the Berry phase:

$$c_n(T) = e^{-\int_0^T E_n(\tau)d\tau}e^{-\oint\langle n|dn\rangle}c_n(\tau = 0). \tag{293}$$

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