A Theorem on Reductions of MPS

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Abstract

Following Appendix B of [1], we trace the proof of a theorem on reductions of MPS.

1 Preliminaries

An MPS $\{A^i\}_{i=1}^d, A^i \in \mathrm{M}_D(\mathbb{C})$ with bond dimension D will be called a D-MPS. When it is unnecessary to specify the bond dimension, we simply write MPS. We say that an MPS $\{A^i\}_i$ is injective when the linear map

$$f_A : X \mapsto \sum_i \mathrm{tr}\,[XA^i]\,|i\rangle \tag{1.1}$$

is injective. Injectivity of an MPS is equivalent to the existence of a left inverse of the matrices A^i,

$$\sum_i [A^i]_{ab}[A^{-1}]^i_{cd} = \delta_{ac}\delta_{bd}. \tag{1.2}$$

Indeed, if a left inverse exists, then

$$\mathrm{tr}\,[XA^i] = 0 \quad \Rightarrow \quad 0 = \sum_i \mathrm{tr}\,[XA^i][A^{-1}]^i_{cd} = \sum_i \sum_{ab} X_{ba}[A^i]_{ab}[A^{-1}]^i_{ab} = \sum_{ab} X_{ba}\delta_{ac}\delta_{bd} = X_{dc}. \tag{1.3}$$

Conversely, if $\{A^i\}_i$ is injective, then, after choosing suitable bases, the map

$$f_A : \mathbb{C}^{D^2} \to \mathbb{C}^d \tag{1.4}$$

can be represented as the matrix

$$f_A = \begin{pmatrix} I_{D^2} \\ O_{d-D^2} \end{pmatrix}. \tag{1.5}$$

The left inverse is then given by

$$\begin{pmatrix} I_{D^2} & * \end{pmatrix}. \tag{1.6}$$

An MPS $\{A^i\}$ is called normal if there exists a natural number $\ell \in \mathbb{N}$ such that the MPS obtained by blocking ℓ sites,

$$\{\tilde{A}^I\}_I = \{A^{i_1} \cdots A^{i_\ell}\}_{i_1,\cdots,i_\ell}, \tag{1.7}$$

is injective.

We abbreviate the length-n vector constructed from the MPS $\{A^i\}_i$ as

$$V_n(A) = \sum_{i_1 \cdots i_n} \mathrm{tr}\,[A^{i_1} \cdots A^{i_n}]\,|i_1 \cdots i_n\rangle\,. \tag{1.8}$$

2 Reduction theorem

Proposition 2.1. *Let A be a normal D-MPS, and let B be a D'-MPS. If*

$$V_n(B) = V_n(A), \quad \forall n \in \mathbb{N} \tag{2.1}$$

holds,[a] *then there exist rectangular matrices $V \in \mathrm{M}_{D\times D'}(\mathbb{C})$ and $W \in \mathrm{M}_{D'\times D}(\mathbb{C})$ such that*

$$VW = I_D, \tag{2.3}$$

$$VB^{i_1}\cdots B^{i_n}W = A^{i_1}\cdots A^{i_n}, \quad \forall (i_1,\ldots,i_n) \in \{1,\ldots,d\}^{\times n}, \quad n \in \mathbb{N}. \tag{2.4}$$

[a] In [1], the statement is that there exists $\lambda \in \mathbb{C}$ such that

$$V_n(B) = \lambda^n V_n(A), \quad \forall n \in \mathbb{N}, \tag{2.2}$$

but λ can be removed by the redefinition $B^i \mapsto \lambda B^i$. Also in the proof of [1], $\lambda = 1$ is implicitly assumed.

Proof. Since A is normal, there exists $L \in \mathbb{N}$ such that $\tilde{A}^I = A^{i_1}\cdots A^{i_L}$ is injective and has a left inverse $\tilde{A}^{-1}$. Put $\tilde{B}^I = B^{i_1}\cdots B^{i_L}$ and consider the Jordan decomposition

$$\sum_I \tilde{B}^I \otimes [\tilde{A}^{-1}]^I = U\left(\bigoplus_k J_{a_k}(\lambda_k)\right)U^{-1}. \tag{2.5}$$

Here

$$J_a(\lambda) = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & \\ & & & \lambda \end{pmatrix} \in \mathrm{M}_a(\mathbb{C}). \tag{2.6}$$

By decomposing each Jordan block as

$$J_a(\lambda) = \begin{pmatrix} \lambda & & & \\ & \lambda & & \\ & & \ddots & \\ & & & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \tag{2.7}$$

we obtain a decomposition

$$\sum_I \tilde{B}^I \otimes [\tilde{A}^{-1}]^I = S + N \tag{2.8}$$

where S is diagonalizable, N is nilpotent, and $[S, N] = O$.

By the assumption that the MPSs A, B generate the same vectors,

$$\sum_{I_1\cdots I_n} \mathrm{tr}\,[\tilde{B}^{I_1}\cdots\tilde{B}^{I_n}]\mathrm{tr}\,[[\tilde{A}^{-1}]^{I_1}\cdots[\tilde{A}^{-1}]^{I_n}] = \sum_{I_1\cdots I_n}\sum_{I_1\cdots I_n} \mathrm{tr}\,[\tilde{A}^{I_1}\cdots\tilde{A}^{I_n}]\mathrm{tr}\,[[\tilde{A}^{-1}]^{I_1}\cdots[\tilde{A}^{-1}]^{I_n}] = D^n. \tag{2.9}$$

On the other hand, the left-hand side is

$$\mathrm{Tr}\,(S+N)^n = \mathrm{Tr}\,S^n + \mathrm{Tr}\,N^n = \mathrm{Tr}\,S^n = \sum_k a_k\lambda_k^n. \tag{2.10}$$

Therefore

$$\sum_{k=1} a_k\lambda_k^n = D^n, \quad \forall n \in \mathbb{N} \tag{2.11}$$

holds. If there are Jordan blocks with the same eigenvalue, we combine their contributions. Then, with distinct nonzero eigenvalues $\lambda_1, \dots, \lambda_M$ and $b_k \in \mathbb{N}$,

$$\sum_{k=1}^{M} b_k \lambda_k^n = D^n, \quad \forall n \in \mathbb{N} \tag{2.12}$$

holds. Using the equations for $n = 1, \dots, M+1$, we get

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & D \\ \vdots & & & \\ \lambda_1^{M+1} & \lambda_2^{M+1} & \cdots & D^{M+1} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \\ -1 \end{pmatrix} = \vec{0}. \tag{2.13}$$

The determinant of the coefficient matrix V (a Vandermonde matrix) is

$$\det V = (\lambda_1 \dots \lambda_M D) \prod_{1 \le i < j \le M+1} (\lambda_j - \lambda_i) \tag{2.14}$$

where we have set $\lambda_{M+1} = D$. If $\lambda_1, \dots, \lambda_M, D$ were all distinct, this determinant would be nonzero, a contradiction. Hence D is equal to one of the λ_k. We may take $\lambda_1 = D$. Then

$$(b_1 - 1)\lambda_1^n + \sum_{k=2}^{M} b_k \lambda_k^n = 0, \quad \forall n \in \mathbb{N} \tag{2.15}$$

holds, and the same argument gives

$$b_1 = 1, \quad b_2 = \dots = b_M = 0. \tag{2.16}$$

Thus the only nonzero eigenvalue of S is D, and there is exactly one 1×1 Jordan block with eigenvalue D:

$$\sum_I \tilde{B}^I \otimes [\tilde{A}^{-1}]^I = U \left((D) \oplus \bigoplus_k J_{a_k}(0) \right) U^{-1}. \tag{2.17}$$

Since S has rank one, there is a decomposition

$$[S]_{ac,bd} = W_{ac} V_{db}, \quad \operatorname{tr} VW = \operatorname{Tr} S = D. \tag{2.18}$$

Moreover,

$$SN = NS = 0 \tag{2.19}$$

holds. In particular, if $n = \min\{k \in \mathbb{N} | N^k = O\}$ is the nilpotency index of N, then

$$(S + N)^n = S^n = D^{n-1} S \tag{2.20}$$

holds. Note that $n \le DD'$.

Now take this n to be the nilpotency index of N and consider the following expression:

$$\sum_{I_1 \cdots I_n} \operatorname{tr} [\tilde{B}^{I_1} \cdots \tilde{B}^{I_n} B^{i_1} \cdots B^{i_m}][[\tilde{A}^{-1}]^{I_1} \cdots [\tilde{A}^{-1}]^{I_n}]_{ab}. \tag{2.21}$$

First, since A, B give the same state, the left inverse of $\tilde{A}$ gives

$$= \sum_{I_1 \cdots I_n} \operatorname{tr} [\tilde{A}^{I_1} \cdots \tilde{A}^{I_n} A^{i_1} \cdots A^{i_m}][[\tilde{A}^{-1}]^{I_1} \cdots [\tilde{A}^{-1}]^{I_n}]_{ab} \tag{2.22}$$

$$= D^{n-1} [A^{i_1} \cdots A^{i_m}]_{ba}. \tag{2.23}$$

On the other hand, using the decomposition (2.8), it is equal to

$$= D^{n-1}[VB^{i_1}\cdots B^{i_m}W]_{ba}. \tag{2.24}$$

In particular, when $m=0$,

$$VW = I_D. \quad \square \tag{2.25}$$

Proposition 2.2. *Let V,W be a reduction from the MPS B to the MPS A. Set*

$$N^i := B^i - WA^iV. \tag{2.26}$$

The algebra generated by $\{N^i\}_i$ is nilpotent. That is, for the algebra

$$\mathcal{A} = \mathrm{Span}\left(\{N^{i_1}\cdots N^{i_n}|\forall(i_1,\ldots,i_n)\in\{1,\ldots,d\}^{\times n},\forall n\in\mathbb{N}\}\right), \tag{2.27}$$

for every element $x\in\mathcal{A}$ there exists $k\in\mathbb{N}$ such that $x^k=0$.

Proof. First we show the following:

$$VN^{i_1}\cdots N^{i_m}W = 0, \quad m\in\mathbb{N}. \tag{2.28}$$

For $m=1$,

$$VN^iW = V(B^i - WA^iV)W = VB^iW - A^i = 0. \tag{2.29}$$

Assume the statement holds up to $m-1$. Then

$$\begin{aligned} VN^{i_1}\cdots N^{i_m}W &= V(B^{i_1}-WA^{i_1}V)N^{i_2}\cdots N^{i_m}W = VB^{i_1}N^{i_2}\cdots N^{i_m}W &(2.30)\\ &= \cdots &(2.31)\\ &= VB^{i_1}B^{i_2}\cdots B^{i_{m-1}}(B^{i_m}-WA^{i_m}V)W = VB^{i_1}\cdots B^{i_m}W - A^{i_1}\cdots A^{i_m} = 0. &(2.32)\end{aligned}$$

Therefore

$$\mathrm{tr}\,[B^{i_1}\cdots B^{i_m}] = \mathrm{tr}\,[(N^{i_1}+WA^{i_1}V)\cdots(N^{i_m}+WA^{i_m}V)] = \mathrm{tr}\,[A^{i_1}\cdots A^{i_m}] + \mathrm{tr}\,[N^{i_1}\cdots N^{i_m}]. \tag{2.33}$$

Since A,B are the same state,

$$\mathrm{tr}\,[N^{i_1}\cdots N^{i_m}] = 0, \quad m\in\mathbb{N}. \tag{2.34}$$

In particular, for any $x\in\mathcal{A}$ and any $n\in\mathbb{N}$, $\mathrm{tr}\,[x^n]=0$. Thus all eigenvalues of x are zero. Hence every element of $\mathcal{A}$ is nilpotent. $\square$

We also show the following.

Proposition 2.3. *Let V,W be a reduction from the MPS B to the MPS A, and set $N^i := B^i - WA^iV$. Then*

$$N^{i_1}\cdots N^{i_{D'}} = 0, \quad \forall(i_1,\ldots,i_{D'})\in\{1,\ldots,d\}^{\times D'}. \tag{2.35}$$

Here D' is the matrix size of B^i.

Proof.[1] First, we prove by contradiction that

$$\bigcap_{i=1}^{d}\ker N^i \neq \{0\}, \tag{2.36}$$

[1]The following proof is due to Akito Shimomura.

namely, that there exists $v \neq 0$ such that $N^i v = 0$ for all $i = 1, \ldots, d$.

Assume that for every $v \neq 0$ there exists i such that $N^i v \neq 0$. Then, starting from any nonzero vector v, we can successively obtain $D' + 1$ nonzero vectors

$$v = v_0, \tag{2.37}$$

$$v_n = N^{i_n} v_{n-1}, \quad n = 1, \ldots, D'. \tag{2.38}$$

We show that $v_0, \ldots, v_{D'}$ are linearly independent. Suppose

$$\alpha_0 v_0 + \cdots \alpha_{D'} v_{D'} = \alpha_0 v + \alpha_1 N^{i_1} v + \alpha_2 N^{i_2} N^{i_1} v + \cdots + \alpha_{D'} N^{i_{D'}} \cdots N^{i_1} v = 0. \tag{2.39}$$

Then

$$-\alpha_0 v = \underbrace{(\alpha_1 N^{i_1} + \alpha_2 N^{i_2} N^{i_1} + \cdots + \alpha_{D'} N^{i_{D'}} \cdots N^{i_1})}_{\in \mathcal{A}} v. \tag{2.40}$$

Since every element of $\mathcal{A}$ has only the eigenvalue zero, we get $\alpha_0 = 0$. Similarly,

$$-\alpha_1 v_1 = \underbrace{(\alpha_2 N^{i_2} + \cdots + \alpha_{D'} N^{i_{D'}} \cdots N^{i_2})}_{\in \mathcal{A}} v_1 \tag{2.41}$$

implies $\alpha_1 = 0$. Repeating the same argument shows $\alpha_0 = \cdots = \alpha_{D'-1} = 0$, and $\alpha_{D'} v_{D'} = 0$ gives $\alpha_{D'} = 0$. Thus $v_0, \ldots, v_{D'}$ are linearly independent, contradicting the fact that the vector space on which N^i is defined has dimension D'.

Hence there exists a vector $e_1 \neq 0$ such that $N^i v = 0$ for all $i = 1, \ldots, d$. Extend it to a basis of $\mathbb{C}^{D'}$ by adding vectors $e_2, \ldots, e_{D'}$. In the basis

$$(e_1, e_2, \ldots, e_{D'}), \tag{2.42}$$

the matrices N^i have the following representation:

$$N^i (e_1, e_2, \ldots, e_{D'}) = (e_1, e_2, \ldots, e_{D'}) \begin{pmatrix} 0 & * \\ 0 & \tilde{N}^i \end{pmatrix}, \quad i = 1, \ldots, d. \tag{2.43}$$

Then the algebra generated by $\{\tilde{N}^i\}_{i=1}^d$ is again nilpotent. Continuing the same argument, we find that $N^1, \ldots, N^d$ are nilpotent and simultaneously triangularizable. □

The length D' is an upper bound coming from the matrix size. We define

$$N_0(V, W) := \min_n \{N^{i_1} \cdots N^{i_{n+1}} = O, \forall (i_1, \ldots, i_{n+1}) \in \{1, \ldots, d\}^{\times (n+1)}\} \tag{2.44}$$

to be the nilpotency length of the reduction (V, W) from the MPS B to A. That is, products of N^i of length $N_0(V, W)$ can be nonzero, while products of length $N_0(V, W) + 1$ are zero.

Theorem 2.4. *Let (V, W) and (V', W') be reductions from the MPS B to the MPS A. Let N_0 be the maximum of the nilpotency lengths of the reductions (V, W) and (V', W'):*

$$N_0 = \max\{N_0(V, W), N_0(V', W')\}. \tag{2.45}$$

Then there exists $\lambda \in \mathbb{C}$ such that, for all $n > 2N_0$,

$$V B^{i_1} \cdots B^{i_n} = \lambda V' B^{i_1} \cdots B^{i_n}, \tag{2.46}$$

$$B^{i_1} \cdots B^{i_n} W = \lambda^{-1} B^{i_1} \cdots B^{i_n} W' \tag{2.47}$$

hold.

Proof. Set

$$N^i = B^i - W A^i V, \quad N'^i = B^i - W' A^i V'. \tag{2.48}$$

First note the following identities, which hold for arbitrary $n \in \mathbb{N}$:

$$\begin{aligned}
B^{i_1}\cdots B^{i_n} &= N^{i_1}\cdots N^{i_n} + \sum_{0\le k<l\le n} N^{i_1}\cdots N^{i_k} W A^{i_{k+1}}\cdots A^{i_l} V N^{i_{l+1}}\cdots N^{i_n} \\
&= N^{i_1}\cdots N^{i_n} + \sum_{0\le k<l\le n, k\le N_0, n-N_0\le l} N^{i_1}\cdots N^{i_k} W A^{i_{k+1}}\cdots A^{i_l} V N^{i_{l+1}}\cdots N^{i_n}, \qquad (2.49)
\end{aligned}$$

$$\begin{aligned}
VB^{i_1}\cdots B^{i_n} &= VN^{i_1}\cdots N^{i_n} + \sum_{0<l\le n} A^{i_1}\cdots A^{i_l} V N^{i_{l+1}}\cdots N^{i_n} = \sum_{0\le l\le n} A^{i_1}\cdots A^{i_l} V N^{i_{l+1}}\cdots N^{i_n} \\
&= \sum_{\max(0,n-N_0)\le l\le n} A^{i_1}\cdots A^{i_l} V N^{i_{l+1}}\cdots N^{i_n}, \qquad (2.50)
\end{aligned}$$

$$\begin{aligned}
B^{i_1}\cdots B^{i_n} W &= N^{i_1}\cdots N^{i_n} W + \sum_{0\le k<n} N^{i_1}\cdots N^{i_k} W A^{i_{k+1}}\cdots A^{i_n} = \sum_{0\le k\le n} N^{i_1}\cdots N^{i_k} W A^{i_{k+1}}\cdots A^{i_n} \\
&= \sum_{0\le k\le \min(n,N_0)} N^{i_1}\cdots N^{i_k} W A^{i_{k+1}}\cdots A^{i_n}. \qquad (2.51)
\end{aligned}$$

The analogous identities also hold for the reduction (V', W').

Let L be the injective length of the normal MPS A, and express

$$VB^{i_1}\cdots B^{i_{2N_0+L}} W' \qquad (2.52)$$

with length $m = 2N_0 + L$ in two ways. From (2.50),

$$= \sum_{N_0+L\le l\le 2N_0+L} A^{i_1}\cdots A^{i_l} V N^{i_{l+1}}\cdots N^{i_{2N_0+L}} W'. \qquad (2.53)$$

From (2.51),

$$= \sum_{0\le k\le N_0} N'^{i_1}\cdots N'^{i_k} W' A^{i_{k+1}}\cdots A^{i_{2N_0+L}}. \qquad (2.54)$$

Contracting the sites $N_0+1, \dots, N_0+L$ with the left inverse $\tilde{A}^{-1}$ gives

$$\begin{aligned}
&\sum_{i_{N_0+1},\dots,i_{N_0+L}} [VB^{i_1}\cdots B^{i_{2N_0+L}} W']_{ab} [\tilde{A}^{-1}]_{cd}^{i_{N_0+1}\cdots i_{N_0+L}} \qquad (2.55) \\
&= [A^{i_1}\cdots A^{i_{N_0}}]_{ac} \sum_{N_0+L\le l\le 2N_0+L} [A^{i_{N_0+L+1}}\cdots A^{i_l} V N^{i_{l+1}}\cdots N^{i_{2N_0+L}} W']_{db} \qquad (2.56) \\
&= \sum_{0\le k\le N_0} [VN'^{i_1}\cdots N'^{i_k} W' A^{i_{k+1}}\cdots A^{i_{N_0}}]_{ac} [A^{i_{N_0+L+1}}\cdots A^{i_{2N_0+L}}]_{db}. \qquad (2.57)
\end{aligned}$$

Therefore there exists $\lambda \in \mathbb{C}$ such that

$$\sum_{0\le k\le N_0} VN'^{i_1}\cdots N'^{i_k} W' A^{i_{k+1}}\cdots A^{i_{N_0}} = \lambda A^{i_1}\cdots A^{i_{N_0}}, \qquad (2.58)$$

$$\sum_{0\le k\le N_0} A^{i_1}\cdots A^{i_k} V N^{i_{k+1}}\cdots N^{i_{N_0}} W' = \lambda^{-1} A^{i_1}\cdots A^{i_{N_0}} \qquad (2.59)$$

hold. Then, using (2.49) for the reduction (V', W'),

$$VB^{i_1}\cdots B^{i_n} = VN'^{i_1}\cdots N'^{i_n} + \sum_{0\le k<l\le n} VN'^{i_1}\cdots N'^{i_k} W' A^{i_{k+1}}\cdots A^{i_l} V' N'^{i_{l+1}}\cdots N'^{i_n}. \qquad (2.60)$$

In the second term, only the terms with $0 \le k \le N_0$ and $n - N_0 \le l \le n$ can remain nonzero. Thus, when $n > 2N_0$,

$$VB^{i_1}\cdots B^{i_n} = \sum_{0\le k\le N_0<n-N_0\le l\le n} VN'^{i_1}\cdots N'^{i_k} W' A^{i_{k+1}}\cdots A^{i_l} V' N'^{i_{l+1}}\cdots N'^{i_n} \quad (n > 2N_0). \qquad (2.61)$$

By (2.58),

$$VB^{i_1}\cdots B^{i_n} = \lambda A^{i_1}\cdots A^{i_{N_0}} \sum_{n-N_0\le l\le n} A^{i_{N_0+1}}\cdots A^{i_l} V' N'^{i_{l+1}}\cdots N'^{i_n} \quad (n>2N_0). \tag{2.62}$$

Using (2.50) for the reduction (V', W'), this becomes

$$VB^{i_1}\cdots B^{i_n} = \lambda V' B^{i_1}\cdots B^{i_n} \quad (n>2N_0). \tag{2.63}$$

Similarly, starting from the expression

$$V' B^{i_1}\cdots B^{i_m} W, \tag{2.64}$$

one obtains

$$B^{i_1}\cdots B^{i_m} W = \lambda^{-1} B^{i_1}\cdots B^{i_m} W'. \tag{2.65}$$

□

Moreover, the following statement holds, guaranteeing that, if sufficiently long MPS B tensors are present on both sides, then the "bulk part" can be replaced by A.[2]

Proposition 2.5. *Let (V, W) be a reduction from the MPS B to A, and let $N_0 = N_0(V, W)$ be its nilpotency length. If $m, m' \geq N_0$, then for any $n \geq 1$,*

$$B^{i_1}\cdots B^{i_m} W A^{i_{m+1}}\cdots A^{i_{m+n}} V B^{i_{m+n+1}}\cdots B^{i_{m+n+m'}} = B^{i_1}\cdots B^{i_{m+n+m'}}, \tag{2.66}$$

and if $m \geq N_0$, then for any $n \geq 0$,

$$B^{i_1}\cdots B^{i_m} W A^{i_{m+1}}\cdots A^{i_{m+n}} = B^{i_1}\cdots B^{i_{m+n}} W, \tag{2.67}$$

$$A^{i_1}\cdots A^{i_n} V B^{i_{n+1}}\cdots B^{i_{n+m}} = V B^{i_1}\cdots B^{i_{n+m}}. \tag{2.68}$$

Proof. From (2.50) and (2.51), for arbitrary m, m', n,

$$B^{i_1}\cdots B^{i_m} W A^{i_{m+1}}\cdots A^{i_{m+n}} V B^{i_{m+n+1}}\cdots B^{i_{m+n+m'}} \tag{2.69}$$

$$= \left(\sum_{0\le k\le \min(m,N_0)} N^{i_1}\cdots N^{i_k} W A^{i_{k+1}}\cdots A^{i_m}\right) A^{i_{m+1}}\cdots A^{i_{m+n}} \tag{2.70}$$

$$\left(\sum_{\max(m+n,m+n+m'-N_0)\le m+n+l\le m+n+m'} A^{i_{m+n+1}}\cdots A^{i_{m+n+l}} V N^{i_{m+n+l+1}}\cdots N^{i_{m+n+m'}}\right). \tag{2.71}$$

This agrees with the rewriting (2.49) of $B^{i_1}\cdots B^{i_{m+n+m'}}$ precisely when

$$m+n+m' > N_0, \quad m < m+n, \quad m \geq N_0, \quad m' \geq N_0, \tag{2.72}$$

that is, when

$$m, m' \geq N_0, \quad n \geq 1. \tag{2.73}$$

Similarly, for arbitrary m, n,

$$B^{i_1}\cdots B^{i_m} W A^{i_{m+1}}\cdots A^{i_{m+n}} \tag{2.74}$$

$$= \left(\sum_{0\le k\le m} N^{i_1}\cdots N^{i_k} W A^{i_{k+1}}\cdots A^{i_m}\right) A^{i_{m+1}}\cdots A^{i_{m+n}}, \tag{2.75}$$

[2] This proposition is not stated explicitly in [1], but it is repeatedly used in papers, including some by later authors, with [1] cited.

and this agrees with the rewriting (2.51) of $B^{i_1}\cdots B^{i_{m+n}}W$ when $m \geq N_0$. The identity (2.68) follows similarly. □

For example, when $N_0 = 1$,

$$
\begin{aligned}
B^iB^jB^k &= (N^i + WA^iV)(N^j + WA^jV)(N^k + WA^kV) && (2.76)\\
&= N^iWA^jVN^k + N^iWA^jA^kV + WA^iA^jVN^k + WA^iA^jA^kV && (2.77)\\
&= (N^i + WA^iV)WA^jV(N^k + WA^kV) && (2.78)\\
&= B^iWA^jVB^k. && (2.79)
\end{aligned}
$$

Summary:

For an MPS A, write

$$V_n(A) = \mathrm{tr}\,[A^{i_1}\cdots A^{i_n}]. \tag{2.80}$$

- For an MPS B and a normal MPS A, if $V_n(B) = V_n(A)$ holds for all $n \in \mathbb{N}$, then there exist rectangular matrices V, W such that

$$VW = I_D, \tag{2.81}$$

$$V\underbrace{B\cdots B}_{\geq 0}W = A\cdots A. \tag{2.82}$$

  The product WV is a projection.

- Let N_0 be the nilpotency length of the algebra generated by the matrices $N^i = B^i - WA^iV$ associated with a reduction (V, W) (the largest length n for which a product $N^{i_1}\cdots N^{i_n}$ can be nonzero; equivalently, for $n > N_0$, $N^{i_1}\cdots N^{i_n} = O$). Then the following hold:

$$\underbrace{B\cdots B}_{\geq N_0}W\underbrace{A\cdots A}_{\geq 1}V\underbrace{B\cdots B}_{\geq N_0} = B\cdots B. \tag{2.83}$$

$$\underbrace{B\cdots B}_{\geq N_0}W\underbrace{A\cdots A}_{\geq 0} = B\cdots BW. \tag{2.84}$$

$$\underbrace{A\cdots A}_{\geq 0}V\underbrace{B\cdots B}_{\geq N_0} = VB\cdots B. \tag{2.85}$$

- Let (V, W) and (V', W') be reductions from B to A, and let N_0, N_0' be the nilpotency lengths of (V, W) and (V', W'), respectively. Then there exists $\lambda \in \mathbb{C}$ such that

$$V\underbrace{B\cdots B}_{\geq 2\max(N_0,N_0')+1} = \lambda V'B\cdots B, \tag{2.86}$$

$$\underbrace{B\cdots B}_{\geq 2\max(N_0,N_0')+1}W = \lambda^{-1}B\cdots BW' \tag{2.87}$$

  hold.

- Is the nilpotency length invariant under blocking? For example, if the nilpotency length is ℓ, can it be made 1 by blocking ℓ sites? Naively,

$$N^{i_1}N^{i_2} = (B^{i_1} - WA^{i_1}V)(B^{i_2} - WA^{i_2}V) \neq B^{i_1}B^{i_2} - WA^{i_1}A^{i_2}V, \tag{2.88}$$

so it seems that the nilpotency length changes under blocking.

- In [2], the term nilpotency length probably refers not to the shortest length for which products of N^i become zero, but to the length of B for which (2.83), (2.84), and (2.85) hold. Since the reduction (V, W) can be chosen unchanged under blocking, by blocking N_0 sites the identities (2.83), (2.84), and (2.85) can be made to hold at length one. In this sense, the claim in [2] that the nilpotency length can always be taken to be 1 by blocking is correct.

References

[1] Andras Molnar, Yimin Ge, Norbert Schuch, J. Ignacio Cirac, *A generalization of the injectivity condition for Projected Entangled Pair States*, arXiv:1706.07329.

[2] Adrian Franco-Rubio, Arkadiusz Bochniak, J. Ignacio Cirac, *Symmetry defects and gauging for quantum states with matrix product unitary symmetries*, arXiv:2502.20257.