Eikonal Approximation

Scattering Amplitude

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = E\psi$$

$$\leftrightarrow \quad \left(-\frac{\hbar^2}{2m}\nabla^2 - E\right)\psi = -V\psi$$

$\therefore$

$$\psi^{(+)}(\mathbf{r}) = N e^{i\vec{k}\cdot\vec{r}} + \int d\mathbf{r}\ G^{(+)}(\mathbf{r},\mathbf{r}')\,V(\mathbf{r}')\,\psi^{(+)}(\mathbf{r}')$$

$$G^{(+)}(\mathbf{r},\mathbf{r}') = \langle \mathbf{r}|\,\frac{1}{-\frac{\hbar^2}{2m}\nabla^2 - E - i\eta}\,|\mathbf{r}'\rangle$$

$$= -\frac{2m}{\hbar^2}\cdot\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi\,|\mathbf{r}-\mathbf{r}'|}$$

$$= N e^{i\vec{k}\cdot\vec{r}} - \frac{2m}{\hbar^2}\cdot\frac{1}{4\pi}\int d\mathbf{r}'\,\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\,V(\mathbf{r}')\,\psi^{(+)}(\mathbf{r}')$$

(note) as $r \to \infty$

$$k|\mathbf{r}-\mathbf{r}'| = k\sqrt{r^2 - 2\mathbf{r}\cdot\mathbf{r}' + r'^2} \sim kr - k\hat{\mathbf{r}}\cdot\mathbf{r}'$$

$\therefore$

$$\psi^{(+)}(\mathbf{r}) \to N e^{i\vec{k}\cdot\vec{r}} - \frac{m}{2\pi\hbar^2}\cdot\frac{e^{ikr}}{r}\int d\mathbf{r}'\, e^{-i\vec{k}'\cdot\vec{r}'}\,V(\mathbf{r}') \times \psi^{(+)}(\mathbf{r}')$$

$$(\mathbf{k}' = k\hat{\mathbf{r}})$$

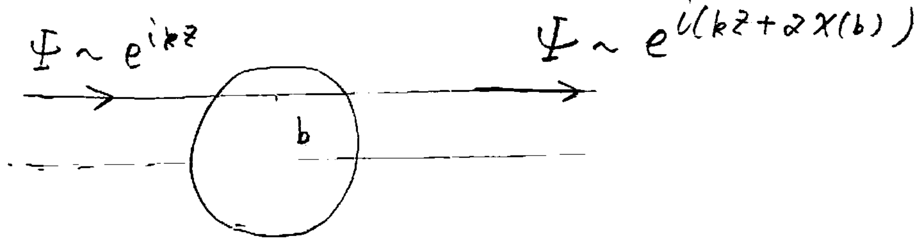
$$\longleftrightarrow \quad \psi(\mathbf{r}) \to N\left(e^{i\mathbf{k}\cdot\mathbf{r}} + f(\hat{\mathbf{r}})\,\frac{e^{ikr}}{r}\right)$$

$$\downarrow \quad \boxed{f(\hat{\mathbf{r}}) = -\frac{m}{2\pi\hbar^2 N}\int d\mathbf{r}'\, e^{-i\mathbf{k}'\cdot\mathbf{r}'}\, V(\mathbf{r}')\, \psi^{(+)}(\mathbf{r}')}$$

(note) Born Approximation:

$$f(\hat{\mathbf{r}}) \sim -\frac{m}{2\pi\hbar^2 \cancel{N}}\int d\mathbf{r}'\, e^{-i\mathbf{k}'\cdot\mathbf{r}'}\, V(\mathbf{r}')\cdot \cancel{N} e^{i\mathbf{k}\cdot\mathbf{r}'}$$

◦ Eikonal Approximation



set $\quad \underbrace{\psi^{(+)}(\mathbf{r}) = e^{i\vec{k}\cdot\vec{r}}\,\phi(\mathbf{r})}$

assume that ϕ is slowly varying

$$\downarrow \quad -\frac{\hbar^2}{2m}\left(\cancel{-k^2} + 2i\vec{k}\cdot\nabla + \underbrace{\nabla^2)\phi}_{\simeq\,0}\right) + \cancel{(V-E)}\phi = 0$$

$$\downarrow \quad ik\frac{\hbar^2}{m}\frac{\partial}{\partial z}\phi(\vec{b},z) = V\phi \qquad (\mathbf{k} = k\,\mathbf{e}_z)$$

(note) $\phi \to 1$ as $z \to -\infty$

$$\downarrow \quad \phi(b,z) = e^{2i\chi(b,z)}$$

$$\boxed{\chi(b,z) = -\frac{m}{2\hbar^2 k}\int_{-\infty}^{z} V(b,z')\,dz'}$$

$$f = -\frac{m}{2\pi\hbar^2}\int d^2b \int_{-\infty}^{\infty} dz \; e^{-i\vec{q}\cdot\vec{r}}\, \underline{V(b,z)\,\phi(b,z)}$$

$$(\vec{q} = \mathbb{k}' - \mathbb{k})$$

(note) $\quad V\phi = ik\frac{\hbar^2}{m}\partial_z\phi$

$\quad q_z \sim 0 \qquad$ (forward angle scattering)

$$\downarrow$$

$$f = -\frac{\cancel{m}}{2\pi\cancel{\hbar^2}}\int d^2b \; e^{-i\vec{q}\cdot\vec{b}} \int_{-\infty}^{\infty} dz \cdot ik\frac{\cancel{\hbar^2}}{\cancel{m}}\partial_z\phi$$

$$= -\frac{ik}{2\pi}\int d^2b \; e^{-i\vec{q}\cdot\vec{b}} \left[\phi(b,z)\right]_{z=-\infty}^{\infty}$$

$$= -\frac{ik}{2\pi}\int d^2b \; e^{-i\vec{q}\cdot\vec{b}} \left(e^{2i\chi(b)} - 1\right)$$

$$\chi(b) \equiv \chi(b, z=\infty) = -\frac{m}{2\hbar^2 k}\int_{-\infty}^{\infty} V(b,z')\,dz'$$

"profile function"

(note) $\int d^2b \; e^{-i\vec{q}\cdot\vec{b}} = \int_0^\infty b\,db \underbrace{\int_0^{2\pi} d\varphi \; e^{-iqb\cos\varphi}}_{\substack{\shortparallel \\ 2\pi J_0(qb)}}$

$\Downarrow$

$$\boxed{f = -ik \int_0^\infty b\,db \; J_0(qb)\left(e^{2i\chi(b)} - 1\right)}$$