Feschbach Formalism

$$\hat{P} = |\phi_0\rangle\langle\phi_0| \quad , \quad \hat{Q} = 1 - \hat{P} = 1 - |\phi_0\rangle\langle\phi_0|$$

(note) $\hat{P}^2 = |\phi_0\rangle\langle\phi_0|\phi_0\rangle\langle\phi_0| = |\phi_0\rangle\langle\phi_0| = \hat{P}$

$$\hat{Q}^2 = (1 - |\phi_0\rangle\langle\phi_0|)(1 - |\phi_0\rangle\langle\phi_0|)$$
$$= 1 - |\phi_0\rangle\langle\phi_0| - |\phi_0\rangle\langle\phi_0| + |\phi_0\rangle\langle\phi_0|$$
$$= \hat{Q}$$

$$\hat{P}\hat{Q} = \hat{Q}\hat{P} = 0$$

$$(\hat{H} - E)|\Psi\rangle = 0$$

$$(\hat{H} - E)(\hat{P} + \hat{Q})|\Psi\rangle = 0$$

$\hat{P}\rightarrow$ $\quad (\hat{P}\hat{H}\hat{P} + \hat{P}\hat{H}\hat{Q} - E\hat{P})|\Psi\rangle = 0$

$\hat{Q}\rightarrow$ $\quad (\hat{Q}\hat{H}\hat{P} + \hat{Q}\hat{H}\hat{Q} - E\hat{Q})|\Psi\rangle = 0$

$$\downarrow \quad (\hat{Q}\hat{H}\hat{Q} - E)\hat{Q}|\Psi\rangle = -\hat{Q}\hat{H}\hat{P}|\Psi\rangle$$

$$\downarrow \quad \hat{Q}|\Psi\rangle = \frac{1}{E - \hat{Q}\hat{H}\hat{Q} + i\eta}\hat{Q}\hat{H}\hat{P}|\Psi\rangle$$

$$\downarrow \quad \underbrace{\left(\hat{P}\hat{H}\hat{P} + \hat{P}\hat{H}\hat{Q}\frac{1}{E - \hat{Q}\hat{H}\hat{Q} + i\eta}\hat{Q}\hat{H}\hat{P} - E\right)}_{\hat{H}_{eff}}\hat{P}|\Psi\rangle = 0$$

(note) $\dfrac{1}{x+i\eta} = P\dfrac{1}{x} - i\pi\delta(x)$

$$\frac{1}{x+i\eta} = \frac{x-i\eta}{x^2+\eta^2} = \frac{1}{x} - i\underbrace{\frac{\eta}{x^2+\eta^2}}_{\pi\cdot\delta(x)}$$