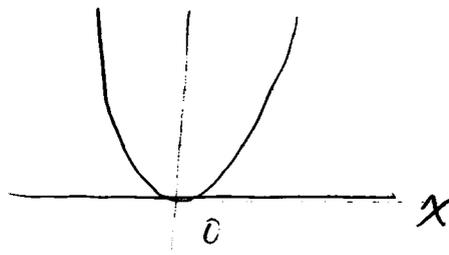
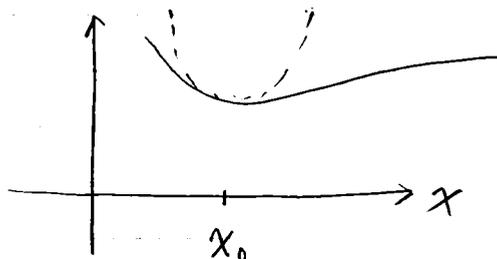


3.8. 調和振動子ポテンシャル

$$V(x) = \frac{1}{2} m \omega^2 x^2$$



(note)



$$V(x) \sim V(x_0) + \frac{1}{2} V''(x_0) (x-x_0)^2 + \dots$$

$$\left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m \omega^2 x^2 \right) \phi(x) = E \phi(x)$$

• $x \rightarrow \pm\infty$ での解の様子:

$$\left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m \omega^2 x^2 \right) \phi(x) \sim 0$$

$$\text{解: } \phi(x) \sim e^{-\frac{m\omega}{2\hbar} x^2}$$

$$\phi'(x) = -\frac{m\omega}{\hbar} x e^{-\frac{m\omega}{2\hbar} x^2}$$

$$\phi''(x) = -\frac{m\omega}{\hbar} e^{-\frac{m\omega}{2\hbar} x^2} + \left(\frac{m\omega}{\hbar}\right)^2 x^2 e^{-\frac{m\omega}{2\hbar} x^2}$$

$$\sim \left(\frac{m\omega}{\hbar}\right)^2 x^2 e^{-\frac{m\omega}{2\hbar} x^2}$$

$\leadsto \phi(x) = f(x) e^{-\frac{m\omega}{2\hbar} x^2}$ とおくと $f(x)$ は $x \rightarrow \pm\infty$ で有限.

$$\rightarrow -\frac{\hbar^2}{2m} f'' + \hbar\omega x f' = \left(E - \frac{\hbar\omega}{2}\right) f$$

or $y = \sqrt{\frac{m\omega}{\hbar}} x$

$$\rightarrow f'' - 2y f' + \left(\frac{2E}{\hbar\omega} - 1\right) f(y) = 0$$

(note) エルミート多項式

$$\left(\frac{d^2}{dx^2} - 2x \frac{d}{dx} + 2n\right) H_n(x) = 0$$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

$$\int_{-\infty}^{\infty} dx H_n(x) H_m(x) e^{-x^2} = \sqrt{\pi} 2^n n! \delta_{m,n}$$

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

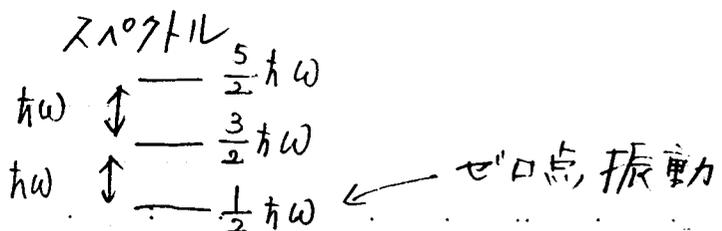
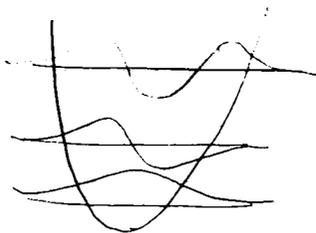
⋮

$$\rightarrow f(y) = H_n(y) \quad ; \quad \frac{2E}{\hbar\omega} - 1 = 2n$$

$$\rightarrow E = \left(n + \frac{1}{2}\right) \hbar\omega$$

規格化まで含めると

$$\Phi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m\omega}{\hbar}} x\right) e^{-\frac{m\omega}{2\hbar} x^2}$$



3.9. 調和振動子ホトシヤル：代数学的解法

$$a^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}} \hat{x} - i \frac{\hat{p}}{\sqrt{2m\omega\hbar}}$$

$$\rightarrow a = (a^{\dagger})^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}} \hat{x} + i \frac{\hat{p}}{\sqrt{2m\omega\hbar}}$$

(note) $[a, a^{\dagger}] = \sqrt{\frac{m\omega}{2\hbar}} \cdot \left(\frac{-i}{\sqrt{2m\omega\hbar}} \right) [x, p]$
 $+ \frac{i}{\sqrt{2m\omega\hbar}} \cdot \sqrt{\frac{m\omega}{2\hbar}} [p, x]$
 $= -\frac{i}{\hbar} [x, p] = 1$

↓

$$[a, (a^{\dagger})^n] = n(a^{\dagger})^{n-1}$$

- $[a, a^{\dagger}] = 1$

- $[a, (a^{\dagger})^n] = n(a^{\dagger})^{n-1}$ とおくと

$$[a, (a^{\dagger})^{n+1}] = (a^{\dagger})^n [a, a^{\dagger}] + [a, (a^{\dagger})^n] a^{\dagger}$$

$$= (a^{\dagger})^n + n(a^{\dagger})^n = (n+1)(a^{\dagger})^n$$

任意の関数 $f(a^{\dagger})$ に対して

$$f(a^{\dagger}) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) (a^{\dagger})^n$$

$$\rightarrow [a, f(a^{\dagger})] = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) \cdot n (a^{\dagger})^{n-1}$$

$$= f'(a^{\dagger})$$

すなわち $a = \frac{d}{da^{\dagger}}$ と見て

同様に $a^{\dagger} = -\frac{d}{da}$; $[a^{\dagger}, g(a)] = -g'(a)$

$$a + a^\dagger = 2 \cdot \sqrt{\frac{m\omega}{2\hbar}} \hat{x} \quad \rightarrow \quad \hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger)$$

$$a^\dagger - a = -2 \cdot \frac{i}{\sqrt{2m\omega\hbar}} \hat{p} \quad \rightarrow \quad \hat{p} = i \sqrt{\frac{m\omega\hbar}{2}} (a^\dagger - a)$$

↓

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2$$

$$= -\frac{1}{2m} \cdot \frac{m\omega\hbar}{2} (a^\dagger - a)^2 + \frac{1}{2} m \omega^2 \cdot \frac{\hbar}{2m\omega} (a + a^\dagger)^2$$

$$= \frac{\hbar\omega}{4} \left(\begin{array}{cccc} -a^\dagger a^\dagger & + a^\dagger a & + a a^\dagger & - a a \\ + a^\dagger a^\dagger & + a^\dagger a & + a a^\dagger & + a a \end{array} \right) = \frac{\hbar\omega}{2} (a a^\dagger + a^\dagger a)$$

$$= \hbar\omega \left(a^\dagger a + \frac{1}{2} \right)$$

(note) $[H, a] = [\hbar\omega a^\dagger, a] a = -\hbar\omega a$

$$[H, a^\dagger] = a^\dagger [\hbar\omega a, a^\dagger] = \hbar\omega a^\dagger$$

↓

$$H |\varphi\rangle = E |\varphi\rangle \quad \varepsilon \text{ 定 } \varepsilon$$

$$H(a |\varphi\rangle) = (-\hbar\omega a + a H) |\varphi\rangle$$

$$= (E - \hbar\omega) \underbrace{(a |\varphi\rangle)}$$

$$H(a^\dagger |\varphi\rangle) = (\hbar\omega a^\dagger + a^\dagger H) |\varphi\rangle$$

$$= (E + \hbar\omega) \underbrace{(a^\dagger |\varphi\rangle)}$$

固有状態

$$\begin{array}{l} \overline{\uparrow a^\dagger} \quad E + \hbar\omega \\ \overline{} \quad E \\ \overline{\downarrow a} \quad E - \hbar\omega \end{array}$$

基底状態 (最低エネルギー状態): $a|0\rangle = 0$

$$\rightarrow H|0\rangle = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)|0\rangle = \frac{1}{2}\hbar\omega|0\rangle$$

固有値

$$a|0\rangle = 0$$

$$\begin{aligned} \rightarrow 0 &= \langle x|a|0\rangle = \langle x|\left(\sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\frac{1}{\sqrt{2m\omega\hbar}}\hat{p}\right)|0\rangle \\ &= \left(\sqrt{\frac{m\omega}{2\hbar}}x + i\frac{1}{\sqrt{2m\omega\hbar}}\cdot\frac{\hbar}{i}\frac{d}{dx}\right)\underbrace{\langle x|0\rangle}_{\phi_0(x)} \end{aligned}$$

$$\begin{aligned} \rightarrow \phi_0'(x) &= -\sqrt{\frac{2m\omega}{\hbar}}\cdot\sqrt{\frac{m\omega}{2\hbar}}x\phi_0(x) \\ &= -\frac{m\omega}{\hbar}x\phi_0(x) \end{aligned}$$

$$\rightarrow \phi_0(x) \propto e^{-\frac{m\omega}{2\hbar}x^2}$$

励起状態: $|n\rangle = A\cdot(a^\dagger)^n|0\rangle$

規格化:

$$1 = \langle n|n\rangle = A^2\langle 0|a^n(a^\dagger)^n|0\rangle$$

$$a(a^\dagger)^n|0\rangle = (a^\dagger)^n a + n(a^\dagger)^{n-1}|0\rangle = n(a^\dagger)^{n-1}|0\rangle$$

$$\leadsto a^2(a^\dagger)^n|0\rangle = na(a^\dagger)^{n-1}|0\rangle = n(n-1)(a^\dagger)^{n-2}|0\rangle$$

$$\rightarrow \text{よって } \langle 0|a^n(a^\dagger)^n|0\rangle = n!|0\rangle$$

$$\leadsto 1 = A^2 \cdot n! \quad \leadsto A = \frac{1}{\sqrt{n!}}$$

すなわち

$$|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle$$

$$\begin{aligned} \rightarrow a|n\rangle &= \frac{1}{\sqrt{n!}} [\cancel{(a^\dagger)^n} a + n (a^\dagger)^{n-1}] |0\rangle \\ &= \sqrt{n} \frac{1}{\sqrt{(n-1)!}} (a^\dagger)^{n-1} |0\rangle = \sqrt{n} |n-1\rangle \end{aligned}$$

$$a^\dagger |n\rangle = \sqrt{n+1} \cdot \frac{(a^\dagger)^{n+1}}{\sqrt{(n+1)!}} |0\rangle = \sqrt{n+1} |n+1\rangle$$

(note) $11 = 0$ a 63 $a|0\rangle = 0, a^\dagger|0\rangle = |1\rangle$

(note) $\hat{n} = a^\dagger a$

$$\hat{n}|n\rangle = a^\dagger a |n\rangle = \sqrt{n} a^\dagger |n-1\rangle = n |n\rangle$$

(note) $\frac{1}{2} m \omega^2 x^2 = \frac{m\omega^2}{2} \cdot \frac{\hbar}{2m\omega} (a^{\dagger 2} + a^2 + \underbrace{aa^\dagger + a^\dagger a}_{a^\dagger a + 1})$

$$\langle n | (a^\dagger)^2 | n \rangle \propto \langle n | n+2 \rangle = 0$$

$$\langle n | a^2 | n \rangle \propto \langle n | n-2 \rangle = 0$$

$$\Downarrow \langle n | \frac{1}{2} m \omega^2 x^2 | n \rangle = \frac{\hbar\omega}{4} \cdot (2n+1) = \frac{1}{2} \cdot \underbrace{\hbar\omega (n + \frac{1}{2})}_{E_n}$$

$$\Downarrow \langle n | \frac{1}{2} m \omega^2 x^2 | n \rangle = \langle n | \frac{p^2}{2m} | n \rangle = \frac{1}{2} E_n$$

(ビリアル定理)

(補足)

$$\langle x | \hat{p} | \phi \rangle = \int dx' \underbrace{\langle x | \hat{p} | x' \rangle}_{\parallel \phi(x')} \langle x' | \phi \rangle$$

(note) $[\hat{x}, \hat{p}] = i\hbar$

$$\begin{aligned} \rightarrow \langle x | \hat{x} \hat{p} - \hat{p} \hat{x} | x' \rangle &= i\hbar \langle x | x' \rangle \\ &= i\hbar \delta(x-x') \\ &= \downarrow \\ &= (x-x') \langle x | \hat{p} | x' \rangle \end{aligned}$$

(note) $(x-x') \frac{d}{dx} \delta(x-x') = -\delta(x-x')$

(証明)

$$-\int dx \delta(x-x') f(x) = -f(x')$$

$$\int dx (x-x') \frac{d}{dx} \delta(x-x') \cdot f(x)$$

$$= -\int dx \frac{d}{dx} ((x-x') f(x)) \delta(x-x')$$

$$= -\int dx (f(x) + (x-x') f'(x)) \delta(x-x')$$

$$= -f(x')$$

$$\leadsto \langle x | \hat{p} | x' \rangle = -i\hbar \frac{d}{dx} \delta(x-x') = \frac{\hbar}{i} \frac{d}{dx} \delta(x-x')$$

$$\leadsto \langle x | \hat{p} | \phi \rangle = \int dx' \frac{\hbar}{i} \frac{d}{dx} \delta(x-x') \phi(x') = \frac{\hbar}{i} \frac{d}{dx} \phi(x)$$

(note) 超対称性量子力学

$$\left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x)\right) \psi_0(x) = 0 \quad (\text{エネルギーが } \lambda \gamma - 1L \text{ 相当} \\ \text{なとき})$$

$$\rightarrow V \psi_0 = \frac{\hbar^2}{2m} \psi_0''$$

$$\rightarrow H = \frac{\hbar^2}{2m} \left(-\frac{d^2}{dx^2} + \frac{\psi_0''}{\psi_0}\right)$$

$$A^\dagger = \frac{\hbar}{\sqrt{2m}} \left(-\frac{d}{dx} - \frac{\psi_0'}{\psi_0}\right), \quad A = \frac{\hbar}{\sqrt{2m}} \left(\frac{d}{dx} - \frac{\psi_0'}{\psi_0}\right)$$

とすると

$$A \psi_0 = \frac{\hbar}{\sqrt{2m}} \left(\psi_0' - \frac{\psi_0'}{\psi_0} \psi_0\right) = 0$$

$$A^\dagger A = \frac{\hbar^2}{2m} \left(-\frac{d^2}{dx^2} + \underbrace{\left(\frac{\psi_0'}{\psi_0}\right)'} + \frac{\psi_0'}{\psi_0} \frac{d}{dx} - \frac{\psi_0'}{\psi_0} \frac{d}{dx} + \left(\frac{\psi_0'}{\psi_0}\right)^2\right) = \frac{\psi_0 \psi_0'' - \psi_0'^2}{\psi_0^2}$$

$$= H$$

$$\tilde{H} = A A^\dagger$$

$$\text{E.L. } H \psi_n = A^\dagger A \psi_n = E_n \psi_n \quad \text{ただし}$$

$$\tilde{H} (A \psi_n) = A A^\dagger A \psi_n = E_n (A \psi_n)$$

