格子カイラルゲージ理論の構成法

— Luscherの積分可能条件とEichten-Preskill-WenのDecoupling条件

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based on :

Y. K. PTEP 2019 (2019) no.11, 113B03, arXiv:1710.11618[hep-lat]

PTEP 2019 (2019) no. 7, 073B02, arXiv:1710.11101[hep-lat]

J.W.Pedersen & Y.K.

arXiv:22mm.xxx[hep-lat] (cf. 数理科学2020.1)

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Chiral gauge theories

- The standard model (SM)
  quarks and leptons belong to complex reps. of the gauge groups

  SU(3)xSU(2)xU(1) xU(1)$_{B-L}$ (**3**, **2**) $_{1/6}$ (**1**, **2**) $_{-1/2}$
  (**3***, **1**) $_{-2/3}$ (**3***, **1**) $_{1/3}$ (**1**, **1**) $_{1}$ (**1**, **1**) $_{0}$

- GUT models are also chiral:

  SU(5) xU(1)$_Q$ (**10**) $_{1}$
  (**5***) $_{-3}$ (**1**) $_{5}$

  SO(10) (**16**)

- Gauge anomaly cancellation
  - local gauge anomalies are cancelled non-trivially, $\Sigma_R Tr_R[\, T^a \{T^b, T^c\} \,]=0$
  - there is no global gauge anomaly [*D. S. Freed (2008)*]

    $\Omega^{spin}{}_5(BSO(10)) = 0$ [*Garcia-Etxebarria-Montero (2018)*]

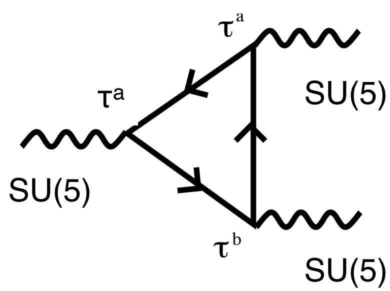
    [*Wang-Wen, Wang-Wen-Witten (2018)*]

- Other properties & Various non-perturbative dynamical effects expected
  - gauge inv. fermion bilinear terms $\psi\varepsilon\psi$ are forbidden
  - fermion number symmetry (B, L) is broken by chiral anomaly
  - realizations of gauge/global (flavor) symmetries
  - even in Euclidean formulation, the action is complex => Sign problem

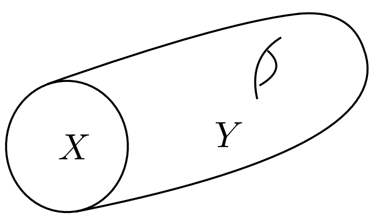
SU(5) x U(1)$_Q$ $\quad$ $\underline{10}_{1} + \underline{5^*}_{-3}$

SU(5) gauge symmetry

$Tr_{\underline{10}}[\,T^a\,\{T^b,T^a\}] + Tr_{\underline{5*}}[\,T^a\,\{T^b,T^a\}] = 0$



$\Omega^{spin}{}_5(BSU(5)) = 0$



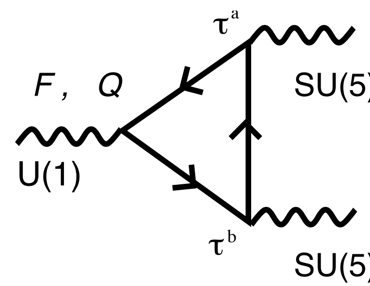
$$\exp\left(2\pi i \eta_Y\right)$$

$$\eta_Y = \frac{1}{2}\left(\sum_{\lambda\neq 0}\mathrm{sign}(\lambda) + \dim\ker(i\mathcal{D}_Y)\right)_{\text{reg.}}$$

$$\exp(2\pi i\,\eta_{Y_1\sqcup\overline{Y_2}}) = \frac{\exp(2\pi i\,\eta_{Y_1})}{\exp(2\pi i\,\eta_{Y_2})}$$

bordism invariant <=> SPT

U(1)$_F$ chiral anomaly



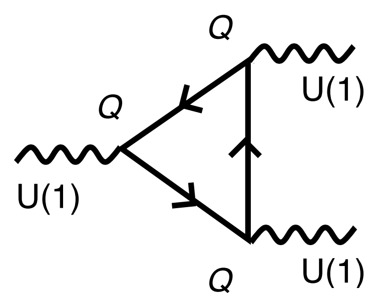
't Hooft vertex

($\underline{10}$ x $\underline{10}$ x $\underline{10}$ x $\underline{5^*}$)$_0$

U(1)$_Q$ "global" symmetry

$(1) \times Tr_{\underline{10}}[\,T^a\,T^b] + (-3) \times Tr_{\underline{5*}}[\,T^a\,T^b] = 0$

't Hooft anomaly



$(-5)^3$

π / SSB of U(1)$_Q$

B_{-5} = ($\underline{10}$ x $\underline{5^*}$ x $\underline{5^*}$)$_{-5}$

$\underline{1}_{5}$ = spectator fermion /SO(10)

Two approaches to non-abelian chiral gauge theory on the lattice with exact gauge invariance

- Overlap Weyl fermion / Neuberger's overlap Dirac operator / Ginsparg-Wilson rel.
- SO(10), SU(5) $\times U(1)_Q$, SU(3)xSU(2)xU(1) $\times U(1)_{B-L}$, …

  no local and global gauge anomalies

  $\Sigma_R \, Tr_R[\, T^a \{T^b, T^c\} \,]=0$, $\Omega^{spin}{}_5(BSO(10)) = 0$
- establish Luscher's integrability for chiral determinant of overlap Weyl fermions
  - Integrability condition with 5-, and 6-dimensional lattice Domain-wall fermions

    cf. bordism inv. / η-invariant, Dai-Freed theorem and APS index theorem
- decouple the mirror d.o.f. of overlap Dirac fermions by Eichten-Preskill-Wen / Kitaev-Fidkowski quartic interaction terms
  - 't Hooft vertex-type SO(10) / SO(7), SO(6) invariant quartic terms,

    which break explicitly the global symmetries with 't Hooft anomaly

    (16 x 16 x 16 x 16) = (10 x 10 x 10 x 5*) $_0$ **+ (10 x 5* x 5*)** $_{-5}$ **x 1** $_5$

    $\lambda \left[\psi^T(x) i\gamma_5 c_d C\Gamma^a \psi(x)\right]^2 \Leftrightarrow y \left[\psi^T(x) i\gamma_5 c_d C\Gamma^a \psi(x)\right] E^a(x)$ $E^a(x)E^a(x) = 1$

    $\pi_d(S^9) = 0 \quad (d = 0, \cdots, 9)$

    cf. gapped boundary of interaction-reduced trivial phase of SPT

0. Basic facts about Lattice Fermions

I. Integrability condition for Chiral determinant of Overlap Weyl fermions

I-I. Topological classification of chiral anomaly of overlap fermions in even dim. non-abelian lattice gauge theories

I-II. Proof of global integrability condition

II. Eichten-Preskill model / Mirror fermion model with overlap fermions

Discussion

- Hamiltoninan formalism

0. Basic facts about lattice Fermions

Ginsparg-Wilson rel. & Overlap Dirac operator as a local covariant sol.

- Ginsparg-Wilson relation defines the chiral limit of lattice fermion action

$$\gamma_5 D_*^{-1} + D_*^{-1}\gamma_5 = 2a\gamma_5\delta_{xy}$$

  - Low energy effective lattice action at IR fixed point of block spin tr.

    [*Ginsparg-Wilson(1982)*]

  - Exact chiral symmetry emerges [*Luscher (1999)*]

$$\delta S = 0 \qquad \delta_\alpha \psi(x) = i\alpha\,\gamma_5(1-2aD)\psi(x), \quad \delta_\alpha\bar\psi(x) = i\alpha\,\bar\psi(x)\gamma_5$$

- overalp Dirac operator is a local, gauge-covariant solution to G-W rel.

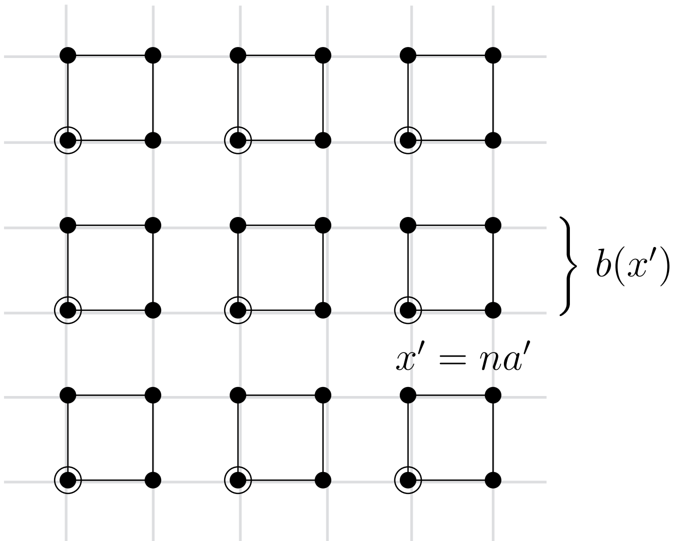
  [*Neuberger (1998)*]

$$D = \frac{1}{2a}\left(1 + X\frac{1}{\sqrt{X^\dagger X}}\right), \quad X = aD_{\rm w} - m_0, \quad X^\dagger = \gamma_5 X\gamma_5$$

$$D_{\rm w} = -\gamma_\mu \frac{1}{2}(\nabla_\mu - \nabla_\mu^\dagger) + \frac{a}{2}\nabla_\mu\nabla_\mu^\dagger$$

  - Low energy effective lattice action of 4+1 dim. DW fermion
  - Index theorem holds true on the lattice (at a finite lattice spacing "*a*")
  - Reflection positivity is satisfied (in the free fermion limit at least)

Block-spin transformation *Ginsparg-Wilson(1982)*



$$\psi'(x') \Leftarrow \frac{Z}{2^4} \sum_{x \in b(x')} \psi(x)$$

$$e^{-S'[\psi',\bar{\psi}']} = \int \prod_x d\psi(x) d\bar{\psi}(x)\ e^{-S_W[\psi,\bar{\psi}]} \times$$
$$\exp\left\{ -\alpha \sum_{x'} \left(\bar{\psi}'(x') - \bar{\Psi}(x';\bar{\psi})\right) \left(\psi'(x') - \Psi(x';\psi)\right) \right\}$$

IR fixed point :

$$S^* = a^4 \sum_x \bar{\psi}(x) D^* \psi(x)$$ local, low-energy effective action

$$\gamma_5 D_*^{-1} + D_*^{-1} \gamma_5 = \frac{2}{\alpha} a \gamma_5 \delta_{xy}$$ GW rel.

Lattice domain-wall fermion

[*Kaplan(1992)*] [*Shamir(1993)*]

$$S_{\rm DW} = \sum_{x,t} \bar\psi(x,t) X_{\rm w}^{(5)} \psi(x,t)$$

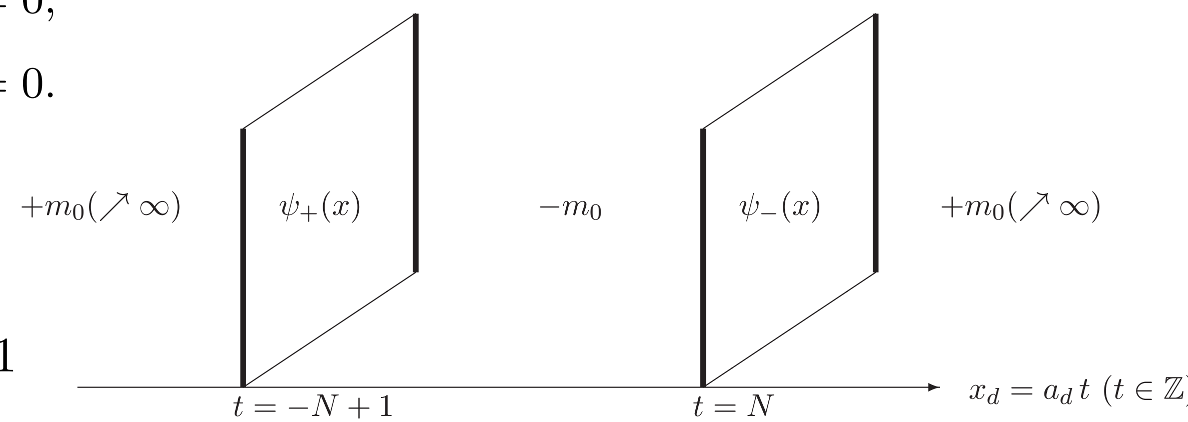
$$X_{\rm w}^{(5)} = D_{\rm w}^{(5)} - m_0 \quad m_0 \in (0,2)$$

$$D_{\rm w} = -\gamma_\mu \frac{1}{2}(\nabla_\mu - \nabla_\mu^\dagger) + \frac{a}{2}\nabla_\mu \nabla_\mu^\dagger$$

Dirichlet b.c. (open b.c.)

$$\psi_-(x,t)|_{t=-N} = 0, \quad \psi_+(x,t)|_{t=N+1} = 0,$$

$$\bar\psi_-(x,t)|_{t=-N} = 0, \quad \bar\psi_+(x,t)|_{t=N+1} = 0.$$



Vector-like set up

$$U(x,t;\mu) = U(x,\mu) \qquad U(x,t;5) = 1$$

$$\lim_{N\to\infty} \frac{\det X_{\rm w}^{(5)}|_{\rm Dir.}}{\det X_{\rm w}^{(5)}|_{\rm AP}} = \det D'_{\rm ov}$$

$$D'_{\rm ov} = \frac{1}{2}\left(1 + X \frac{1}{\sqrt{X^\dagger X}}\right) \qquad X = X_{\rm w} \frac{1}{1 + a_5 X_{\rm w}/2}$$

[*Neuberger(1998)*]

$$\det X_{\rm w}^{(5)}|_{\rm Dir.} = \det D'_{\rm ov} \det X_{\rm w}^{(5)}|_{\rm AP} \quad (N \to \infty)$$

rigorous result at a finite lattice spacing "*a*"

cf. the bd effective theory in the continuum

$$\hat H_{\rm 3D}^{\rm (bd)} = \sum_{i=1}^{\nu} \int d^3x\, \hat\psi_i(x)^\dagger \Big\{ \sum_{l=1}^{3} (-i)\sigma_l \partial_l \Big\} \hat\psi_i(x)$$

Relation to 4D TI/TSC and SPT phase

- Hamiltonian formalism of 4+1 dim. lattice DWF

= Free Fermion 4D TI/TSC (with Time-reversal symmetry, cf. 2+1dim. IQHE)

[*Creutz, Horvath(1994)*] [*Qi, Hughes, Zhang (2008)*]
[*Wen(2013), You-BenTov-Xu(2014),You-Xu (2015),Wang-Wen (2018)*]

$$\hat{H}_{\rm 4DTI} = \sum_{i=1}^{\nu}\sum_{p} \hat{a}_i(p)^\dagger \Big\{ \sum_{k=1}^{4} \alpha_k \sin(p_k) + \beta\Big(\big[\sum_{k=1}^{4} \cos(p_k) - 4 \big] + m \Big) \Big\} \hat{a}_i(p)$$

- classification of 4D TI/TSC, SPT phase

free fermion case:

- TI (with U(1)) AII type classified by v in $\mathbb{Z}$
- TSC (without U(1), Majorana mass) DIII type with only a trivial vacuum

interacting case:

- v=16, **16** of SO(10), U(1) broken by interaction
  (for **16** of SO(9), one can breaks U(1) by Majorana mass —> "DIII")

  $\Omega^{\rm spin}_5(BSO(10)) = 0$ [*Garcia-Etxebarria & Montero (2018), Wang-Wen-Witten(2018)*]

- "AII" is trivialized by a certain SO(10)-invariant and U(1)-breaking interaction, and the boundary edge modes may be gapped completely without symmetry breaking
- No obstruction for SO(9) => SO(10) symmetry restoration: $\Pi_d(S^9) = 0$ (d =0, …,9)

$$\lambda\left[\psi^T(x) i\gamma_5 c_d C\Gamma^a \psi(x)\right]^2 \Leftrightarrow y\left[\psi^T(x) i\gamma_5 c_d C\Gamma^a \psi(x)\right] E^a(x) \quad E^a(x)E^a(x) = 1$$

[*Wen(2013)*]

Admissibility condition, locality, topology of lattice gauge fields

$$\mathfrak{U} = \left\{ \{U(x,\mu)\} \,\middle|\, \| 1 - P_{\mu\nu}(x) \| < \epsilon \ \ {}^{\forall}(x,\mu,\nu) \right\}, \quad \epsilon < \frac{2}{5d(d-1)} = \begin{cases} \frac{1}{75} & (d=6) \\ \frac{1}{30} & (d=4) \\ \frac{1}{5} & (d=2) \end{cases}$$

$$\gamma_5 D + D\gamma_5 = 2aD\gamma_5 D$$

$$\delta_\alpha \psi(x) = i\alpha\,\gamma_5(1-2aD)\psi(x), \quad \delta_\alpha \bar{\psi}(x) = i\alpha\,\bar{\psi}(x)\gamma_5$$

Locality of D $\Longrightarrow$ GW rel. $\gamma_5 D + D\gamma_5 = 2D\gamma_5 D$ $\Longrightarrow$ Index of D

$\Uparrow$ (Admissibility cond. $\Rightarrow$ Locality of D)

GW rel. $\searrow$ Local chiral anomaly $\sum_x -\mathrm{tr}\gamma_5 D(x,x)$ $\searrow$ Topological charge

Index of D $\Downarrow$ Index theorem on the lattice $\Uparrow$ Topological charge

Admissibility cond. $\|1 - U_\square(x)\| < \epsilon$ $\Longrightarrow$ Topological structure of lattice gauge fields $\Longrightarrow$ Topological charge $Q = -\frac{1}{2}\sum_x \mathrm{tr}\gamma_5 D$

$\Longrightarrow$ Mass gap of 4+1 dim. Domain-wall fermions (w/o bdy)

$\Longrightarrow$ Symmetry Protected Topological phase in 4+1 dim. bulk

Weyl Fermion on the lattice (cf. NN theorem)

- Overlap Weyl fermions

$$\hat\gamma_5 = \gamma_5(1-2aD) \qquad {\hat\gamma_5}^2 = \mathbb{I} \qquad \hat P_\pm = \left(\frac{1\pm\hat\gamma_5}{2}\right), \quad P_\pm = \left(\frac{1\pm\gamma_5}{2}\right)$$

$$\hat\gamma_5\psi_\pm(x) = \pm\,\psi_\pm(x), \qquad \bar\psi_\pm(x)\gamma_5 = \mp\,\bar\psi_\pm(x)$$

$$\begin{aligned} S &= a^4\sum_x \bar\psi(x)\,D\,\psi(x) \\ &= a^4\sum_x\left\{\bar\psi(x)P_+D\hat P_-\psi(x) + \bar\psi(x)P_-D\hat P_+\psi(x)\right\} \end{aligned}$$

$$D = P_+\hat P_- + P_+\hat P_+$$

- Path integral measure and Chiral determinant

$$\psi_-(x) = \sum_i v_i(x)c_i \qquad \bar\psi_-(x) = \sum_i \bar c_i\bar v_i(x)$$

$$\{v_i(x)\mid\hat\gamma_5 v_i(x) = -v_i(x)\ (i=1,\cdots,N_-)\}$$
$$\{\bar v_i(x)\mid\bar v_i(x)\gamma_5 = +\bar v_i(x)\ (i=1,\cdots,\bar N_-)\}$$

$$Z = \int\mathcal{D}[\psi_-]\mathcal{D}[\bar\psi_-]\,\mathrm{e}^{-a^4\sum_x\bar\psi_-(x)D\psi_-(x)}$$

$$Z = \int\prod_i dc_i\prod_j d\bar c_j\,\mathrm{e}^{-\sum_{ij}\bar c_j M_{ji}c_i} = \det M_{ji}$$

$$M_{ji} = a^4\sum_x \bar v_j D v_i(x)$$

$(N_-\times\bar N_-$ rectangular matrix)

"chiral determinant as a vacuum overlap"

[*Narayanan-Neuberger (1993)*]

- Gauge anomaly of overlap Weyl fermion

$$\mathfrak{U} = \left\{ \{U(x,\mu)\} \,\middle|\, \| 1 - U_{\mu\nu}(x) \| < \epsilon \;{}^{\forall}(x,\mu,\nu) \right\}$$

$$\mathcal{O} \subset \mathfrak{U}[Q] \qquad \tilde{v}_i(x) = v_l(x) \left(\mathcal{Q}^{-1}\right)_{li}, \quad \tilde{c}_j = \sum_l \mathcal{Q}_{jl} c_l$$

$$\mathcal{D}[\psi_-] \to \mathcal{D}[\psi_-] \,\det \mathcal{Q} \qquad \det M_{ji} \to \det M_{ji} \,\det \mathcal{Q}$$

$$\delta_\eta U(x,\mu) = ia\eta_\mu(x)\, U(x,\mu)$$

$$\delta_\eta \ln \det M_{ji} = \mathrm{Tr}\{\delta_\eta D \hat{P}_- D^{-1} P_+\} + \sum_i (v_i, \delta_\eta v_i)$$

$$= i\mathrm{Tr}\omega\gamma_5\,(1-D) - i\sum_i (v_i, \delta_\omega v_i) \qquad \eta_\mu(x) = -i\nabla_\mu \omega(x)$$

$$\mathrm{tr}\{T^a\gamma_5(1-aD)(x,x)\} \xrightarrow{a\to 0} \frac{-1}{128\pi^2} d^{abc}\epsilon_{\mu\nu\rho\sigma} F^b_{\mu\nu}(x) F^c_{\rho\sigma}(x) + O(a)$$ **gauge anomaly!**

$$\sum_R d^{abc} = \sum_R 2i\mathrm{tr}\{T^a[T^bT^c + T^cT^b]\}$$

- Determinant line bundle over $\mathfrak{U}$

$$\mathcal{O}^a \cap \mathcal{O}^b \qquad v_i^b(x) = \sum_l v_l^a(x)\tau(a\to b)_{li} \qquad g_{ab} \equiv \det \tau(a \to b)$$

$$\tilde{v}_i^a(x) = v_l^a(x) h^a_{lj} \qquad \tilde{g}_{ab} = \det h_a \, g_{ab} \,\det h_b^{-1}$$

$$\mathcal{O}^a \cap \mathcal{O}^b \cap \mathcal{O}^c \qquad g_{ac} = g_{ab}\, g_{bc}$$

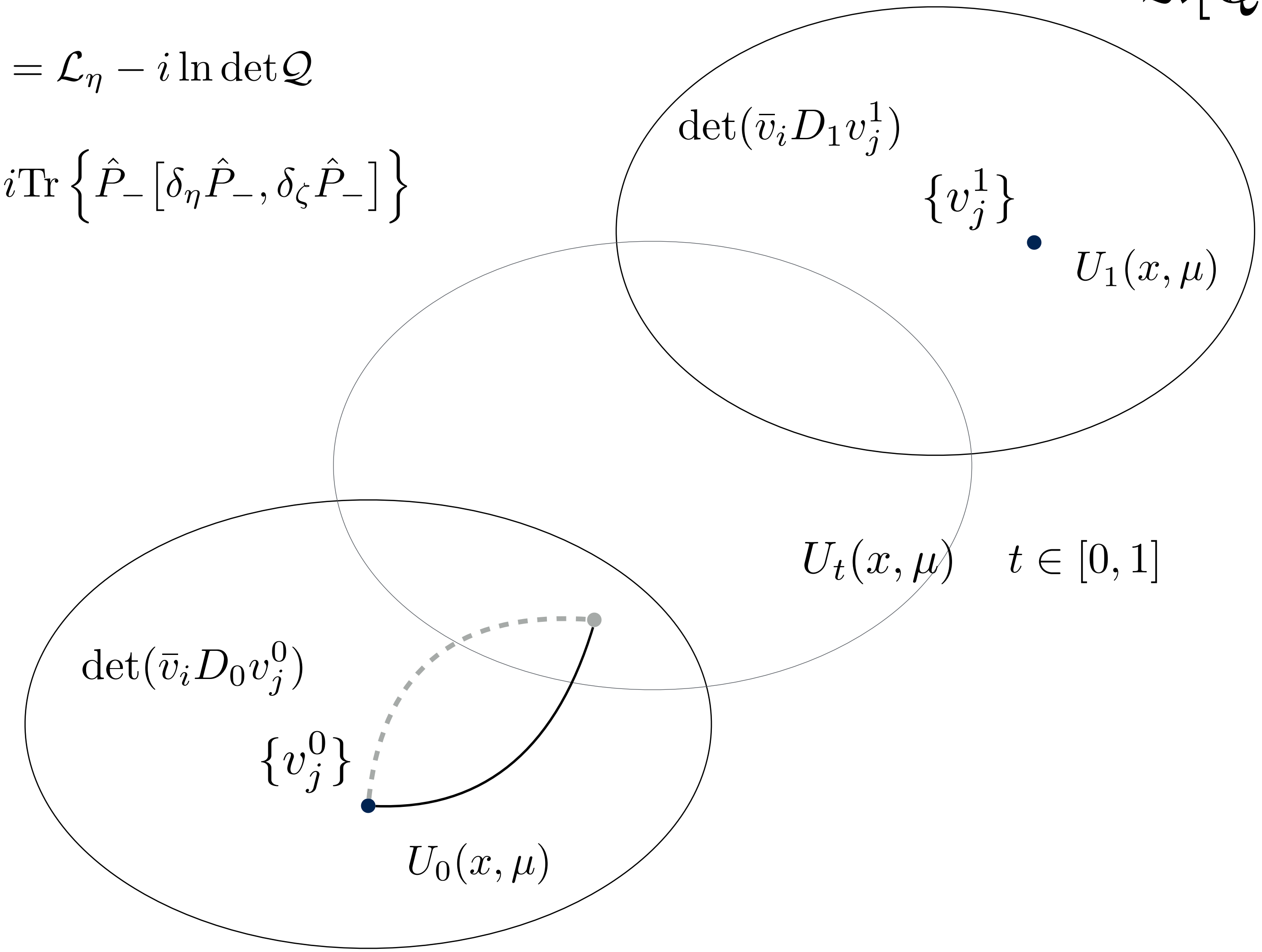
1. Integrability condition for Chiral determinant of Overlap Weyl fermions

$\mathfrak{U}[Q]$

$$\mathcal{L}_\eta = i\sum_i (v_i, \delta_\eta v_i) \qquad \tilde{\mathcal{L}}_\eta = \mathcal{L}_\eta - i \ln\det\mathcal{Q}$$

$$\delta_\eta \mathcal{L}_\zeta - \delta_\zeta \mathcal{L}_\eta + a\mathcal{L}_{[\eta,\zeta]} = i\mathrm{Tr}\left\{\hat{P}_-\left[\delta_\eta \hat{P}_-, \delta_\zeta \hat{P}_-\right]\right\}$$

$$W = \exp\left\{ i\int_0^1 dt\, \mathfrak{L}_\eta \right\}$$



$$P_t = Q_t P_0 Q_t^{-1}, \qquad P_t = \hat{P}_L|_{U=U_t} \equiv \left(\frac{1-\hat{\gamma}_5}{2}\right)\bigg|_{U=U_t}$$

$$\det(\bar{v} D_{\mathrm{ov}} v^\diamond)|_{t=1}\ \det(\bar{v} D_{\mathrm{ov}} v^\diamond)^*|_{t=0}$$

$$= \det(1 - P_+ + P_+ \hat{P}_1 Q_1 \hat{P}_0)\, W_1{}^{-1}$$

$$v_i^\diamond|_t = \begin{cases} Q_t\, v_i^\diamond|_{t=0}\, W_t^{-1} & (i=1) \\ Q_t\, v_i^\diamond|_{t=0} & (i\neq 1) \end{cases}$$

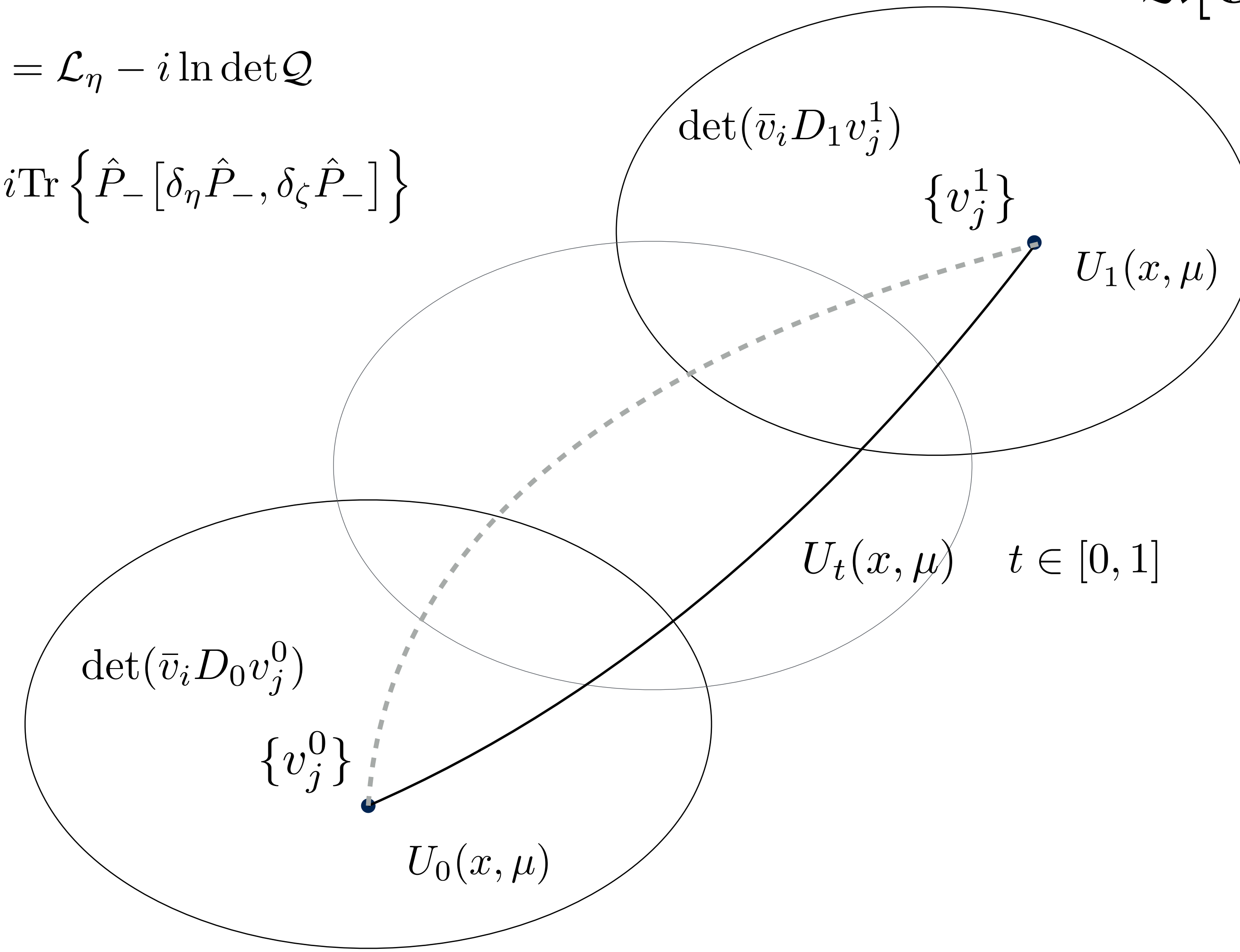
$$W_t = \mathrm{e}^{i\int_0^t ds \mathcal{L}_{\eta_s}}$$

$\mathfrak{U}[Q]$

$$\mathcal{L}_\eta = i\sum_i (v_i, \delta_\eta v_i) \qquad \tilde{\mathcal{L}}_\eta = \mathcal{L}_\eta - i\ln\det\mathcal{Q}$$

$$\delta_\eta\mathcal{L}_\zeta - \delta_\zeta\mathcal{L}_\eta + a\mathcal{L}_{[\eta,\zeta]} = i\mathrm{Tr}\left\{\hat{P}_-\left[\delta_\eta\hat{P}_-, \delta_\zeta\hat{P}_-\right]\right\}$$

$$W = \exp\left\{i\int_0^1 dt\, \mathfrak{L}_\eta\right\}$$



$$P_t = Q_t P_0 Q_t^{-1}, \qquad P_t = \hat{P}_L|_{U=U_t} \equiv \left(\frac{1-\hat{\gamma}_5}{2}\right)\Bigg|_{U=U_t}$$

$$\det(\bar{v}D_{\mathrm{ov}}v^\diamond)|_{t=1}\, \det(\bar{v}D_{\mathrm{ov}}v^\diamond)^*|_{t=0}$$

$$= \det(1 - P_+ + P_+\hat{P}_1 Q_1 \hat{P}_0)\, {W_1}^{-1}$$

$$v_i^\diamond|_t = \begin{cases} Q_t\, v_i^\diamond|_{t=0}\, W_t^{-1} & (i=1) \\ Q_t\, v_i^\diamond|_{t=0} & (i\neq 1)\end{cases}$$

$$W_t = \mathrm{e}^{i\int_0^t ds \mathcal{L}_{\eta_s}}$$

- Integrability condition

[*Luscher (1999)*]

- the line bundle must be trivial with a global section

$j^a_\mu(x)[U]$ smooth & local over entire $\mathfrak{U}$

$$\mathcal{L}_\eta = i\sum_i (v_i, \delta_\eta v_i) = \sum_x \eta^a_\mu(x) j^a_\mu(x)$$

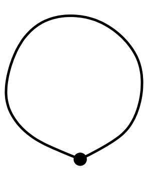
section/local patch $\mathcal{O} \subset \mathfrak{U}[Q]$ global section over $\mathfrak{U}$

- global integrability

$$P_t = Q_t P_0 Q_t^{-1}, \qquad P_t = \hat{P}_L|_{U=U_t} \equiv \left(\frac{1-\hat{\gamma}_5}{2}\right)\Bigg|_{U=U_t}$$

$$v_i^\diamond|_t = \begin{cases} Q_t\, v_i^\diamond|_{t=0}\, W_t^{-1} & (i=1) \\ Q_t\, v_i^\diamond|_{t=0} & (i \neq 1) \end{cases}$$

$$W_t = \mathrm{e}^{i\int_0^t ds \mathcal{L}_{\eta_s}}$$



$$\det\left\{1 - P_0 + P_0 Q\right\} W^{-1} = 1 \qquad {}^\forall\, l \in \mathfrak{U}$$

invariant for the deformation of *l*

there may be a non-contractible loop in $\mathfrak{U}$

cf. $\mathrm{e}^{i2\pi\eta} = 1$

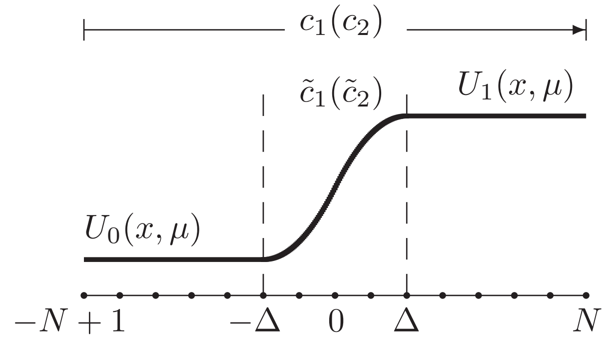
Lattice domain-wall fermion (chiral case) [Aoyama-*YK (1999)*] [*YK (2002)*]

Chiral set up

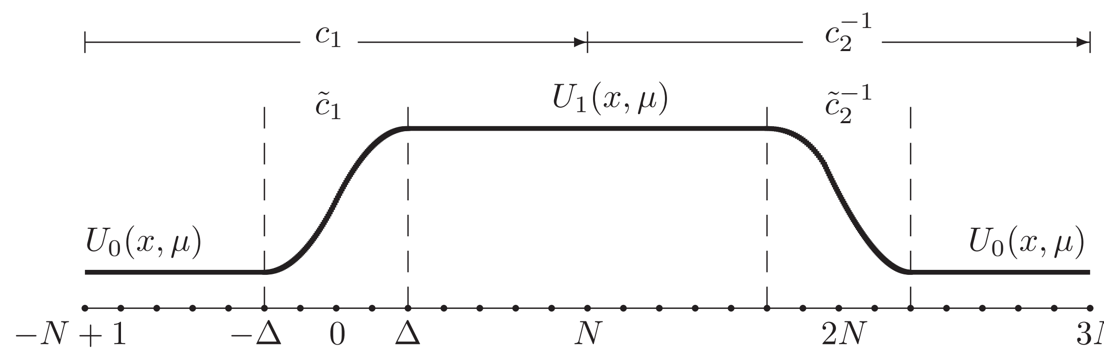
$$U(x,t,\mu)|_{t=-N+1} = U_0(x,\mu)$$

$$U(x,t,\mu)|_{t=N} = U_1(x,\mu)$$

Dirichlet b.c.



AP b.c.



$$\frac{\det X_{\rm w}^{(5)}\big|^{c}_{\rm Dir.}}{\sqrt{\det X_{\rm w}^{(5)}\big|^{c\cdot c^{-1}}_{\rm AP}}} = \det(\bar{v} D'_{\rm ov} v^1)\det(\bar{v} D'_{\rm ov} v^0)^* \frac{\det(v^{1\dagger}\prod_{t\in\tilde{c}} T_t v^0)}{\left|\det(v^{1\dagger}\prod_{t\in\tilde{c}} T_t v^0)\right|} \qquad (N\to\infty)$$

$$\frac{\det X_{\rm w}^{(5)}\big|^{c_1 c_2^{-1}}_{\rm AP}}{\left|\det X_{\rm w}^{(5)}\big|^{c_1 c_2^{-1}}_{\rm AP}\right|} = \frac{\det(v^{1\dagger}\prod_{t\in\tilde{c}_1} T_t v^0)}{\left|\det(v^{1\dagger}\prod_{t\in\tilde{c}_1} T_t v^0)\right|} \Bigg/ \frac{\det(v^{1\dagger}\prod_{t\in\tilde{c}_2} T_t v^0)}{\left|\det(v^{1\dagger}\prod_{t\in\tilde{c}_2} T_t v^0)\right|} \qquad (N\to\infty)$$

- Integrability condition using 5-, 6-dim. lattice DWFs

[*YK (2002)*]

[*Pedersen-YK (2019)*]

$$q^{(6)}(z) = \partial^*_\mu k_\mu(z)$$

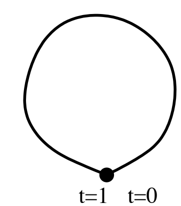
$$q^{(6)}(z) \equiv -\frac{1}{2}\mathrm{tr}\left\{ \left. \frac{H_\mathrm{w}^{(6)}}{\sqrt{{H_\mathrm{w}^{(6)}}^2}} \right|_\mathrm{AP} \right\}(z,z)$$

$\Sigma_R \, Tr_R[\, T^a \, \{T^b, T^c\} \,]=0$

local, topological field in 6-dim. lattice AS index theorem

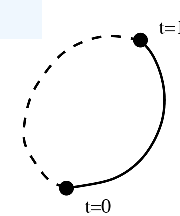
$$\frac{\det X_\mathrm{w}^{(5)}\big|_\mathrm{AP}^{c_1 c_2^{-1}}}{\left|\det X_\mathrm{w}^{(5)}\big|_\mathrm{AP}^{c_1\cdot c_2^{-1}}\right|} \; \mathrm{e}^{-i2\pi \sum_{y\in c_1 c_2^{-1}} k_6(z)\big|_{\mathbb{Y}|_\mathrm{AP}^{c_1 c_2^{-1}}}} = 1 \qquad {}^\forall c_1 c_2^{-1} \subset \mathfrak{U}^{(5)} \qquad (N\to\infty)$$

invariant for the deformation $c_1' {c_2'}^{-1} \simeq c_1 c_2^{-1}$



$$\Rightarrow \quad \det(\bar v D'_\mathrm{ov} v^1) \; \frac{\det({v^1}^\dagger \prod_{t\in\tilde c} T_t v^0)}{\left|\det({v^1}^\dagger \prod_{t\in\tilde c} T_t v^0)\right|} \; \mathrm{e}^{-i2\pi \sum_{y\in c_1} k_6(z)\big|_{\mathbb{Y}|_\mathrm{AP}^{c_1 c_2^{-1}}}}$$

Integrable !

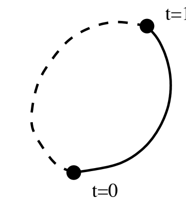


Integrability condition by 5-, 6-dim. lattice DWFs

[*YK (2002)*]
[*Pedersen-YK (2019)*]

$\mathbb{X}_0 = \{\mathbb{L}^4; U_0(x,\mu) \in \mathfrak{U}\}$ periodic in x

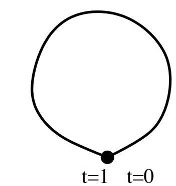
$\mathbb{X}_1 = \{\mathbb{L}^4; U_1(x,\mu) \in \mathfrak{U}\}$ periodic in x ← $\det(\bar{v}_i D'_{\rm ov} v^1_j)$



$\mathbb{L}^5 \ (\{x,t\} = y)$ anti-periodic or Dirichlet in t

$\mathbb{Y}|_{\rm Dir} = \{\mathbb{L}^4 \times \mathbb{L} \ ; c : U_t(x,\mu) \in \mathfrak{U}^{(5)}\}$ $\quad \partial\mathbb{Y}|_{\rm Dir} = \mathbb{X}_1 \cup \bar{\mathbb{X}}_0$

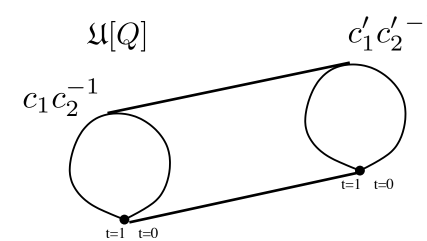
$\mathbb{Y}|_{\rm AP} = \{\mathbb{L}^4 \times 2\mathbb{L} \, ; l : U_t(x,\mu) \in \mathfrak{U}^{(5)}\}$ $\quad \mathbb{Y}|^{c_1}_{\rm Dir} \cup \bar{\mathbb{Y}}|^{c_2}_{\rm Dir} = \mathbb{Y}|^{c_1 c_2^{-1}}_{\rm AP}$ $\quad \partial\mathbb{Y}|_{\rm AP} = \emptyset$

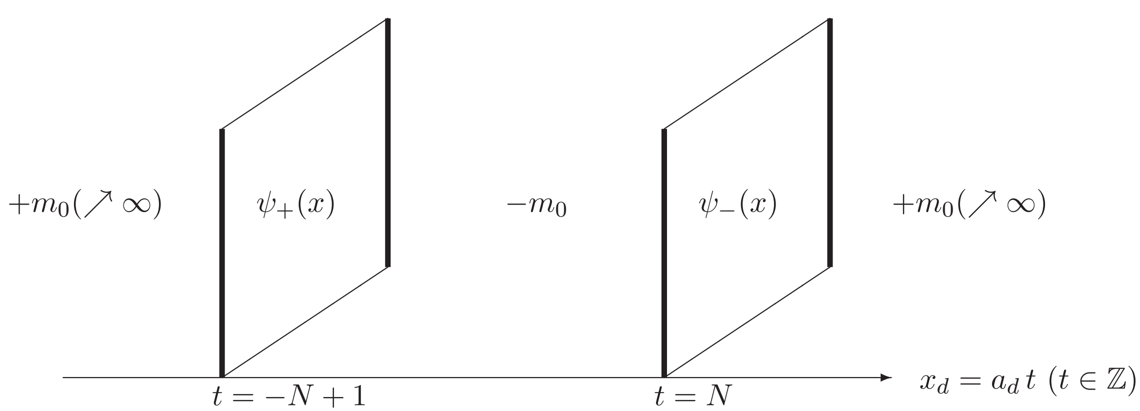


$\mathbb{L}^6 \ (\{x,t,s\} = \{y,s\} = z)$ anti-periodic in t, anti-periodic or Dirichlet in s

$\mathbb{Z}|_{\rm Dir} = \{\mathbb{L}^5 \times \mathbb{L} \ ; c : U_s(y,\mu) \in \mathfrak{U}^{(6)}\}$ $\quad \partial\mathbb{Z}|_{\rm Dir} = \mathbb{Y}_1|_{\rm AP} \cup \bar{\mathbb{Y}}_0|_{\rm AP}$

$\mathbb{Z}|_{\rm AP} = \{\mathbb{L}^5 \times 2\mathbb{L} \, ; l : U_s(y,\mu) \in \mathfrak{U}^{(6)}\}$ $\quad \mathbb{Z}|^{c_1}_{\rm Dir} \cup \bar{\mathbb{Z}}|^{c_2}_{\rm Dir} = \mathbb{Z}|^{c_1 c_2^{-1}}_{\rm AP}$ ← $q(z)^{(6)}$





1-1. Topological classification of chiral anomaly of overlap fermions in even dim. non-abelian lattice gauge theories

$$q(x) = -\frac{1}{2}\mathrm{tr}\left\{\frac{H_{\mathrm{w}}}{\sqrt{H_{\mathrm{w}}^2}}\right\}(x,x) \qquad H_{\mathrm{w}} = \gamma_{2n+1}X_{\mathrm{w}}^{(2n)}, \qquad X_{\mathrm{w}}^{(2n)} = D_{\mathrm{w}}^{(2n)} - m_0$$

$$\mathfrak{U} = \left\{\{U(x,\mu)\}\,\Big|\, \| 1 - P_{\mu\nu}(x) \| < \epsilon \;\; {}^{\forall}(x,\mu,\nu)\right\}, \quad \epsilon < \frac{2}{5d(d-1)} = \begin{cases} \frac{1}{75} & (d=6) \\ \frac{1}{30} & (d=4) \\ \frac{1}{5} & (d=2) \end{cases}$$

$$\sum_z \delta_\eta q(z) = 0 \qquad \delta_\eta U(x,\mu) = i\eta(x,\mu)U(x,\mu)$$

$d = 6$ $\quad \sum_r \mathrm{Tr}_r\{T^a[T^b,T^c]\} = 0$

$$q(x) = \partial_\mu^* k_\mu(x)$$

$d = 4$

$$q(x) = c_2^{(L)}(x) + \partial_\mu^* k_\mu(x)$$

$$q_L(x) = q(x) + \Delta q(x)$$

$$D_L(x,y) = D(x,y) + \sum_{n\in\mathbb{Z}^4, n\neq 0} D(x, y+nL)$$

$$D_L(x,y) = D(x,y) + \mathrm{O}(\mathrm{e}^{-L/\varrho})$$

$$|\Delta q(x)| \leq \kappa L^\sigma \mathrm{e}^{-L/\varrho} \qquad \sum_x \Delta q(x) = 0 \qquad \Delta q(x) = \partial_\mu^* \Delta k_\mu(x)$$

$d = 6$ $\Sigma_R \, Tr_R[T^a \{T^b, T^c\}] = 0$

$$\begin{aligned} q(x) &= \sum_{n=5}^{\infty} \frac{1}{n!} q^{(n)}_{[5]}(x) \\ &= \sum_{n=5}^{\infty} \frac{1}{n!} \frac{1}{2^n} \theta^{(n)}_{[5]}(x, y_1, \cdots, y_n)^{a_1, \cdots, a_n}_{\mu_1\nu_1, \cdots, \mu_n\nu_n} \hat{F}^{a_1}(y_1, \mu_1, \nu_1; x) \cdots \hat{F}^{a_n}(y_n, \mu_n, \nu_n; x) \\ &= \sum_{n=5}^{\infty} \frac{1}{n!} \ \theta^{(n)}_{[5]}(x, \{y\}) \cdot \left\{ \frac{\hat{F}(y)}{2} \right\}^n \Big|_x . \end{aligned}$$

there exist a gauge-invarinat local current $k_\mu(x)$ such that

$$q(x) = \partial^*_\mu k_\mu(x),$$

which admits a convergent series expansion as

$$\begin{aligned} k_\mu(x) &= \sum_{n=5}^{\infty} \frac{1}{n!} k^{(n)}_\mu(x) \\ &= \sum_{n=5}^{\infty} \frac{1}{n!} \frac{1}{2^n} \tau^{(n)}_\mu(x, y_1, \cdots, y_n)^{a_1, \cdots, a_n}_{\mu_1\nu_1, \cdots, \mu_n\nu_n} \hat{F}^{a_1}_s(y_1, \mu_1, \nu_1; x) \cdots \hat{F}^{a_n}_s(y_n, \mu_n, \nu_n; x) \\ &= \sum_{n=5}^{\infty} \frac{1}{n!} \ \tau^{(n)}_\mu(x, \{y\}) \cdot \left\{ \frac{\hat{F}(y)}{2} \right\}^n \Big|_x . \end{aligned}$$

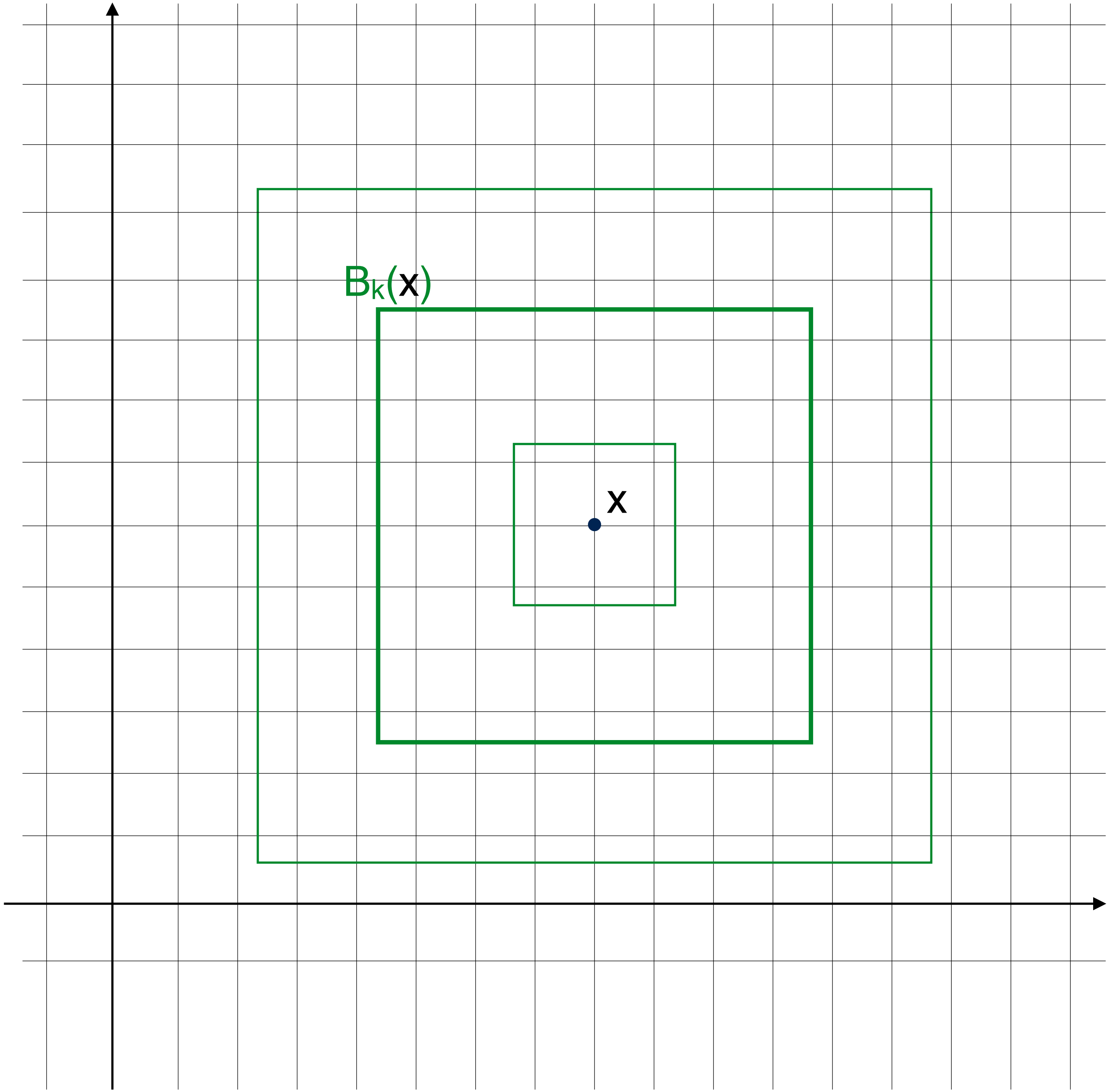
$$
\begin{aligned}
& v \le H_{\rm w}^2 \le u, \\
& \cosh\theta = \frac{v-u}{v+u}, \quad t = \mathrm{e}^{-\theta}, \quad \kappa = \frac{4t}{v-u}, \\
& z = \frac{v+u-2H_{\rm w}^2}{v-u} \quad (-1 \le z \le 1),
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\sqrt{H_{\rm w}^2}}(y,x) &= \kappa \frac{1}{\sqrt{1-2tz+t^2}}(y,x) \\
&= \kappa \sum_{k=0}^{\infty} t^k P_k(z)(y,x),
\end{aligned}
$$

$$
\begin{aligned}
q(x) &= -\frac{1}{2}\mathrm{tr}\left\{ H_{\rm w} \frac{1}{\sqrt{H_{\rm w}^2}} \right\}(x,x) \\
&= \sum_{k=0}^{\infty} t^k \left(-\frac{\kappa}{2}\mathrm{tr}\left\{ H_{\rm w}\, P_k(z) \right\}(x,x) \right)
\end{aligned}
$$

Here $\mathrm{tr}\{H_{\rm w}P_k(z)\}(x,x)$ is localized strictly in a finite rectangular block of lattice points $\mathbb{B}_k(x)$ centred at x with side-lengths proportional to k, and is a smooth function of the link variables within the block. In fact, it is analytic and expandable in the Tayler series with respect to the parameter s,

$$
\mathrm{tr}\{H_{\rm w}P_k(z)\}(x,x) = \sum_{n=0}^{\infty} \frac{s^n}{n!}\, \mathrm{tr}\{H_{\rm w}P_k^{(n)}(z)\}(x,x).
$$

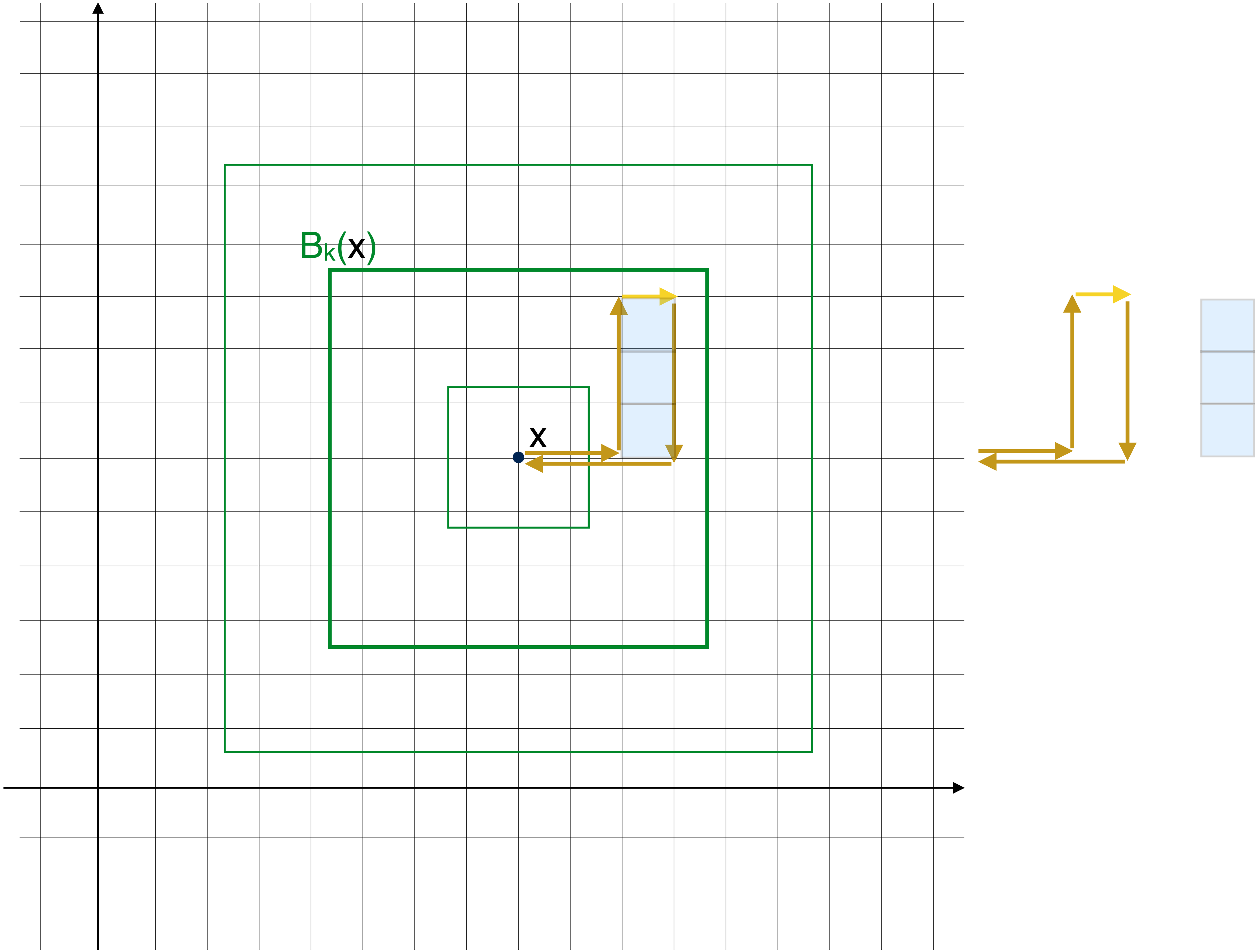


Bk(x)
x

$$
\begin{aligned}
g(x;x^{(0)}) \;=\; & \prod_{y_1=x_1^{(0)}}^{x_1-1} U(y,1) \times \prod_{y_2=x_2^{(0)}}^{x_2-1} U(y,2)\Big|_{y_1=x_1} \\
& \times \prod_{y_3=x_3^{(0)}}^{x_3-1} U(y,3)\Big|_{y_1=x_1,y_2=x_2} \times \prod_{y_4=x_4^{(0)}}^{x_4-1} U(y,4)\Big|_{y_1=x_1,y_2=x_2,y_3=x_3} \\
& \times \prod_{y_5=x_5^{(0)}}^{x_5-1} U(y,5)\Big|_{y_1=x_1,y_2=x_2,y_3=x_3,y_4=x_4} \\
& \times \prod_{y_6=x_6^{(0)}}^{x_6-1} U(y,6)\Big|_{y_1=x_1,y_2=x_2,y_3=x_3,y_4=x_4,y_5=x_5}.
\end{aligned}
$$

$$
\hat{U}(x,\mu;x^{(0)}) \;=\; g(x;x^{(0)})\,U(x,\mu)\,g(x+\hat{\mu};x^{(0)})^{-1}
$$

$$
\begin{aligned}
\hat{P}(x,\mu,\nu;x^{(0)}) \;&=\; \hat{U}(x,\mu;x^{(0)})\hat{U}(x+\hat{\mu},\nu;x^{(0)})\hat{U}(x+\hat{\nu},\mu;x^{(0)})^{-1}\hat{U}(x,\nu;x^{(0)})^{-1} \\
&=\; g(x;x^{(0)})\,P(x,\mu,\nu)\,g(x;x^{(0)})^{-1}.
\end{aligned}
$$



$B_k(x)$
x

$$
\begin{aligned}
q(x)\Big|_{U(y,\mu)\to\hat{U}_s(y,\mu;x)} &= -\frac{\kappa}{2}\sum_{k=0}^{\infty} t^k \,\mathrm{tr}\left\{H_{\mathrm{w}} P_k(z)\right\}(x,x)\Big|_{U(y,\mu)\to\hat{U}_s(y,\mu;x)} \\
&= -\frac{\kappa}{2}\sum_{k=0}^{\infty} t^k \sum_{n=0}^{\infty}\frac{s^n}{n!}\,\mathrm{tr}\left\{H_{\mathrm{w}} P_k^{(n)}(z)\right\}(x,x) \\
&= -\frac{\kappa}{2}\sum_{n=0}^{\infty}\frac{s^n}{n!}\sum_{k=0}^{\infty} t^k \,\mathrm{tr}\left\{H_{\mathrm{w}} P_k^{(n)}(z)\right\}(x,x) \\
&= \sum_{n=0}^{\infty}\frac{s^n}{n!} q^{(n)}(x) \\
&= \sum_{n=0}^{\infty}\frac{s^n}{n!}\,\frac{1}{2^n}\theta^{(n)}_{[5]}(x,y_1,\cdots,y_n)^{a_1,\cdots,a_n}_{\mu_1\nu_1,\cdots,\mu_n\nu_n}\times \\
&\qquad\qquad \hat{F}^{a_1}(y_1,\mu_1,\nu_1;x)\cdots\hat{F}^{a_n}(y_n,\mu_n,\nu_n;x)
\end{aligned}
$$

Convergence

$$\|\,\hat{F}(y)\,\| \le \tilde{\epsilon}$$

$$|q(x)| \le \sum_{n=5}^{\infty}\frac{1}{n!}\sum_{\{y\}}\left|\theta^{(n)}_{[5]}(x,\{y\})\right|\tilde{\epsilon}^n = \sum_{n=5}^{\infty} a^{(n)}\tilde{\epsilon}^n$$

$$\lim_{n\to\infty}(a^{(n)})^{\frac{1}{n}} = \frac{1}{\tilde{\varrho}}$$

$$\tilde{\varrho} \quad \Leftrightarrow \quad t$$

$$\frac{t+t^{-1}}{2} = \frac{v-u}{v+u}$$

$$u > 1-\frac{5}{2}d(d-1)\epsilon$$

Leading term **topological field in a linearized abelian theory**

$$\hat{F}^a_{\mu\nu}(y;x) \to \check{F}^a_{\mu\nu}(y) = \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x)$$

$$\check{q}(x) \equiv \frac{1}{5!}\,\theta^{(5)}_{[5]}(x,\{y\}) \cdot \left\{\frac{\check{F}(y)}{2}\right\}^5$$

$$\delta_\omega \check{q}(x) = 0 \qquad \delta_\omega A^a_\mu = \partial_\mu \omega^a(x)$$

$$\sum_x \delta_\eta \check{q}(x) = 0 \qquad \delta_\eta A^a_\mu = \eta^a_\mu(x)$$

$$\Longrightarrow \quad \theta^{(5)}_{[5]}(x,\{y\}) = \partial^*_\mu \tau^{(5)}_\mu(x,\{y\})$$

"local cohomology analysis for abelian gauge theory"

[*Luscher (1999)*]

[*Fujiwara-Suzuki-Wu(1999)*]

[*Kadoh-YK(2013)*]

$$\check{q}(x) = \frac{1}{5!}\,\partial^*_\mu \tau^{(5)}_\mu(x,\{y\}) \cdot \left\{\frac{\check{F}}{2}\right\}^5$$

$$= \partial^*_\mu \left\{ \frac{1}{5!}\tau^{(5)}_\mu(x,\{y\}) \cdot \left\{\frac{\check{F}}{2}\right\}^5 \right\}$$

$$q(x) = \frac{1}{5!}\partial_\mu^* \tau_\mu^{(5)}(x,\{y\}) \cdot \left\{\frac{\hat{F}}{2}\right\}^5\Big|_x + \sum_{n=6}^{\infty} \frac{1}{n!}\, \theta_{[5]}^{(n)}(x) \cdot \left\{\frac{\hat{F}}{2}\right\}^n\Big|_x$$

$$= \partial_\mu^* \left\{ \frac{1}{5!}\tau_\mu^{(5)}(x,\{y\}) \cdot \left\{\frac{\hat{F}}{2}\right\}^5\Big|_x \right\} + \sum_{n=6}^{\infty} \frac{1}{n!}\, \theta_{[6]}^{(n)}(x) \cdot \left\{\frac{\hat{F}}{2}\right\}^n\Big|_x .$$

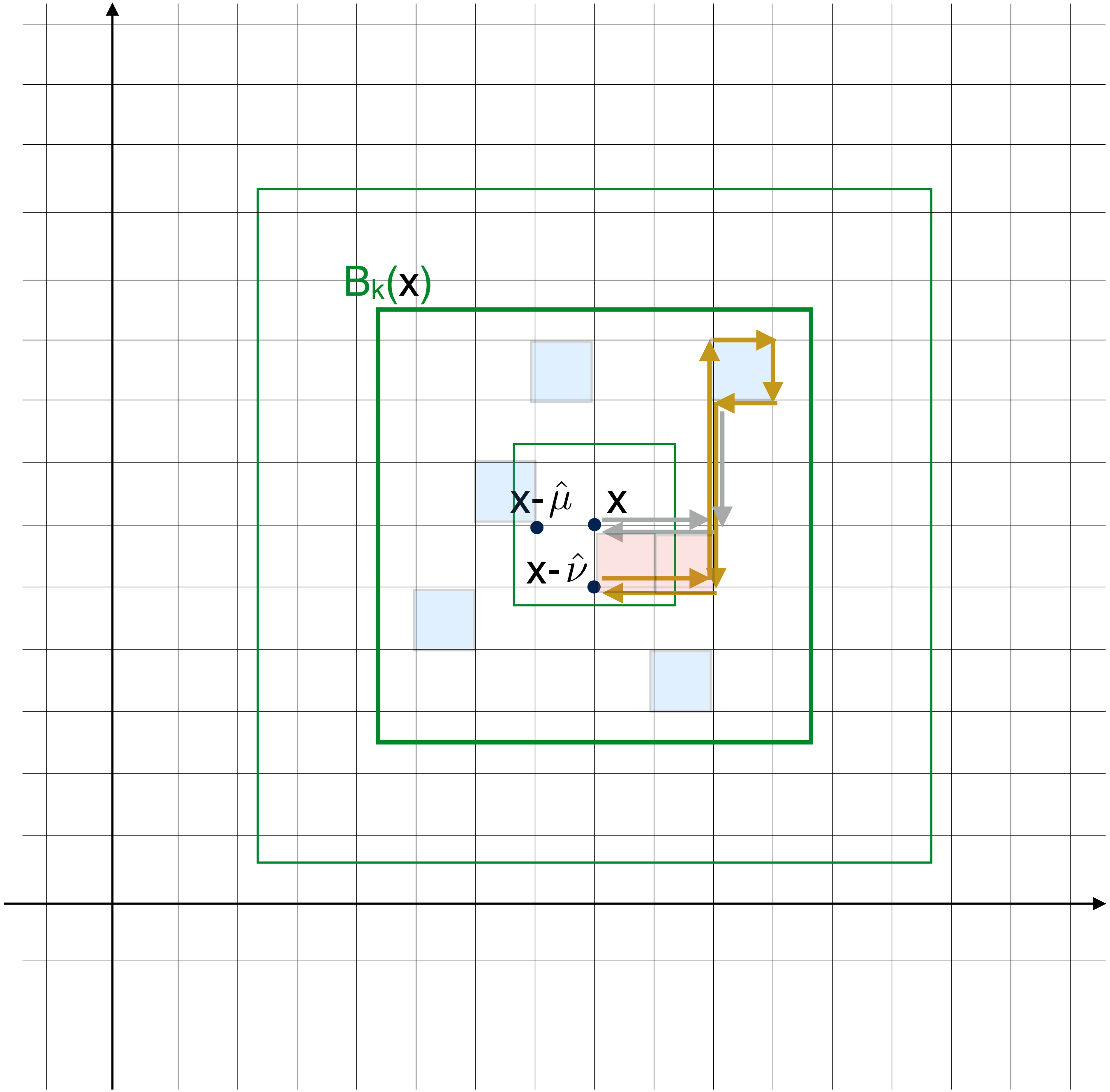
$$\partial_\mu^* \tau_\mu^{(5)}(x,\{y\}) \cdot \left\{\frac{\hat{F}}{2}\right\}^5\Big|_x = \partial_\mu^* \left\{ \tau_\mu^{(5)}(x,\{y\}) \cdot \left\{\frac{\hat{F}}{2}\right\}^5\Big|_x \right\}$$
$$+\tau_\mu^{(5)}(x-\hat{\mu},\{y\}) \left[\left\{\frac{\hat{F}}{2}\right\}^5\Big|_{x-\hat{\mu}} - \left\{\frac{\hat{F}}{2}\right\}^5\Big|_x \right]$$

$$\frac{1}{5!}\left[\left\{\frac{\hat{F}(y)}{2}\right\}^5\Big|_{x-\hat{\mu}} - \left\{\frac{\hat{F}(y)}{2}\right\}^5\Big|_x \right] = \sum_{n=6}^{\infty} \frac{1}{n!}\, c_{\mu\,[5]}^{\ (n)}(\{y\},\{y'\}) \cdot \left\{\frac{\hat{F}(y)}{2}\right\}^5\Big|_x \left\{\frac{\hat{F}(y')}{2}\right\}^{n-5}\Big|_x$$

$$\theta_{[5]}^{(n)}(x,\{y\}) \to \theta_{[6]}^{(n)}(x,\{y\}) = \theta_{[5]}^{(n)}(x,\{y\}) + \tau_\mu^{(5)}(x,\{y\})\, c_{\mu\,[5]}^{\ (n)}(\{y\})$$

cf. [*Luscher (2000)*]

"local cohomology analysis in Non-abelian gauge theory
to all orders of weak gauge-coupling expansion"



$B_k(\mathbf{x})$
$\mathbf{x}$- $\hat{\mu}$
$\mathbf{x}$
$\mathbf{x}$- $\hat{\nu}$

$$q(x) = \partial_\mu^* \left\{ \sum_{n=5}^{n'-1} \frac{1}{n!} \tau_\mu^{(n)}(x,\{y\}) \cdot \left\{ \frac{\hat{F}}{2} \right\}^n \Big|_x \right\} + \sum_{n=n'}^{\infty} \frac{1}{n!}\, \theta_{[n']}^{(n)}(x) \cdot \left\{ \frac{\hat{F}}{2} \right\}^n \Big|_x$$

$$\theta_{[n']}^{(n')}(x,\{y\}) = \partial_\mu^* \tau_\mu^{(n')}(x,\{y\})$$

$$\theta_{[n'+1]}^{(n)}(x,\{y\}) = \theta_{[n']}^{(n)}(x,\{y\}) + \tau_\mu^{(n')}(x,\{y\})\, c_{[n']}^{(n)}(\{y'\}) \quad (n \geq n'+1)$$

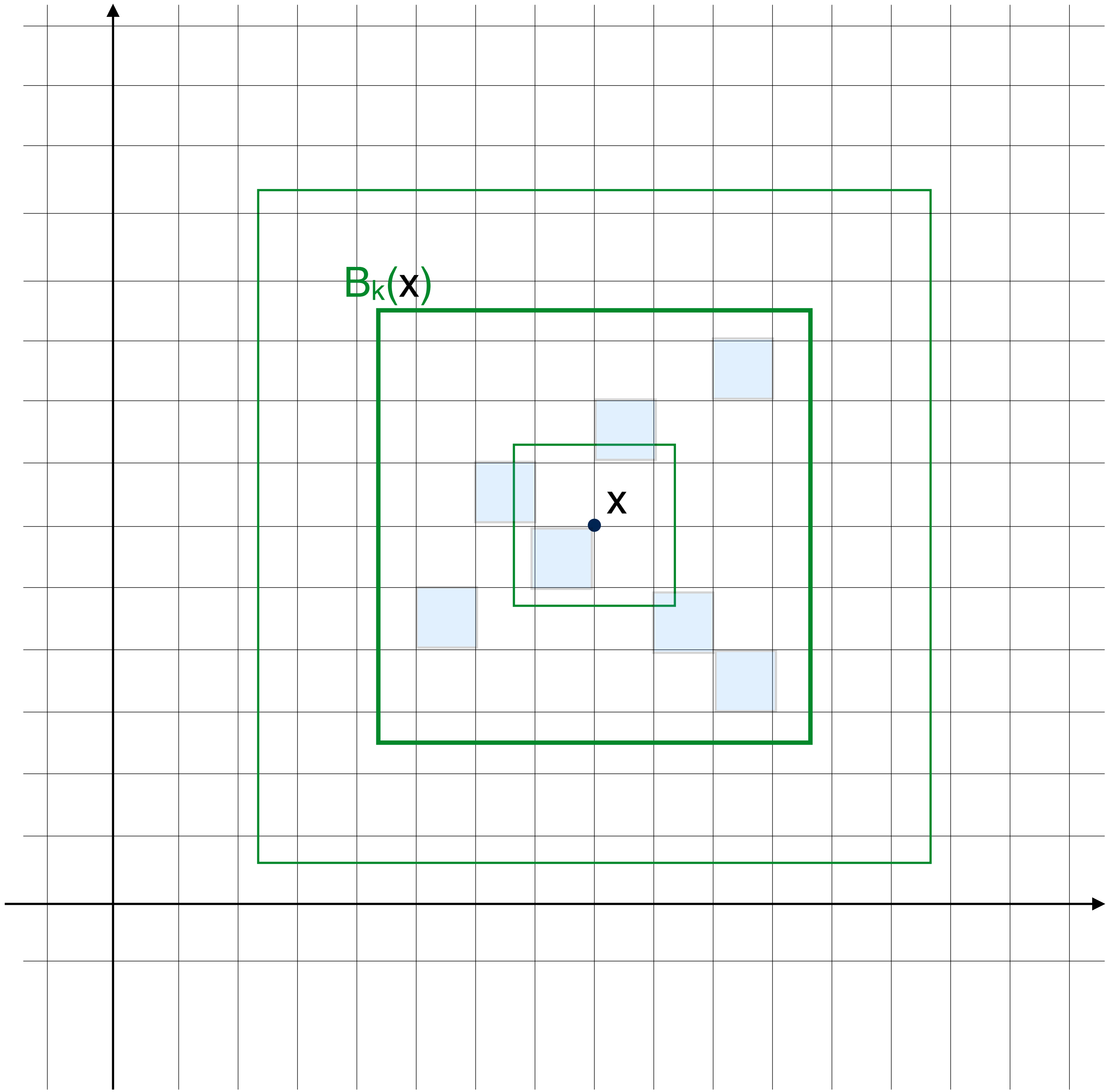
Convergence

$$\frac{1}{n!} \sum_{\{y\}} \left| \theta_{[n']}^{(n)}(x,\{y\}) \right| \;\leq\; c\; 2^{n'} (1 + (n')^\sigma)\, \frac{1}{n!} \sum_{\{y\}} \left| \theta_{[5]}^{(n)}(x,\{y\}) \right|$$

$$n' \to 0$$

$$\sum_{n=n'}^{\infty} \frac{1}{n!} \left| \sum_{\{y\}} \theta_{[n']}^{(n)}(x,\{y\}) \right| \tilde{\epsilon}^n \;\to\; 0 \qquad\qquad \frac{2\tilde{\epsilon}}{\tilde{\varrho}} \;<\; 1$$

$$
\begin{array}{ccccc}
\partial\left\{\frac{1}{5!}\,\partial^{-1}\theta^{(5)}_{[5]}\cdot\hat{F}^{5}\right\} & \partial\left\{\frac{1}{6!}\,\partial^{-1}\theta^{(6)}_{[5]}\cdot\hat{F}^{6}\right\} & \frac{1}{7!}\,\theta^{(7)}_{[5]}\cdot\hat{F}^{7} & \cdots & \frac{1}{n!}\,\theta^{(n)}_{[5]}\cdot\hat{F}^{n} \\
 & \partial\left\{\frac{1}{6!}\,\partial^{-2}\theta^{(5)}_{[5]}\,c^{(6)}_{[5]}\cdot\hat{F}^{6}\right\} & \frac{1}{7!}\,\partial^{-1}\theta^{(5)}_{[5]}\,c^{(7)}_{[5]}\cdot\hat{F}^{7} & \cdots & \frac{1}{n!}\,\partial^{-1}\theta^{(5)}_{[5]}\,c^{(n)}_{[5]}\cdot\hat{F}^{n} \\
 & & \frac{1}{7!}\,\partial^{-1}\theta^{(6)}_{[5]}\,c^{(7)}_{[6]}\cdot\hat{F}^{7} & \cdots & \frac{1}{n!}\,\partial^{-1}\theta^{(6)}_{[5]}\,c^{(n)}_{[6]}\cdot\hat{F}^{n} \\
 & & \frac{1}{7!}\,\partial^{-2}\theta^{(5)}_{[5]}\,c^{(6)}_{[5]}\,c^{(7)}_{[6]}\cdot\hat{F}^{7} & \cdots & \frac{1}{n!}\,\partial^{-2}\theta^{(5)}_{[5]}\,c^{(6)}_{[5]}\,c^{(n)}_{[6]}\cdot\hat{F}^{n}
\end{array}
$$



Bk(x)
x

$$q(x) = \partial_\mu^* \left\{ \sum_{n=5}^{n'-1} \frac{1}{n!} \tau_\mu^{(n)}(x, \{y\}) \cdot \left\{ \frac{\hat{F}}{2} \right\}^n \Big|_x \right\} + \sum_{n=n'}^{\infty} \frac{1}{n!} \; \theta_{[n']}^{(n)}(x) \cdot \left\{ \frac{\hat{F}}{2} \right\}^n \Big|_x$$

$$\theta_{[n']}^{(n')}(x, \{y\}) = \partial_\mu^* \tau_\mu^{(n')}(x, \{y\})$$

$$\theta_{[n'+1]}^{(n)}(x, \{y\}) = \theta_{[n']}^{(n)}(x, \{y\}) + \tau_\mu^{(n')}(x, \{y\})\, c_{[n']}^{(n)}(\{y'\}) \quad (n \geq n' + 1)$$

Convergence

$$\frac{1}{n!} \sum_{\{y\}} \left| \theta_{[n']}^{(n)}(x, \{y\}) \right| \;\leq\; c\; 2^{n'} (1 + (n')^\sigma)\, \frac{1}{n!} \sum_{\{y\}} \left| \theta_{[5]}^{(n)}(x, \{y\}) \right|$$

$$n' \to 0$$

$$\sum_{n=n'}^{\infty} \frac{1}{n!} \Big| \sum_{\{y\}} \theta_{[n']}^{(n)}(x, \{y\}) \Big| \, \tilde{\epsilon}^n \;\to\; 0 \qquad\qquad \frac{2\tilde{\epsilon}}{\tilde{\varrho}} \;<\; 1$$

$d = 6$ $\Sigma_R \, Tr_R[T^a \{T^b, T^c\}] = 0$

$$\begin{aligned} q(x) &= \sum_{n=5}^{\infty} \frac{1}{n!} q_{[5]}^{(n)}(x) \\ &= \sum_{n=5}^{\infty} \frac{1}{n!} \frac{1}{2^n} \theta_{[5]}^{(n)}(x, y_1, \cdots, y_n)^{a_1, \cdots, a_n}_{\mu_1\nu_1, \cdots, \mu_n\nu_n} \hat{F}^{a_1}(y_1, \mu_1, \nu_1; x) \cdots \hat{F}^{a_n}(y_n, \mu_n, \nu_n; x) \\ &= \sum_{n=5}^{\infty} \frac{1}{n!} \; \theta_{[5]}^{(n)}(x, \{y\}) \cdot \left\{ \frac{\hat{F}(y)}{2} \right\}^n \Big|_x . \end{aligned}$$

there exist a gauge-invarinat local current $k_\mu(x)$ such that

$$q(x) = \partial_\mu^* k_\mu(x),$$

which admits a convergent series expansion as

$$\begin{aligned} k_\mu(x) &= \sum_{n=5}^{\infty} \frac{1}{n!} k_\mu^{(n)}(x) \\ &= \sum_{n=5}^{\infty} \frac{1}{n!} \frac{1}{2^n} \tau_\mu^{(n)}(x, y_1, \cdots, y_n)^{a_1, \cdots, a_n}_{\mu_1\nu_1, \cdots, \mu_n\nu_n} \hat{F}_s^{a_1}(y_1, \mu_1, \nu_1; x) \cdots \hat{F}_s^{a_n}(y_n, \mu_n, \nu_n; x) \\ &= \sum_{n=5}^{\infty} \frac{1}{n!} \; \tau_\mu^{(n)}(x, \{y\}) \cdot \left\{ \frac{\hat{F}(y)}{2} \right\}^n \Big|_x . \end{aligned}$$

$$q(x) = -\frac{1}{2}\mathrm{tr}\left\{\frac{H_\mathrm{w}}{\sqrt{H_\mathrm{w}^2}}\right\}(x,x) \qquad H_\mathrm{w} = \gamma_{2n+1}X_\mathrm{w}^{(2n)}, \qquad X_\mathrm{w}^{(2n)} = D_\mathrm{w}^{(2n)} - m_0$$

$$\mathfrak{U} = \left\{\{U(x,\mu)\} \,\Big|\, \| 1 - P_{\mu\nu}(x) \| < \epsilon \quad {}^\forall(x,\mu,\nu)\right\}, \quad \epsilon < \frac{2}{5d(d-1)} = \begin{cases} \frac{1}{75} & (d=6) \\ \frac{1}{30} & (d=4) \\ \frac{1}{5} & (d=2) \end{cases}$$

$$\sum_z \delta_\eta q(z) = 0 \qquad \delta_\eta U(x,\mu) = i\eta(x,\mu)U(x,\mu)$$

$d = 6$ $\qquad \sum_r \mathrm{Tr}_r\{T^a[T^b, T^c]\} = 0$

$$q(x) = \partial_\mu^* k_\mu(x)$$

$d = 4$

$$q(x) = c_2^{(L)}(x) + \partial_\mu^* k_\mu(x)$$

(1) x $Tr_{\underline{10}}[T^a T^b]$ + (-3) x $Tr_{\underline{5*}}[T^a T^b]$ = 0

SU(5) x U(1)$_Q$

$$q_L(x) = q(x) + \Delta q(x)$$

$$D_L(x,y) = D(x,y) + \sum_{n\in\mathbb{Z}^4, n\neq 0} D(x, y+nL)$$

$$D_L(x,y) = D(x,y) + \mathrm{O}(\mathrm{e}^{-L/\varrho})$$

$$|\Delta q(x)| \leq \kappa L^\sigma \mathrm{e}^{-L/\varrho} \qquad \sum_x \Delta q(x) = 0 \qquad \Delta q(x) = \partial_\mu^* \Delta k_\mu(x)$$

- Integrability condition using 5-, 6-dim. lattice DWFs

[*YK (2002)*]
[*Pedersen-YK (2019)*]

$$q^{(6)}(z) = \partial^*_\mu k_\mu(z)$$

$$q^{(6)}(z) \equiv -\frac{1}{2}\mathrm{tr}\left\{ \left.\frac{H_{\rm w}^{(6)}}{\sqrt{{H_{\rm w}^{(6)}}^2}}\right|_{\rm AP} \right\}(z,z)$$

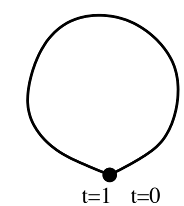
$\Sigma_R \, Tr_R[\, T^a \, \{T^b, T^c\} \,]=0$

local, topological field in 6-dim. lattice AS index theorem

$$\frac{\det X_{\rm w}^{(5)}\big|_{\rm AP}^{c_1 c_2^{-1}}}{\left|\det X_{\rm w}^{(5)}\big|_{\rm AP}^{c_1\cdot c_2^{-1}}\right|} \; \mathrm{e}^{-i2\pi \sum_{y\in c_1 c_2^{-1}} k_6(z)\big|_{\mathbb{Y}|_{\rm AP}^{c_1 c_2^{-1}}}} = 1 \qquad {}^\forall c_1 c_2^{-1} \subset \mathfrak{U}^{(5)}$$

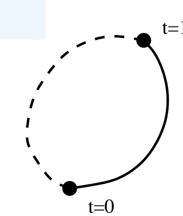
$$(N\to\infty)$$

invariant for the deformation $c_1' {c_2'}^{-1} \simeq c_1 c_2^{-1}$



$$\Rightarrow \quad \det(\bar v D'_{\rm ov} v^1) \; \frac{\det(v^{1\dagger} \prod_{t\in\tilde c} T_t v^0)}{\left|\det(v^{1\dagger} \prod_{t\in\tilde c} T_t v^0)\right|} \; \mathrm{e}^{-i2\pi \sum_{y\in c_1} k_6(z)\big|_{\mathbb{Y}|_{\rm AP}^{c_1 c_2^{-1}}}}$$

Integrable !



I-II. “Proof” of global integrability condition

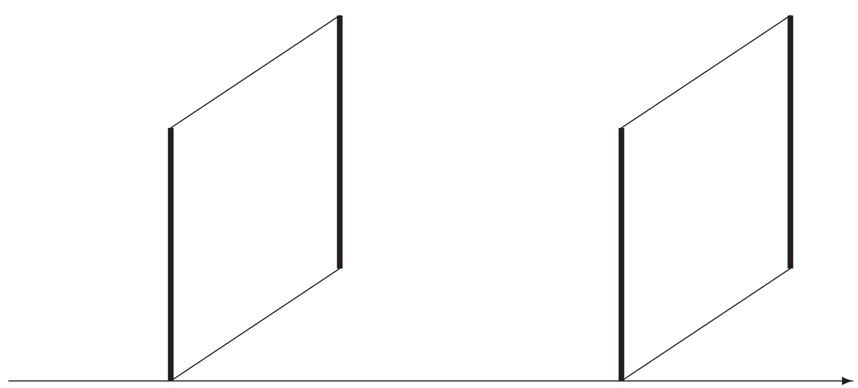
$$\frac{\det X_{\rm w}^{(5)}\big|_{\rm AP}^{c_1 c_2^{-1}}}{\det X_{\rm w}^{(5)}\big|_{\rm AP}^{c_1\cdot c_2^{-1}}\big|} \;\; \mathrm{e}^{-i2\pi \sum_{y\in c_1 c_2^{-1}} k_6(z)\big|_{\mathbb{Y}|_{\rm AP}^{c_1 c_2^{-1}}}} \qquad \forall c_1 c_2^{-1} \subset \mathfrak{U}^{(5)}$$

invariant for the deformation $c_1' c_2'^{-1} \simeq c_1 c_2^{-1}$

$$\frac{\det X_{\rm w}^{(5)}\big|_{\rm AP}^{c_1 c_2^{-1}}}{\left|\det X_{\rm w}^{(5)}\big|_{\rm AP}^{c_1 c_2^{-1}}\right|} \Bigg/ \frac{\det X_{\rm w}^{(5)}\big|_{\rm AP}^{c'_1 c'^{-1}_2}}{\left|\det X_{\rm w}^{(5)}\big|_{\rm AP}^{c'_1 c'^{-1}_2}\right|} = \mathrm{e}^{i2\pi \sum_{y,t\in c} q^{(6)}(z)} \qquad (N \to \infty)$$

$$q^{(6)}(z) = \partial^*_\mu k_\mu(z)$$

$$q_L(x) = q(x) + \Delta q(x)$$



$x_d = a_d s \; (s \in \mathbb{Z})$

$s = -N+1$ $\quad$ $s = N$

$$\partial \mathbb{Z}|_{\rm Dir} = \mathbb{Y}_1|_{\rm AP} \cup \bar{\mathbb{Y}}_0|_{\rm AP}$$

$$\mathbb{Z}|_{\rm Dir}^{c_1} \cup \bar{\mathbb{Z}}|_{\rm Dir}^{c_2} = \mathbb{Z}|_{\rm AP}^{c_1 c_2^{-1}}$$

$U(x,\mu)[c_1 c_2^{-1}]$ $\qquad$ $U(x,\mu)[c_1' c_2'^{-1}]$

$$\mathbb{Z}|_{\rm Dir} = \{\mathbb{L}^5 \times \mathbb{L} \;\; ; c : U_s(y,\mu) \in \mathfrak{U}^{(6)}\}$$

5dim. Link fields

$$\mathfrak{U}^{(5)} = \left\{\{U(x,\mu)\}\middle| \parallel 1 - P_{\mu\nu}(x) \parallel < \epsilon \ {}^{\forall}(x,\mu,\nu)\right\}$$

$$\mathbb{L}^5 = \mathbb{L}^3 \times \mathbb{L}_4 \times \mathbb{L}_5$$

$$Q_4 = \sum_{\mathbb{L}^3 \times \mathbb{L}_4} c_2^{(L)}(x)$$ x_5 -independent

$$\pi_3(S^3) = \mathbb{Z}$$

$$Q_5 = \sum_{\mathbb{L}^3 \times \mathbb{L}_5} c_2^{(L)}(x)$$ x_4 -independent

interpolated transition functions on the faces of hypercubes

[*Luscher (1982)*]

interpolated vector potential in the hypercubes

transversely continuous, non-periodic on the boundary faces

[*Hernandez-Sundrum (1996)*]

$U(x,\mu) \neq -1$ (measure zero)

interpolated vector potential

[*Hernandez-Sundrum (1996)*]

$U(x,\mu) \neq -1$ (measure zero)

$$U(x,\mu)$$

$$a_\mu(y) \qquad y = x + \sum_\nu s_\nu\,\hat{\nu}$$

$$U(x,\mu) = P\exp\left(i\int_x^{x+\hat{\mu}} ds_\mu\, a_\mu(x+s_\mu\hat{\mu})\right)$$

$$P(x,\mu,\nu) = P\exp\left(i\int_x^{x+\hat{\mu}} dy_\mu \int_x^{x+\hat{\nu}} dy_\nu\, g(y;x) f_{\mu\nu}(y) g^{-1}(x;y)\right)$$

$$Q = \sum_x c_2^{(L)}(x) = \frac{1}{16\pi^2}\int d^4y \mathrm{Tr}\left[f_{\mu\nu}\tilde{f}_{\mu\nu}(y)\right]$$

$$a_\mu^{(x)}\left(x+\nu+\sum_{\lambda\neq\nu} s_\lambda\hat{\lambda}\right) = a_\mu^{(x+\hat{\nu})}\left(x+\nu+\sum_{\lambda\neq\nu} s_\lambda\hat{\lambda}\right)$$

transversely continuous

$$a_\mu(x_k,0,x_5) = a_\mu^{g_4}(x_k,L_4,x_5) \qquad g_4(x_k,L_4,x_5) = h_4\big(x_k^{(4)} + \sum s_k\hat{k}\big)\delta_{x_k,x_k^{(4)}}$$

$$a_\mu(x_k,x_4,0) = a_\mu^{g_5}(x_k,L_4,L_5) \qquad g_5(x_k,x_4,L_5) = h_5\big(x_k^{(5)} + \sum s_k\hat{k}\big)\delta_{x_k,x_k^{(5)}}$$

non-periodic on the boundary faces

$U(x,\mu) \qquad U'(x,\mu) \qquad \in \mathfrak{U}^{(5)}$ same Q_4, Q_5

$a_\mu(y) \qquad a'_\mu(y)$

gauge transformation to resolve the non-periodicity

$$a_\mu(y)_s{}^h = \left[(1-s)a_\mu(y)^h + s\, a'_\mu(y)^h\right]$$

$$s \in [0,1]$$

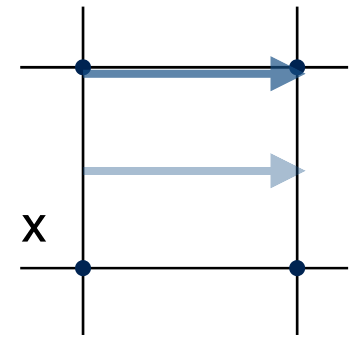
$$U(x,\mu)_s = P\,\exp\left(i\int ds_\mu\, a_\mu(x+s_\mu\hat{\mu})_s\right)$$

2dim. U(1)

$U(x,\mu) \neq -1 \qquad A_\mu(x) = \frac{1}{i}\ln U(x,\mu)$

links around a plaquette $\qquad y = x + s_1\hat{1} + s_2\hat{2}$

$$g(y) = \begin{cases} U(x,1)^{s_1} \\ U(x,1)U(x+\hat{1},2)^{s_2} \\ U(x,2)^{s_2} \\ P(x,1,2)^{s_1}U(x,2)U(x+\hat{2},1)^{s_1} \end{cases}$$



$$i\,\bar{a}_\mu(y) = \begin{cases} s_2 \ln P(x,2,1) \\ 0 \end{cases}$$

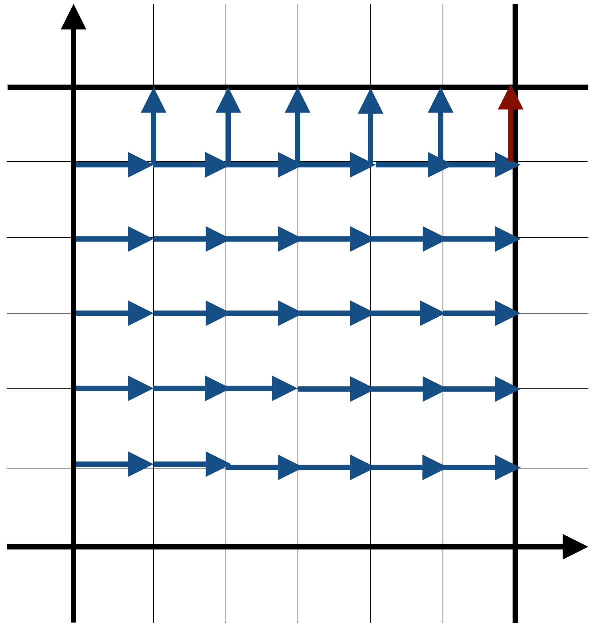
$$\pi_1(S^1) = \mathbb{Z}$$

$$2\pi n(x) = \oint_x dy_\mu \frac{1}{i}\partial_\mu g(y)\, g(y)^{-1}$$

$$= [A_1(x) + A_2(x+\hat{1}) - A_1(x+\hat{2}) - A_2(x)] - \frac{1}{i}\ln P(x,1,2)$$

$$Q = \sum_x n(x) = \sum_x \frac{i}{2\pi}\ln P(x,1,2)$$

$$n(x) = \sum_{\mu<\nu} \{\partial_\mu m_\nu(x) - \partial_\nu m_\mu(x)\}$$



$$\sigma(y) = \begin{cases} \mathrm{e}^{i2\pi m_1(x) s_1} \\ \mathrm{e}^{i2\pi m_2(x+\hat{1}) s_2} \\ \mathrm{e}^{i2\pi m_2(x) s_2} \\ \mathrm{e}^{i2\pi m_1(x+\hat{2}) s_1} \end{cases}$$

$$\begin{aligned} \tilde{g}(y) &= g(y)\sigma(y) \\ &= \tilde{U}(x,1)^{s_1} \left(\tilde{U}(x,1)^{-s_1} P(x,1,2)^{s_1} \tilde{U}(x,2) \tilde{U}(x+\hat{2},1)^{s_1} \right)^{s_2} \end{aligned}$$

$$\tilde{U}(x,\mu) = U(x,\mu)\,\mathrm{e}^{i2\pi m_\mu(x)}$$

$$i\,a_\mu(y) = i\,\bar{a}_\mu(y) + \tilde{g}(y)^{-1}\partial_\mu \tilde{g}(y)$$

$$a_2(L, L-1+s_2\hat{2}) = a_2(0, L-1+s_2\hat{2}) + 2\pi m_2(L, L-1)$$

$$m_2(L, L-1) = Q\ (= m_{12})$$

$$N(x) = -\frac{1}{24\pi^2}\epsilon_{\mu\nu\lambda\rho}\int_{\partial\mathrm{HC}(x)} dS_\mu \mathrm{Tr}(g^{-1}\partial_\nu g)(g^{-1}\partial_\lambda g)(g^{-1}\partial_\rho g)$$

$$N[U] = \sum_x N(x)$$

$$N(x) = \sum_\mu \{N_\mu(x+\hat{\mu}) - N_\mu(x)\}$$

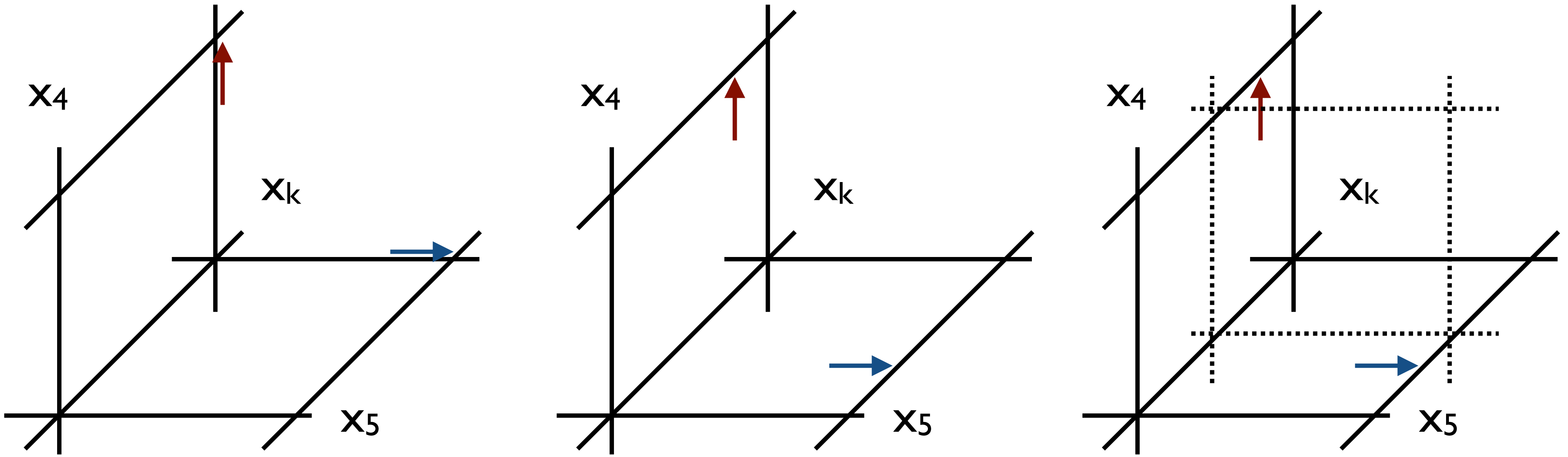
$$\tilde{g}(y) = g(y)\,\sigma(y)^{N_\mu(x)} \qquad y = x + s_1\hat{\nu} + s_2\hat{\lambda} + s_3\hat{\rho}$$

$$\sigma(x + s_1\hat{\nu} + s_2\hat{\lambda} + s_3\hat{\rho}) = \frac{(\sum_k \tau^k \cot(\pi s_k) + i\epsilon_{\mu\nu\lambda\rho})^2}{\sum_k \cot^2(\pi s_k) + 1}$$

$$N_k(x + L\hat{k}) = N_k(x) \quad (k = 1, 2, 3)$$

$$N_4(x_k, L_4) = N_4(x_k, 0) + N[U]\delta_{x_1,0}\delta_{x_2,0}\delta_{x_3,0}$$

$$i\,a_k(s, L) = i\,g_\sigma(s)^{-1}a_k(s,0)\,g_\sigma(s) + g_\sigma(s)^{-1}\partial_k\,g_\sigma(s) \qquad g_\sigma(s) = \sigma(s)^{N[U]}$$



$$\left.\frac{\det X_{\rm w}^{(5)}|_{\rm AP}}{|\det X_{\rm w}^{(5)}|_{\rm AP}|}\right|_{\mathbb{Y}} = \left.\frac{\det X_{\rm w}^{(5)}|_{\rm AP}}{|\det X_{\rm w}^{(5)}|_{\rm AP}|}\right|_{\mathbb{Y}_1} \times \left.\frac{\det X_{\rm w}^{(5)}|_{\rm AP}}{|\det X_{\rm w}^{(5)}|_{\rm AP}|}\right|_{\mathbb{Y}_2} \qquad (L_k \to \infty)$$

mass gap in

Transfer matrix in k-th direction

cf. gluing property of η invariant

5dim. Link fields

$$\mathfrak{U}^{(5)} = \Big\{ \{U(x,\mu)\} \Big| \parallel 1 - P_{\mu\nu}(x) \parallel < \epsilon \ ^{\forall}(x,\mu,\nu) \Big\}$$

$$\mathbb{L}^5 = \mathbb{L}^3 \times \mathbb{L}_4 \times \mathbb{L}_5$$

$$Q_4 = \sum_{\mathbb{L}^3 \times \mathbb{L}_4} c_2^{(L)}(x)$$ **x5 -independent**

$$\pi_3(S^3) = \mathbb{Z}$$

$$Q_5 = \sum_{\mathbb{L}^3 \times \mathbb{L}_5} c_2^{(L)}(x)$$ **x4 -independent**

$$U(x,t,\mu) = U(x,\mu) \in \mathfrak{U}[Q_4]$$ **x5 -independent**

$$U(x,t,\mu) = U(x_k,t,\mu) \in \mathfrak{U}[Q_5]$$ **x4 -independent**

cf. [*Witten-Yonekura (2019)*]

$$\det X_{\rm w}^{(5)}|_{U(x,t,\mu)=U(x,\mu)} = \prod_{p_5=\frac{2\pi}{N}(m+1/2)} \det(X_{\rm w}^{(4)} + i\gamma_5 \sin p_5 (1-\cos p_5))$$

$$= \prod_{p_5>0} \det(X_{\rm w}^{(4)} + i\gamma_5 \sin p_5 + (1-\cos p_5)) \det(X_{\rm w}^{(4)} - i\gamma_5 \sin p_5 + (1-\cos p_5))$$

$$\times \det(X_{\rm w}^{(4)} + 2)$$

$$\det X_{\rm w}^{(5)}|_{U(x,t,\mu)=U(x_k,t,\mu)} = \prod_{p_4=\frac{2\pi}{N}m} \det(X_{\rm w}^{(4)} + i\gamma_4 \sin p_4 + (1-\cos p_4))$$

$$= \prod_{p_4>0} \det(X_{\rm w}^{(4)} + i\gamma_4 \sin p_4 + (1-\cos p_4)) \det(X_{\rm w}^{(4)} - i\gamma_4 \sin p_4 + (1-\cos p_4))$$

$$\times \det(X_{\rm w}^{(4)}) \times \det(X_{\rm w}^{(4)} + 2)$$

$$\frac{\det X_{\rm w}^{(4)}|_{\rm P}}{|\det X_{\rm w}^{(4)}|_{\rm P}|} = (-1)^Q \qquad Q = \frac{1}{2}{\rm Tr}\left\{\frac{H_{\rm w}^{(4)}}{\sqrt{H_{\rm w}^{(4)}}}\right\} = 4n$$ (AS index theorem)

SO(10), SU(5) xU(1)$_{\rm Q}$, SU(3)xSU(2)xU(1) xU(1)$_{\rm B\text{-}L}$

cf. for SU(2) doublet (Witten anomaly)

$$\left.\frac{\det X_{\rm w}^{(5)}|_{\rm AP}}{|\det X_{\rm w}^{(5)}|_{\rm AP}|}\right|_{U(x,t,\mu)=U(x_k,t,\mu)} = -1$$

SU(2) doublet

- Integrability condition using 5-, 6-dim. lattice DWFs

[*YK (2002)*]

[*Pedersen-YK (2019)*]

$$q^{(6)}(z) = \partial^*_\mu k_\mu(z)$$

$$q^{(6)}(z) \equiv -\frac{1}{2} \mathrm{tr} \left\{ \left. \frac{H_{\mathrm{w}}^{(6)}}{\sqrt{{H_{\mathrm{w}}^{(6)}}^2}} \right|_{\mathrm{AP}} \right\} (z, z)$$

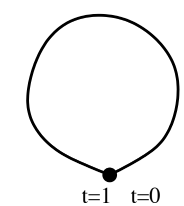
$\Sigma_R Tr_R[\, T^a \{T^b, T^c\} \,]=0$

local, topological field in 6-dim. lattice AS index theorem

$$\frac{\det X_{\mathrm{w}}^{(5)}\big|_{\mathrm{AP}}^{c_1 c_2^{-1}}}{\left| \det X_{\mathrm{w}}^{(5)}\big|_{\mathrm{AP}}^{c_1 \cdot c_2^{-1}} \right|} \; \mathrm{e}^{-i 2\pi \sum_{y \in c_1 c_2^{-1}} k_6(z)\big|_{\mathbb{Y}|_{\mathrm{AP}}^{c_1 c_2^{-1}}}} = 1 \qquad {}^{\forall} c_1 c_2^{-1} \subset \mathfrak{U}^{(5)}$$

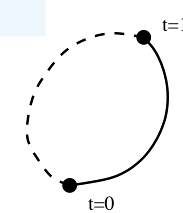
$$(N \to \infty)$$

invariant for the deformation $c_1' {c_2'}^{-1} \simeq c_1 c_2^{-1}$



$$\Rightarrow \quad \det(\bar{v} D'_{\mathrm{ov}} v^1) \; \frac{\det({v^1}^\dagger \prod_{t \in \tilde{c}} T_t v^0)}{\left| \det({v^1}^\dagger \prod_{t \in \tilde{c}} T_t v^0) \right|} \; \mathrm{e}^{-i 2\pi \sum_{y \in c_1} k_6(z)\big|_{\mathbb{Y}|_{\mathrm{AP}}^{c_1 c_2^{-1}}}}$$

Integrable !



II. Eichten-Preskill model / Mirror fermion model with overlap fermions

J. Preskill, e-Print: 1811.10085 [hep-lat], PoS LATTICE2018 (2018) 024

"Simulating quantum field theory with a quantum computer"

"This long-standing problem may be nearing a resolution, guided in part by recent insights regarding symmetry-protected topological phases of matter. Two old ideas are:

(1) To realize a D-dimensional chiral theory on the lattice, we can introduce an extra spatial dimension, so that the left-handed and right-handed fermions live on two different D-dimensional edges of a (D + 1)- dimensional bulk [45].

(2) To realize a D-dimensional chiral theory on the lattice, we can introduce strong interactions for the express purpose of removing the unwanted right-handed fermions (by giving them large masses) while preserving the massless left-handed fermions [46].

It seems likely that (1) and (2) together work more effectively than either (1) or (2) by itself [47]. That's because separating the two edges with a higher-dimensional bulk makes it easier to apply the strong interactions to one chirality without affecting the other.

The efficacy of this method still needs to be demonstrated convincingly, but if it works that will settle the longstanding open question whether quantum field theories with chiral fermions really exist, and will also open the door for classical and quantum studies of the rich dynamics of strongly-coupled chiral gauge theories, with potential applications to physics beyond the standard model."

Let me add …

In view of the fact that "the overlap fermion is nothing but the low energy effective (local & lattice) theory for the edge chiral modes of DWF infinitely separated", the mirror fermion models with overlap fermions (1) and (refined) Eichten-Presskill interaction terms (2) are the simplest-possible effective framework to construct & test CGT on the lattice !

Early attempt for chiral gauge theory on the lattice by Eichten-Preskill

[*Eichten-Preskill(1986)*]

$$
\begin{aligned}
S_{\rm EP} \;=\; & \sum_x \Big\{ \bar\psi(x)\gamma_\mu P_- \frac{1}{2}(\nabla_\mu - \nabla_\mu^\dagger)\psi(x) \\
& -\frac{\lambda}{24}\left[\psi_-(x)^T i\gamma_5 C_D \mathrm{T}^a \psi_-(x)\right]^2 - \frac{\lambda}{24}\left[\bar\psi_-(x) i\gamma_5 C_D \mathrm{T}^a \bar\psi_-(x)^T\right]^2 \\
& -\frac{\lambda}{48}\triangle\left[\psi_-(x)^T i\gamma_5 C_D \mathrm{T}^a \psi_-(x)\right]^2 - \frac{\lambda}{48}\triangle\left[\bar\psi_-(x) i\gamma_5 C_D \mathrm{T}^a \bar\psi_-(x)^T\right]^2 \Big\}
\end{aligned}
$$

generalized Wilson-term

$$
\begin{aligned}
& \Delta\{A(x)B(x)C(x)D(x)\} \\
& \equiv +\frac{1}{2}\sum_\mu \Big\{ \big(\nabla_\mu\nabla_\mu^\dagger A(x)\big)B(x)C(x)D(x) + A(x)\big(\nabla_\mu\nabla_\mu^\dagger B(x)\big)C(x)D(x) \\
& \qquad + A(x)B(x)\big(\nabla_\mu\nabla_\mu^\dagger C(x)\big)D(x) + A(x)B(x)C(x)\big(\nabla_\mu\nabla_\mu^\dagger D(x)\big) \Big\}.
\end{aligned}
$$

- break explicitly
  - staggered symmetry $(-1)^{x_\mu}$ • global chiral symmetries with 't Hooft anomaly

  (<u>16</u> x <u>16</u> x <u>16</u> x <u>16</u>) = (<u>10</u> x <u>10</u> x <u>10</u> x <u>5*</u>) $_0$ **+ (<u>10</u> x <u>5*</u> x <u>5*</u>)** $_{-5}$ **x <u>1</u>** $_5$
- resolve the degenerated physical(chiral) and species-doubling modes

  { **(<u>16</u>)**- + **(<u>16</u>)**+ } x 8 —> light **(<u>16</u>)**- + heavy { **(<u>16</u>)**- x 7 + **(<u>16</u>)**+ x 8}
- 't Hooft vertex-type SO(10) invariant quartic terms,

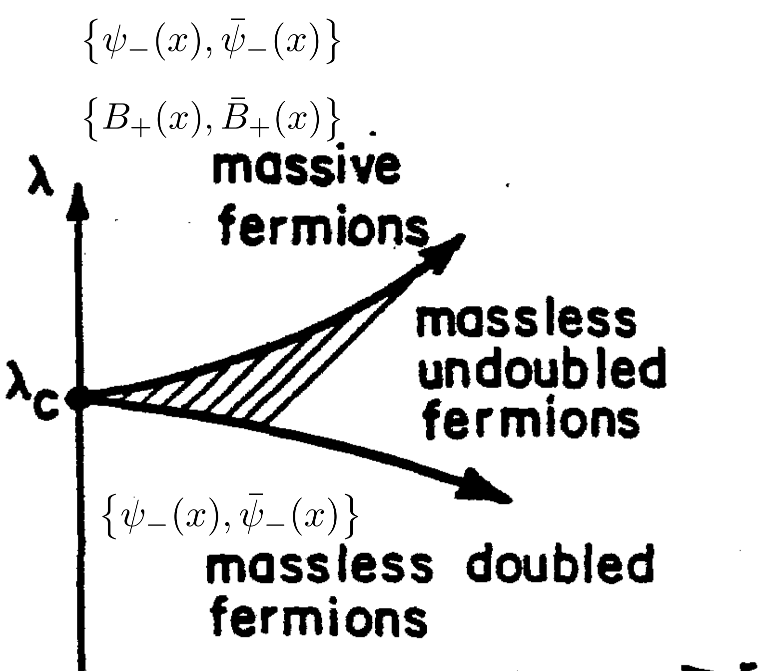
  No obstruction for SO(9) => SO(10) symmetry restoration: $\Pi_d(S^9) = 0$ (d =0, …,9) (Wen)

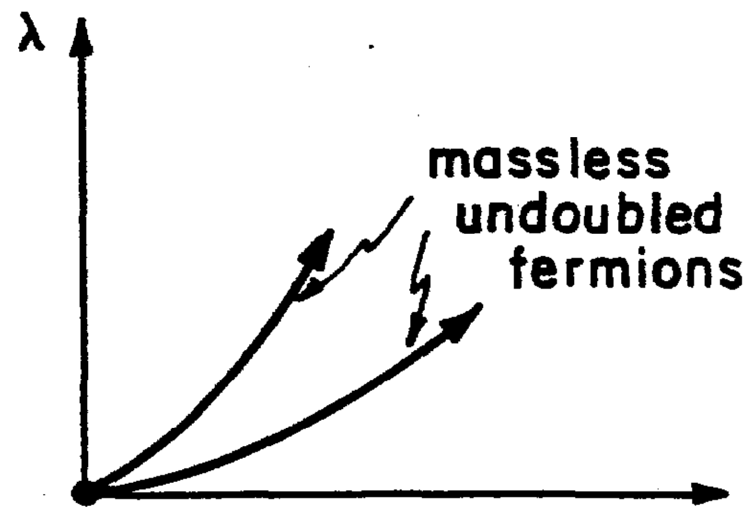
$$
\lambda\left[\psi^T(x) i\gamma_5 c_d \mathrm{C}\Gamma^a \psi(x)\right]^2 \;\Leftrightarrow\; y\left[\psi^T(x) i\gamma_5 c_d \mathrm{C}\Gamma^a\psi(x)\right]E^a(x) \qquad E^a(x)E^a(x) = 1
$$

Phase structure and Fine tuning of the couplings ?

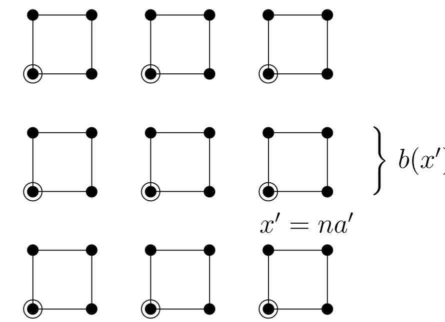
- resolve the degenerated physical and species-doubling modes
  { (**16**)- + (**16**)+ } x 8 —> light (**16**)- + heavy { (**16**)- x 7 + (**16**)+ x 8}
- fine-tune λ (λ1, λ2) to the massless limit within a SO(10)[SU(5)] -symmetric phase
- global anomaly cancellation
  $\Omega^{spin}_5(BSO(10)) = 0$ <=> trivial phase of SPT
- ν=16, "All" is trivialized by a certain SO(10)-invariant and U(1)-breaking interaction, and the boundary edge modes may be gapped completely without symmetry breaking

Study of fixed point structure in EP model by Block spin transformation ?!





Block spin transformation for Wilson-Dirac fermions



$$B_{x'} = \{x \in \mathbb{Z}^d \,\big|\, x = x' + \hat{\mu}, \mu = 1, \cdots, d\},\; x' \in (2\mathbb{Z})^d$$

$$\mathrm{e}^{-S'[\psi'(x'),\bar{\psi}'(x')]} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\,\mathcal{D}\eta'\mathcal{D}\bar{\eta}'\,\mathrm{e}^{-S[\psi(x),\bar{\psi}(x)]} \times$$
$$\mathrm{e}^{\sum_{x'}\left\{a\,\bar{\eta}'(x')\eta'(x')+\bar{\eta}'(x')\left(\psi'(x')-\frac{b}{2d}\sum_{x\in B_{x'}}\psi(x)\right)+\eta'(x')\left(\bar{\psi}'(x')-\frac{b}{2d}\sum_{x\in B_{x'}}\bar{\psi}(x)\right)\right\}}$$

$$\mathrm{e}^{W[\eta(x),\bar{\eta}(x)]}\,Z = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\;\mathrm{e}^{-S[\psi(x),\bar{\psi}(x)]}\,\mathrm{e}^{-\sum\left\{\bar{\eta}(x)\psi(x)+\eta(x)\bar{\psi}(x)\right\}}$$

$$\mathrm{e}^{W'[\eta'(x'),\bar{\eta}'(x')]} = \mathrm{e}^{W[\eta(x),\bar{\eta}(x)]}\Big|_{\eta(x),\bar{\eta}(x)\to\frac{b}{2d}\eta'(x'),\,\frac{b}{2d}\bar{\eta}'(x'),\,x\in B_{x'}} \times \mathrm{e}^{a\sum_{x'}\bar{\eta}'(x')\eta'(x')}$$

$$\mathrm{e}^{W[\eta(x),\bar{\eta}(x)]} = \mathrm{e}^{-\sum_{x,y}\bar{\eta}(x)D_W{}^{-1}(x,y)\eta(y)}$$

Fixed point theory

$$\mathrm{e}^{W_{D^*}[\eta(x),\bar{\eta}(x)]} = \mathrm{e}^{-\sum_{x,y}\bar{\eta}(x){D_*}^{-1}(x,y)\eta(y)}$$

$$b = 2^{\frac{d-1}{2}}$$

$${D_*}^{-1}(x,y) = \int \frac{d^dk}{(2\pi)^4}\,\mathrm{e}^{ik(x-y)}\left\{-i\gamma_\mu\alpha_\mu(k)+R\right\}, \quad {D_c}^{-1}(x,y) = \int \frac{d^dk}{(2\pi)^4}\,\mathrm{e}^{ik(x-y)}\left\{-i\gamma_\mu\alpha_\mu(k)\right\}$$

$$\alpha_\mu(k) = \sum_{l\in\mathbb{Z}^d}\prod_\nu\left(\frac{2\sin(k_\nu/2)}{k_\nu+2\pi l_\nu}\right)^2\frac{k_\mu+2\pi l_\mu}{(k_\mu+2\pi l_\mu)^2},$$

$$R = \frac{a}{\left(1-\frac{b^2}{2^d}\right)} = 2a.$$

$$D_*\gamma_5+\gamma_5D_* = 2R\,D_*\gamma_5D_*$$

$$\begin{aligned} S_{D^*} &= \sum_x \bar{\psi}(x)D_*\psi(x) \\ &= \sum_x \left\{\bar{\psi}_+(x)D_*\psi_+(x)+\bar{\psi}_-(x)D_*\psi_-(x)\right\} \end{aligned}$$

relevant mass perturbations which breaks parity(chirality) & fermion number

$$-M_+ \left\{ \psi_+(x)^T i\gamma_5 c_D \bar{\psi}_+(x) + \bar{\psi}_+(x) i\gamma_5 c_D \bar{\psi}_+(x)^T \right\}$$

$$\mathrm{e}^{W_{M_-^*}[\eta'(x'),\bar{\eta}'(x')]} = \mathrm{e}^{-\sum_{x'} \frac{1}{2} \left(\bar{\eta}' \;\; \eta'^T \right) \mathcal{D}_*^{-1} \begin{pmatrix} \bar{\eta}'^T \\ \eta' \end{pmatrix}}$$

M_-^*

$$b_- = 2^{\frac{d-1}{2}}$$

$$b_+ = 2^{\frac{d}{2}}$$

$$\mathcal{D}_*{}^{-1} = \begin{pmatrix} -P_+ i\gamma_5 c_D P_+^T & P_- D_c^{-1} P_+ + R \\ -(P_- D_c^{-1} P_+ + R)^T & -P_-^T i\gamma_5 c_D P_- \end{pmatrix}$$

$$R = \frac{a}{\left(1 - \frac{b_- b_+}{2^d}\right)} = \frac{a}{\left(1 - \frac{1}{\sqrt{2}}\right)}.$$

Block spin kernel which breaks parity(chirality) & fermion number

$$e^{W'[\eta'(x'),\bar\eta'(x')]} = e^{W[\eta(x),\bar\eta(x)]}\Big|_{\eta(x),\bar\eta(x)\to\frac{b}{2^d}\eta'(x'),\frac{b}{2^d}\bar\eta'(x'),\,x\in B_{x'}} \times$$
$$e^{a\sum_{x'}\bar\eta'(x')\eta'(x')} \times e^{c\sum_{x'}\left\{\bar\eta'(x')P_+ i\gamma_5 c_D P_+^T\bar\eta'(x')^T+\eta'(x')^T P_-^T i\gamma_5 c_D P_-\eta'(x')^T\right\}}.$$

M_-^*

$$b_- = 2^{\frac{d-1}{2}}$$
$$b_+ < 2^{\frac{d}{2}}$$

for ν=16, SO(9) symmetric kernel
for <u>16</u> , but not for SO(10)

$$\mathcal{D}_*{}^{-1} = \begin{pmatrix} -\tilde R\,P_+ i\gamma_5 c_D P_+^T & P_- D_c^{-1}P_+ + R \\ -(P_- D_c^{-1}P_+ + R)^T & -\tilde R\,P_-^T i\gamma_5 c_D P_- \end{pmatrix},$$
$$R = \frac{a}{\left(1-\frac{b_-b_+}{2^d}\right)},$$
$$\tilde R = \frac{c}{\left(1-\frac{b_+^2}{2^d}\right)}.$$

$$\mathcal{D}_*\Gamma_5+\Gamma_5\mathcal{D}_* = 2\,\mathcal{D}_*\Gamma_5\begin{pmatrix} -\tilde R\,P_+ i\gamma_5 c_D P_+^T & R \\ -R & -\tilde R\,P_-^T i\gamma_5 c_D P_- \end{pmatrix}\mathcal{D}_*$$
$$\mathcal{D}_*\Gamma+\Gamma\mathcal{D}_* = 2\,\mathcal{D}_*\Gamma\begin{pmatrix} -\tilde R\,P_+ i\gamma_5 c_D P_+^T & \\ & -\tilde R\,P_-^T i\gamma_5 c_D P_- \end{pmatrix}\mathcal{D}_*,$$

$$S_{M_-^*} = \sum_x \frac{1}{2}\begin{pmatrix}\psi'^T & \bar\psi'\end{pmatrix}\mathcal{D}_*\begin{pmatrix}\psi' \\ \bar\psi'^T\end{pmatrix}$$

$$\Gamma_5 = \begin{pmatrix}\gamma_5 & 0 \\ 0 & \gamma_5\end{pmatrix},\quad \Gamma = \begin{pmatrix}\mathbb{I} & 0 \\ 0 & -\mathbb{I}\end{pmatrix}.$$

SO(9) symmetric kernel for <u>16</u>
SO(10) exact symmetry
(but, non on-site)

Block spin transformation for Eichten-Preskill model

$$
\begin{aligned}
S_{\mathrm{EP}} &= \sum_x \Big\{ \bar\psi(x)\gamma_\mu P_- \frac{1}{2}(\nabla_\mu - \nabla_\mu^\dagger)\psi(x) \\
&\quad - \frac{\lambda}{24}\left[\psi_-(x)^T i\gamma_5 C_D \mathrm{T}^a \psi_-(x)\right]^2 - \frac{\lambda}{24}\left[\bar\psi_-(x) i\gamma_5 C_D \mathrm{T}^a \bar\psi_-(x)^T\right]^2 \\
&\quad - \frac{\lambda}{48}\triangle\left[\psi_-(x)^T i\gamma_5 C_D \mathrm{T}^a \psi_-(x)\right]^2 - \frac{\lambda}{48}\triangle\left[\bar\psi_-(x) i\gamma_5 C_D \mathrm{T}^a \bar\psi_-(x)^T\right]^2 \Big\}
\end{aligned}
$$

$$
\begin{aligned}
S'_{EP} = \sum_x \Big\{ & \bar\psi(x)\gamma_\mu \frac{1}{2}(\partial_\mu - \partial_\mu^\dagger)P_-\psi(x) + \bar\psi(x)\Big(\frac{w}{2}\partial_\mu\partial_\mu^\dagger + m_0\Big)\psi(x) \\
& -y\left(\psi(x)^T i\gamma_5 c_D T^a P_+\psi(x) + \bar\psi(x)P_- i\gamma_5 c_D T^{a\dagger}\bar\psi(x)^T\right) E^a(x)\Big\}
\end{aligned}
\qquad E^a(x)E^a(x) = 1
$$

SO(10)-invariant Block spin kernel which breaks parity(chirality) & fermion number (need a non-bilinear kernel)

$$\mathrm{e}^{W'[\eta'(x'),\bar\eta'(x')]} = \mathrm{e}^{W[\eta(x),\bar\eta(x)]}\Big|_{\eta(x),\bar\eta(x)\to\frac{b}{2^d}\eta'(x'),\,\frac{b}{2^d}\bar\eta'(x'),\,x\in B_{x'}} \times$$

$$\mathrm{e}^{a\sum_{x'}\bar\eta'(x')\eta'(x')} \times \mathrm{e}^{\frac{c}{2}\sum_{x'}\left\{\bar\eta'(x')P_+ i\gamma_5 c_D T^{a\dagger} P_+^T \bar\eta'(x')^T + \eta'(x')^T P_-^T i\gamma_5 c_D T^a P_- \eta'(x')^T\right\}^2}$$

Fixed point theory exists !

$$\mathrm{e}^{W_{\mathrm{SO(10)}}[\eta'(x'),\bar\eta'(x')]} = \mathrm{e}^{-\sum_{x'}\frac{1}{2}\left(\bar\eta' \;\; \eta'^T\right)\mathcal{D}_*^{-1}\begin{pmatrix}\bar\eta'^T\\ \eta'\end{pmatrix}} \times$$

$$\mathrm{e}^{\frac{\tilde R}{2}\sum_{x'}\left\{\bar\eta'(x')P_+ i\gamma_5 c_D T^{a\dagger} P_+^T \bar\eta'(x')^T + \eta'(x')^T P_-^T i\gamma_5 c_D T^a P_- \eta'(x')^T\right\}^2}$$

$$b_- = 2^{\frac{d-1}{2}},$$
$$b_+ < 2^{\frac{d+1}{2}} \quad (< 2^{\frac{3d}{4}})$$

$$\mathcal{D}_*{}^{-1} = \begin{pmatrix} 0 & P_- D_c^{-1} P_+ + R \\ -(P_- D_c^{-1} P_+ + R)^T & 0 \end{pmatrix}$$

$$R = \frac{a}{\left(1-\frac{b_- b_+}{2^d}\right)},$$

$$\tilde R = \frac{c}{\left(1-\frac{b_+^4}{(2^d)^3}\right)}.$$

correlation functions

The correlation functions at the fixed point:

$$
\begin{aligned}
\langle \psi(x)\bar{\psi}(y)\rangle &= P_- D_c^{-1}(x,y) P_+ + R\,\delta_{xy}, \\
\langle \psi(x)\psi(y)\rangle &= 0, \\
\langle \bar{\psi}(x)\bar{\psi}(y)\rangle &= 0, \\
\langle \psi(x_1)\psi(x_2)\psi(x_3)\psi(x_4)\rangle &= 4\tilde{R}\Big[\{P_+ i\gamma_5 c_D P_+^T\}_{s_1,s_2}\{T^a\}_{i_1,i_2}\{P_+ i\gamma_5 c_D P_+^T\}_{s_3,s_4}\{T^a\}_{i_3,i_4}\delta_{x_1,x_2}\delta_{x_3,x_4}\delta_{x_1,x_3} \\
&\quad -\{P_+ i\gamma_5 c_D P_+^T\}_{s_1,s_3}\{T^a\}_{i_1,i_3}\{P_+ i\gamma_5 c_D P_+^T\}_{s_2,s_4}\{T^a\}_{i_2,i_4}\delta_{x_1,x_3}\delta_{x_2,x_4}\delta_{x_1,x_2} \\
&\quad +\{P_+ i\gamma_5 c_D P_+^T\}_{s_1,s_4}\{T^a\}_{i_1,i_4}\{P_+ i\gamma_5 c_D P_+^T\}_{s_2,s_3}\{T^a\}_{i_2,i_3}\delta_{x_1,x_4}\delta_{x_2,x_3}\delta_{x_1,x_2}\Big]
\end{aligned}
$$

$$
\begin{aligned}
\langle \bar{\psi}(x_1)\bar{\psi}(x_2)\bar{\psi}(x_3)\bar{\psi}(x_4)\rangle &= 4\tilde{R}\Big[\{P_-^T i\gamma_5 c_D P_-\}_{s_1,s_2}\{T^a\}_{i_1,i_2}\{P_-^T i\gamma_5 c_D P_-\}_{s_3,s_4}\{T^a\}_{i_3,i_4}\delta_{x_1,x_2}\delta_{x_3,x_4}\delta_{x_1,x_3} \\
&\quad -\{P_-^T i\gamma_5 c_D P_-\}_{s_1,s_3}\{T^a\}_{i_1,i_3}\{P_-^T i\gamma_5 c_D P_-\}_{s_2,s_4}\{T^a\}_{i_2,i_4}\delta_{x_1,x_3}\delta_{x_2,x_4}\delta_{x_1,x_2} \\
&\quad +\{P_-^T i\gamma_5 c_D P_-\}_{s_1,s_4}\{T^a\}_{i_1,i_4}\{P_-^T i\gamma_5 c_D P_-\}_{s_2,s_3}\{T^a\}_{i_2,i_3}\delta_{x_1,x_4}\delta_{x_2,x_3}\delta_{x_1,x_2}\Big]
\end{aligned}
$$

$$
\langle X^a(x) X^b(y)\rangle = R_X\,\delta^{ab}\,\delta_{xy},
$$

In the Strong coupling limit of the Eichten-Preskill model

The saturation of lattice fermion measures due to 't Hooft vertices

$$\psi_+(x) = P_+\psi(x) \quad \bar{\psi}_+(x) = \bar{\psi}(x)P_- \quad \textit{in } \underline{\mathbf{16}}$$

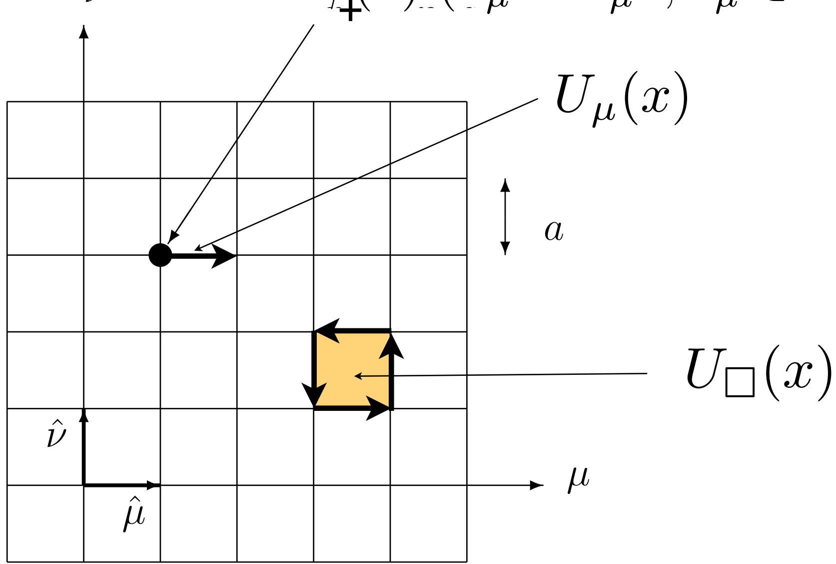
$$\int \prod_{x\in\Lambda}\prod_{\alpha=1}^{2}\prod_{s=1}^{16} d\psi_{\alpha s}(x) \prod_{x\in\Lambda} \frac{4!}{8!12!}\left\{\frac{1}{2}\,\psi(x)^{\mathrm{T}}P_+ i\gamma_5 C_D \mathrm{T}^a\psi(x) \quad \psi(x)^{\mathrm{T}}P_+ i\gamma_5 C_D \mathrm{T}^a\psi(x)\right\}^8 = 1$$

$$\int \prod_{x\in\Lambda}\prod_{\alpha=3}^{4}\prod_{s=1}^{16} d\bar{\psi}_{\alpha s}(x) \prod_{x\in\Lambda} \frac{4!}{8!12!}\left\{\frac{1}{2}\,\bar{\psi}(x)P_- i\gamma_5 C_D \mathrm{T}^a\bar{\psi}(x)^{\mathrm{T}}\,\bar{\psi}(x)P_- i\gamma_5 C_D \mathrm{T}^a\bar{\psi}(x)^{\mathrm{T}}\right\}^8 = 1$$

32-components at a site !

for the **naive Weyl fermion** with species doublers

$$\psi_+(x) \;(x_\mu = n_\mu a, n_\mu \in \mathbb{Z}) \quad \textit{in } \underline{\mathbf{16}}$$



$$\psi_+(x) = \hat{P}_+\psi(x) \quad \bar{\psi}_+(x) = \bar{\psi}(x)P_-$$

$$\hat{P}_\pm = \left(\frac{1\pm\hat{\gamma}_5}{2}\right), \quad P_\pm = \left(\frac{1\pm\gamma_5}{2}\right)$$

An explicit model which flows to IR fixed point

$$Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}E^a \, \mathrm{e}^{-S[\psi,\bar{\psi},E^a]}$$

$$S[\psi,\bar{\psi},E^a] = -\sum_x \frac{1}{2} \begin{pmatrix} \psi^T & \bar{\psi} \end{pmatrix} \mathcal{D}[E^a] \begin{pmatrix} \psi \\ \bar{\psi}^T \end{pmatrix} \qquad E^a(x)E^a(x) = 1$$

$$\mathcal{D}[E^a]^{-1} = \begin{pmatrix} -P_+ i\gamma_5 c_D T^{a\dagger} E^a(x) P_+^T & P_- D_c^{-1} P_+ + R \\ -\left(P_- D_c^{-1} P_+ + R\right)^T & -P_-^T i\gamma_5 c_D T^a E^a(x) P_- \end{pmatrix}$$

$$b_- = 2^{\frac{d-1}{2}}, \qquad b_+ < 2^{\frac{d+1}{2}} \quad (< 2^{\frac{3d}{4}}),$$

$$D_c{}^{-1}(x,y) = \int \frac{d^dk}{(2\pi)^4} \, \mathrm{e}^{ik(x-y)} \left\{ -i\gamma_\mu \alpha_\mu(k) \right\},$$

$$\alpha_\mu(k) = \sum_{l\in\mathbb{Z}^d} \prod_\nu \left(\frac{\sin(k_\nu/2)}{k_\nu/2 + \pi l_\nu} \right)^2 \frac{k_\mu + 2\pi l_\mu}{(k_\mu + 2\pi l_\mu)^2}$$

$$R = \frac{a}{\left(1 - \frac{b_- b_+}{2^d}\right)},$$

The saddle point method to O(n) vector model applicable
—> Phase structure, IR fixed point, Correlation functions

Gauge-invariant model —> use of Overlap Dirac operator

Eichten-Preskill-Wen / Mirror fermion model with overlap fermions

$$Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}E^a \, \mathrm{e}^{-S[\psi,\bar{\psi},E^a]} \qquad E^a(x)E^a(x) = 1$$

$$S[\psi,\bar{\psi},E^a] = -\sum_x \frac{1}{2} \begin{pmatrix} \psi^T & \bar{\psi} \end{pmatrix} \mathcal{D}[E^a] \begin{pmatrix} \psi \\ \bar{\psi}^T \end{pmatrix}$$

$$\mathcal{D}[E^a]^{-1} = \begin{pmatrix} -P_+ i\gamma_5 c_D T^{a\dagger} E^a(x) P_+^T & P_- D_c^{-1} P_+ + R \\ -(P_- D_c^{-1} P_+ + R)^T & -P_-^T i\gamma_5 c_D T^a E^a(x) P_- \end{pmatrix} \qquad D_{\mathrm{ov}}^{-1} = D_{\mathrm{c}}^{-1} + 1$$

$$\mathcal{D}[E] = \frac{\mathcal{D}_c[E]}{1 + \mathcal{R}\mathcal{D}_c[E]} \qquad \text{cf.} \quad D = \frac{D_c}{1 + R D_c}$$

$$S_{\mathrm{Ov}}[\psi,\bar{\psi},E^a,\bar{E}^a] = \sum_{x\in\Lambda} \bar{\psi}_-(x) D \psi_-(x) - \sum_{x\in\Lambda} \{ E^a(x) \psi_+^{\mathrm{T}}(x) i\gamma_5 C_D \mathrm{T}^a \psi_+(x) + \bar{E}^a(x) \bar{\psi}_+(x) i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar{\psi}_+(x)^T \}$$

Chiral determinant and 't Hooft vertex pfaffians

$$\frac{1}{2}\sum_{x\in\Lambda}\begin{pmatrix}\psi^T & \bar\psi\end{pmatrix}(x)\begin{pmatrix}-\hat P_+^T i\gamma_5 C_D \mathrm{T}^a E^a \hat P_+ & -\hat P_-^T D^T P_+^T \\ P_+ D \hat P_- & -P_- i\gamma_5 C_D \mathrm{T}^{a\dagger}\bar E^a P_-^T\end{pmatrix}\begin{pmatrix}\psi \\ \bar\psi^T\end{pmatrix}(x)$$

$$\psi_-(x) = \sum_j v_j(x) c_j, \quad \bar\psi_-(x) = \sum_k \bar c_k \bar v_k(x)$$

$$\psi_+(x) = \sum_j u_j(x) b_j, \quad \bar\psi_+(x) = \sum_k \bar b_k \bar u_k(x)$$

$$\mathrm{pf}\begin{pmatrix}-\hat P_+^T i\gamma_5 C_D \mathrm{T}^a E^a \hat P_+ & -\hat P_-^T D^T P_+^T \\ P_+ D \hat P_- & -P_- i\gamma_5 C_D \mathrm{T}^{a\dagger}\bar E^a P_-^T\end{pmatrix}$$

$$= \mathrm{pf}\begin{pmatrix}-(u^T i\gamma_5 C_D \mathrm{T}^a E^a u) & 0 & 0 & 0 \\ 0 & 0 & 0 & -(v^T D^T \bar v^T) \\ 0 & 0 & -(\bar u\, i\gamma_5 C_D \mathrm{T}^{a\dagger}\bar E^a \bar u^T) & 0 \\ 0 & (\bar v D v) & 0 & 0\end{pmatrix}$$

$$Z \equiv \int \mathcal{D}[U]\, \mathrm{e}^{-S_G[U]+\Gamma_W[U]}$$

$$\mathcal{D}[E] = \prod_{x\in\Lambda}(\pi^5/12)^{-1}\prod_{a=1}^{10} dE^a(x)\delta(\sqrt{E^b(x)E^b(x)}-1)$$

$$\mathcal{D}[\bar E] = \prod_{x\in\Lambda}(\pi^5/12)^{-1}\prod_{a=1}^{10} d\bar E^a(x)\delta(\sqrt{\bar E^b(x)\bar E^b(x)}-1)$$

$$\mathrm{e}^{\Gamma_W[U]} \equiv \int \mathcal{D}[\psi_-]\mathcal{D}[\bar\psi_-]\, \mathrm{e}^{-S_W[\psi_-,\bar\psi_-]}$$

$$= \int \mathcal{D}[\psi]\mathcal{D}[\bar\psi] \prod_{x\in\Lambda} F(T_+(x)) \prod_{x\in\Lambda} F(\bar T_+(x))\, \mathrm{e}^{-S_W[\psi_-,\bar\psi_-]}$$

$$= \int \mathcal{D}[\psi]\mathcal{D}[\bar\psi]\, \mathcal{D}[E]\mathcal{D}[\bar E]\; \mathrm{e}^{-S_W[\psi_-,\bar\psi_-]+\sum_{x\in\Lambda}\{E^a(x)V_+^a(x)+\bar E^a(x)\bar V_+^a(x)\}[\psi_+,\bar\psi_+]}$$

$$= \det(\bar v D v) \times \int \mathcal{D}[\bar E]\, \mathrm{pf}(u^T i\gamma_5 C_D T^a E^a u) \int \mathcal{D}[\bar E]\, \mathrm{pf}(\bar u\, i\gamma_5 C_D \mathrm{T}^{a\dagger}\bar E^a \bar u^T)$$

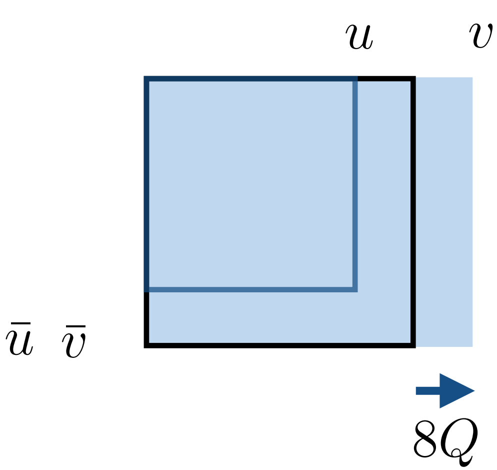
Chiral determinant and 't Hooft vertex pfaffians

$$e^{\Gamma_W[U]} = \det(\bar{v}Dv) \times \int \mathcal{D}[\bar{E}]\,\mathrm{pf}(u^T i\gamma_5 C_D T^a E^a u) \int \mathcal{D}[\bar{E}]\,\mathrm{pf}(\bar{u}\, i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar{E}^a \bar{u}^T)$$

$(\bar{v}Dv)_{ki}$ $\quad (k = 1, \cdots, n/2; i = 1, \cdots, n/2 + 8Q)$ **[variable, rectangular]**

$\left(u^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a E^a u\right)_{ij}$ $\quad (i, j = 1, \cdots, n/2 - 8Q)$ **[variable, square]**

$\left(\bar{u} i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar{E}^a \bar{u}^{\mathrm{T}}\right)_{kl}$ $\quad (k, l = 1, \cdots, n/2)$ **[fixed, square]**



—> Left-handed parts: correct behavior of `chiral determinant'

- overlap formula for the chiral determinant *Narayanan-Neuberger(1997)*
  - Zero modes
  - VEV of 't Hooft vertex >> Fermion # non-conservation
- gauge-invarinat formulation *Luscher (1999)*
  - measure term (local counter terms) $-i\mathfrak{L}_\eta = \sum_j (v_j, \delta_\eta v_j)$

—> Right-handed parts : non-vanishing in all topological sec. !?
"Saturation of the right-handed part of fermion measure"

$$\sum_j (u_j, \delta_\eta u_j) + \sum_j (v_j, \delta_\eta v_j) = 0$$

The saturation of the Right-handed measures due to 't Hooft vertices (the anti-fields)

the part of the anti-field: $\mathcal{D}_\star[\bar\psi_+]$

$$\int \mathcal{D}[\bar E]\,\mathrm{pf}\big(\bar u\, i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar E^a \bar u^{\mathrm{T}}\big) = 1$$ *for all topological sectors*

$$(\bar u i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar E^a \bar u^{\mathrm{T}})_{kl} = i\,\epsilon_{\sigma\sigma'}\delta_{xx'}\big(\mathrm{T}^{a\dagger}\mathrm{P}_+\big)_{tt'}\bar E^a(x')$$

$$\begin{aligned}\mathrm{pf}\big(\bar u i\gamma_5 C_D \mathrm{T}^{a} \bar E^a \bar u^{\mathrm{T}}\big) &= \prod_x \det\big(\mathrm{P}_- + \mathrm{P}_+ i\,\mathrm{T}^{a\dagger}\bar E^a(x)\big)\\ &= \prod_x \det\big(i\,\check{\mathrm{T}}^{a\dagger}\bar E^a(x)\big)\\ &= \prod_x \det\big(i\check C^\dagger[E^{10}(x) + i\check\Gamma^{a'}\bar E^{a'}(x)]\big)\\ &= 1.\end{aligned}$$

$$\begin{aligned}1 &= \int \mathcal{D}_\star[\bar\psi_+] F\big(\bar T_+(x)[\bar\psi_+]\big)\\ &= \int \prod_{x\in\Lambda}\prod_{\alpha=3}^{4}\prod_{s=1}^{16} d\bar\psi_{\alpha s}(x)\,\prod_{x\in\Lambda}\frac{4!}{8!12!}\left\{\frac{1}{2}\,\bar\psi(x)P_- i\gamma_5 C_D\mathrm{T}^a\bar\psi(x)^{\mathrm{T}}\,\bar\psi(x)P_- i\gamma_5 C_D\mathrm{T}^a\bar\psi(x)^{\mathrm{T}}\right\}^8\end{aligned}$$

cf. [*Eichten-Preskill(1986)*]

The saturation of the Right-handed measures due to 't Hooft vertices (the fields)

the part of the field: $\mathcal{D}_\star[\psi_+]$

$$\int \mathcal{D}[E]\,\mathrm{pf}\left(u^{\mathrm{T}}\, i\gamma_5 C_D \mathrm{T}^a E^a u\right) = c\,[U(x,\mu)] \neq 0$$

for all topological sectors

$$\left(u^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a E^a u\right)_{ij} \equiv \sum_{x\in\Lambda} u_i(x)^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a E^a(x) u_j(x)$$

$$(i, j = 1, \cdots, n/2 - 8Q)$$

- In general, the pfaffian of the matrix does not vanish identically
- We checked numerically that the pfaffian stays real, positive-semi definite for randomly generated SO(10) vector spin fields with several background gauge link fields
- These results suggest the non-vanishing pfaffian path-integral

More on the saturation of the Right-handed measures due to ’t Hooft vertices (cont’d)

$$U(x,\mu) = 1$$

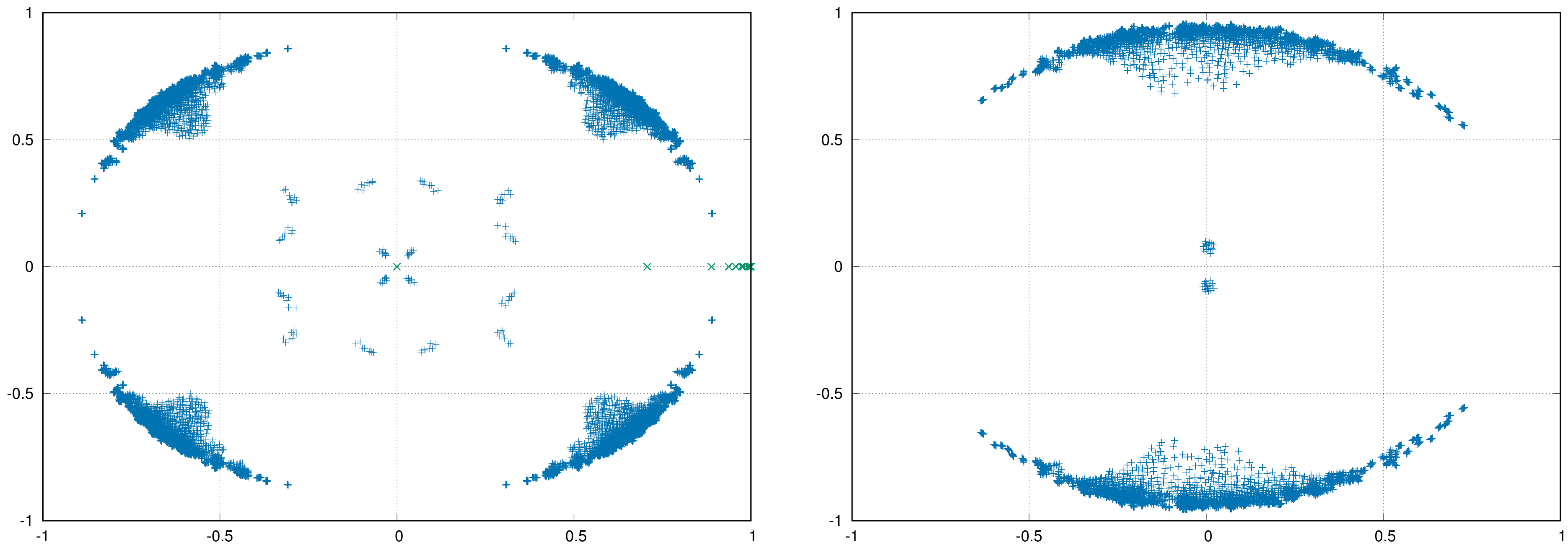


Figure 2. The eigenvalue spectra of the matrices $\left(u^{\mathrm{T}}\, i\gamma_5 C_D \mathrm{T}^a E^a u\right)$ and $\left(u^{\dagger}\Gamma^{10}\Gamma^a E^a u\right)$ with a randomly generated spin-field configuration for the case of the trivial link field. The lattice size is $L = 4$ and the boundary condition for the fermion field is periodic. For reference, the eigenvalue spectrum of the matrix $(\bar{v}_k D v_i)$ is also shown with green x symbol for the same boundary condition.

cf. $\{\lambda, \lambda, \lambda^*, \lambda^*\}$

More on the saturation of the Right-handed measures due to 't Hooft vertices (cont'd)

$$U(x,\mu) = \mathrm{e}^{i\theta_{12}(x,\mu)\Sigma^{12}}$$

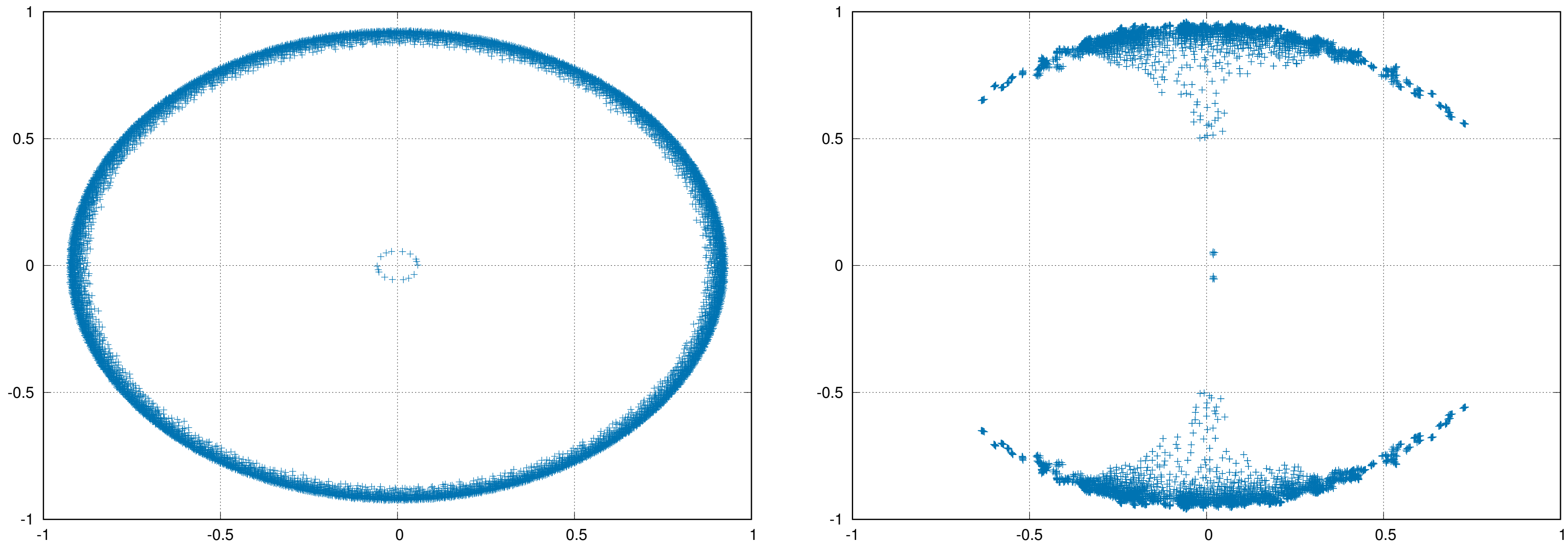


Figure 5. The eigenvalue spectra of the matrices $\left(u^{\mathrm{T}}\, i\gamma_5 C_D \mathrm{T}^a E^a u\right)$ and $\left(u^{\dagger}\Gamma^{10}\Gamma^a E^a u\right)$ with a randomly generated spin-field configuration for the case of the representative SU(2) link field of the topological sector with $Q = -2$. The lattice size is $L = 4$ and the boundary condition for the fermion field is periodic.

cf. $\{\lambda, \lambda, \lambda^*, \lambda^*\}$

the right-handed sector is a gapped system !   locality issue ?

- **Schwinger-Dyson eq. for link field:** local operator insertions!

$$\Big\langle \Big[-\delta_\eta S_G[U] - \sum_{x\in\Lambda} \bar\psi(x) P_+ \delta_\eta D \psi(x) + 2 \sum_{x\in\Lambda} \psi^{\rm T} \hat P_+^T i\gamma_5 C_D {\rm T}^a E^a \delta_\eta \hat P_+ \psi(x) \Big] \Big\rangle = 0$$

- **Fermion two-point correlation functions:** short-range in the right-handed sector!

$$\langle \psi_-(x)\, \bar\psi_-(y) \rangle_F = \hat P_- D^{-1} P_+(x,y) \langle 1 \rangle_F, \qquad \mathbf{16_- \quad 16_-^*}$$

$$\Big\langle \psi_+(y) \left[\psi_+^{\rm T} i\gamma_5 C_D {\rm T}^a E^a \hat P_+(x) \right] \Big\rangle_F = -\frac{1}{2} \hat P_+(y,x) \langle 1 \rangle_F, \qquad \mathbf{16_+}$$

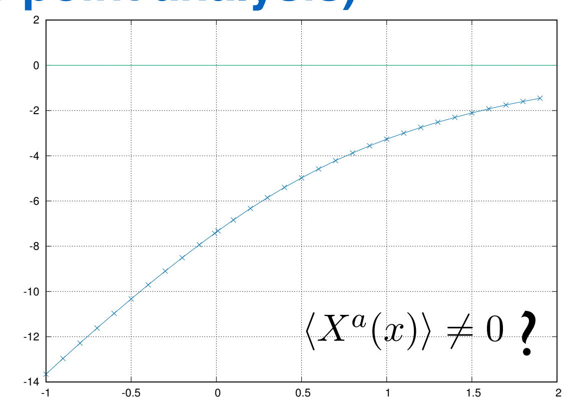
$$\Big\langle \left[P_- i\gamma_5 C_D {\rm T}^{a\dagger} \bar E^a \bar\psi_+^{\rm T}(x) \right] \bar\psi_+(y) \Big\rangle_F = -\frac{1}{2} P_- \delta_{xy} \langle 1 \rangle_F, \qquad \mathbf{16_+^*}$$

- **SO(10)-vector spin field dynamics:** disordered! (in a saddle point analysis)

$$\langle 1 \rangle_E = \int \mathcal{D}[E]\, {\rm pf}\big(u^{\rm T} i\gamma_5 C_D {\rm T}^a E^a u\big)$$

$$= \int \mathcal{D}[X]\mathcal{D}[\lambda]\, {\rm pf}\big(u^{\rm T} i\gamma_5 C_D {\rm T}^a X^a u\big)\, {\rm e}^{i\sum_x \lambda(x)(X^a(x)X^a(x)-1)}$$

$$f(m_0) \equiv 1 - \frac{9}{32}\frac{1}{V}\sum_{k\neq 0} \frac{4}{-\tilde D(k)+2} \quad \le\ 0 \ \ {\rm for}\ \ m_0 \ <\ 2$$



- **Link-field dependence of Effective action:** should be local in the right-handed sector

$$\delta_\eta \Gamma_W[U] = {\rm Tr}\{P_+ \delta_\eta D D^{-1}\} - i\mathfrak{T}_\eta$$

$$-i\mathfrak{T}_\eta \equiv -{\rm Tr}\big\{\delta_\eta \hat P_+ \langle \psi_+ [\psi_+^{\rm T} i\gamma_5 C_D {\rm T}^a E^a] \rangle_F \big\} / \langle 1 \rangle_F$$

cf. measure term
(local counter terms)

$$-i\mathfrak{L}_\eta = \sum_j (v_j, \delta_\eta v_j)$$

- 2dim. 21(-1)^3 chiral Schwinger model

$21(\text{-}1)^3$ chiral gauge model

U(1)$_A$ x SO(6)(=Spin(6)=SU(4))

$$Q = +\frac{1}{2} + \Sigma^{12} + \Sigma^{34} + \Sigma^{56}$$
$$Q' = -\frac{1}{2} + \Sigma^{12} + \Sigma^{34} + \Sigma^{56}$$

$$Q = \mathrm{diag}(q_1, q_2, q_3, q_4) = \mathrm{diag}(+2,\ \ 0,\ \ 0,\ \ 0),$$
$$Q' = \mathrm{diag}(q'_1, q'_2, q'_3, q'_4) = \mathrm{diag}(+1, -1, -1, -1).$$

$$S_W = \sum_{x\in\Gamma} \bar{\psi}(x) P_+ D \psi(x) + \sum_{x\in\Gamma} \bar{\psi}'(x) P_- D' \psi'(x)$$

$$S_M = \sum_{x\in\Gamma} \left\{ \psi_+(x)^T i\gamma_3 c_D \mathrm{T}^a E^a(x) \psi'_-(x) + \bar{\psi}_+(x) i\gamma_3 c_D \mathrm{T}^{a\dagger} \bar{E}^a(x) \bar{\psi}'_-(x)^T \right\}$$

$z/h \to 0$

$\kappa \to 0$

	+	+	−	−	(mixed) gauge anomaly	chiral anomaly
$\mathrm{U}(1)_g$	2	0	1	-1	matched (gauged)	—
SU(3)	$\underline{1}$	$\underline{3}$	$\underline{1}$	$\underline{3}$	matched (can be gauged)	anomaly free
$\mathrm{U}(1)_b$	0	1	0	1	not matched	anomalous
$\mathrm{U}(1)_a$	0	1	0	-1	not matched	anomalous
$\mathrm{U}(1)_{b-3l}$	-3	1	-3	1	matched (can be gauged)	anomaly free

Table 4. Fermionic continuous symmetries in the mirror sector of the $21(\text{-}1)^3$ model and their would-be gauge anomalies

- the relation to ν=8 1D Majorana chain with SO(7)-invariant int.

$$\hat{H} = \sum_{\alpha=1}^{8} \hat{H}_\alpha + \hat{V}$$

[*Fidkowski-Kitaev 2010)*]

$$\hat{H}_\alpha = \frac{i}{2}\left(u \sum_{l=1}^{n} \hat{c}^\alpha_{2l-1}\hat{c}^\alpha_{2l} + v \sum_{l=1}^{n-1} \hat{c}^\alpha_{2l}\hat{c}^\alpha_{2l+1} \right)$$

$$\hat{V} = \sum_{l=1}^{n} \left(\hat{W}_{2l-1} + \hat{W}_{2l} \right) \qquad \hat{W}_m = \hat{W}\big|_{\hat{c}^\alpha \to \hat{c}^\alpha_m}$$

SO(7)-invariant int.

$$\begin{aligned}\hat{W} = \; & \hat{c}^1\hat{c}^2\hat{c}^3\hat{c}^4 + \hat{c}^5\hat{c}^6\hat{c}^7\hat{c}^8 + \hat{c}^1\hat{c}^2\hat{c}^5\hat{c}^6 + \hat{c}^3\hat{c}^4\hat{c}^7\hat{c}^8 - \hat{c}^2\hat{c}^3\hat{c}^6\hat{c}^7 \\ & -\hat{c}^1\hat{c}^4\hat{c}^5\hat{c}^8 + \hat{c}^1\hat{c}^3\hat{c}^5\hat{c}^7 + \hat{c}^3\hat{c}^4\hat{c}^5\hat{c}^6 + \hat{c}^1\hat{c}^2\hat{c}^7\hat{c}^8 - \hat{c}^2\hat{c}^3\hat{c}^5\hat{c}^8 \\ & -\hat{c}^1\hat{c}^4\hat{c}^6\hat{c}^7 + \hat{c}^2\hat{c}^4\hat{c}^6\hat{c}^8 - \hat{c}^1\hat{c}^3\hat{c}^6\hat{c}^8 - \hat{c}^2\hat{c}^4\hat{c}^5\hat{c}^7,\end{aligned}$$

$$\hat{W} = -\frac{1}{4!}\Big(\sum_{a=1}^{7} \hat{c}^T\gamma^a\hat{c} \; \hat{c}^T\gamma^a\hat{c} - 16 \Big)$$

[*Y.-Z. You & C. Xu 2015)*]

$$\hat{\psi}^\alpha_l = (\hat{c}^\alpha_{2l}, \hat{c}^\alpha_{2l-1})^T \qquad \alpha = -\sigma_1,\, \beta = \sigma_2 \qquad \tilde{P}_\pm = (1 \mp i\beta\alpha)/2 = (1 \pm \sigma_3)/2$$

$$\hat{H}_\alpha = \sum_{l=1}^{n} \frac{z}{2}\, \hat{\psi}^{\alpha T}_l \left\{ \alpha \frac{1}{2i}(\nabla - \nabla^\dagger) + \beta \frac{1}{2}\nabla\nabla^\dagger + \beta m_0 \right\} \hat{\psi}^\alpha_l$$

Majorana-Wilson fermion in Hamiltonian formalism

$$\hat{V} = -\frac{1}{4!}\sum_{l=1}^{n} \Big\{ \sum_{a=1}^{7} \big(\hat{\psi}^T_l \tilde{P}_+ \gamma^a \hat{\psi}_l\big)^2 + \sum_{a=1}^{7} \big(\hat{\psi}^T_l \tilde{P}_- \gamma^a \hat{\psi}_l\big)^2 - 32 \Big\}$$

Wilson-Dirac fermion / overlap Dirac fermion $D = D_{\mathrm{w,ov}}$

$$S'_{8MC/SO(6)} = \sum_x \Big\{ z\,\bar\psi(x)\big(D+m_0\big)\psi(x) - \frac{1}{12}\sum_{a=1}^{6}\big(\psi_+(x) i\gamma_3 c_D \check{\mathrm{T}}^a \psi_-(x) - \bar\psi_+(x) i\gamma_3 c_D \check{\mathrm{T}}^{a\dagger} \bar\psi_-(x)^T\big)^2 \Big\}$$

the mirror sector of our model

<=>

ν=8 1D Majorana chain with SO(6)-invariant int.

in Euclidean (covariant) Path-integral formalism

Gapped by SO(7)-invariant int.

Refinement of free fermion classification of TI/TCI due to interactions ($Z \to Z_8, Z_{16}$)

[*Fidkowski-Kitaev 2010)*]

Also true for reduced SO(6)-invariant int.

[*Y.-Z.You & C. Xu 2015)*]

$$
\begin{aligned}
\frac{1}{L^2}\sum_k \tilde{\eta}_\mu(-k)\,\tilde{\Pi}_{\mu\nu}(k)\,\tilde{\zeta}_\nu(k) &= \delta_\zeta\left[\mathrm{Tr}\{P_+\delta_\eta D D^{-1}\} + \mathrm{Tr}\{P_-\delta_\eta D' D'^{-1}\}\right]\Big|_{U(x,\mu)\to 1} \\
&= \left[\mathrm{Tr}\{\delta_\zeta\delta_\eta D D^{-1}\} - \mathrm{Tr}\{\delta_\eta D D^{-1}\delta_\zeta D D^{-1}\}\right]\Big|_{U(x,\mu)\to 1}
\end{aligned}
$$

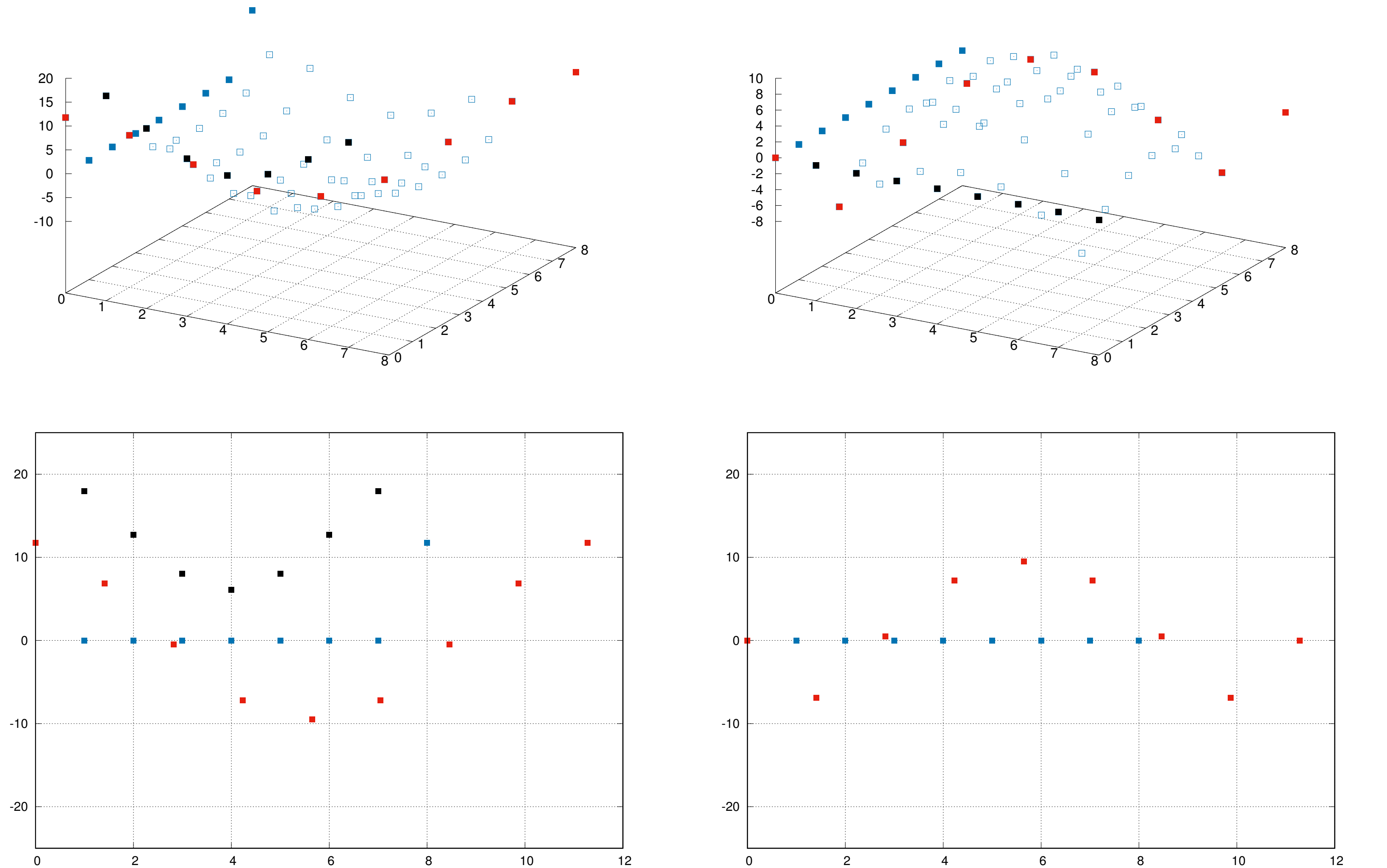


Figure 12. $(1/4)L^2\,\tilde{\Pi}_{00}(k)$ [left] and $(1/4)L^2\,\tilde{\Pi}_{01}(k)$ [right] vs. $|k|_2 \equiv \sqrt{k_0^2+k_1^2}$. The lattice size is $L=8$. The anti-periodic boundary condition is assumed for the fermion fields. The black-, blue-, red-symbol plots are along the spacial momentum axis ($k_0=0$), the temporal momentum axis ($k_1=0$) and the diagonal momentum axis ($k_0=k_1$), respectively.

$$\frac{1}{L^2}\sum_k \tilde{\eta}_\mu(-k)\,\tilde{\Pi}'_{\mu\nu}(k)\,\tilde{\zeta}_\nu(k) = \delta_\zeta\left[\langle -\delta_\eta S_M\rangle_M/\langle 1\rangle_M\right]\Big|_{U(x,\mu)\to 1}$$

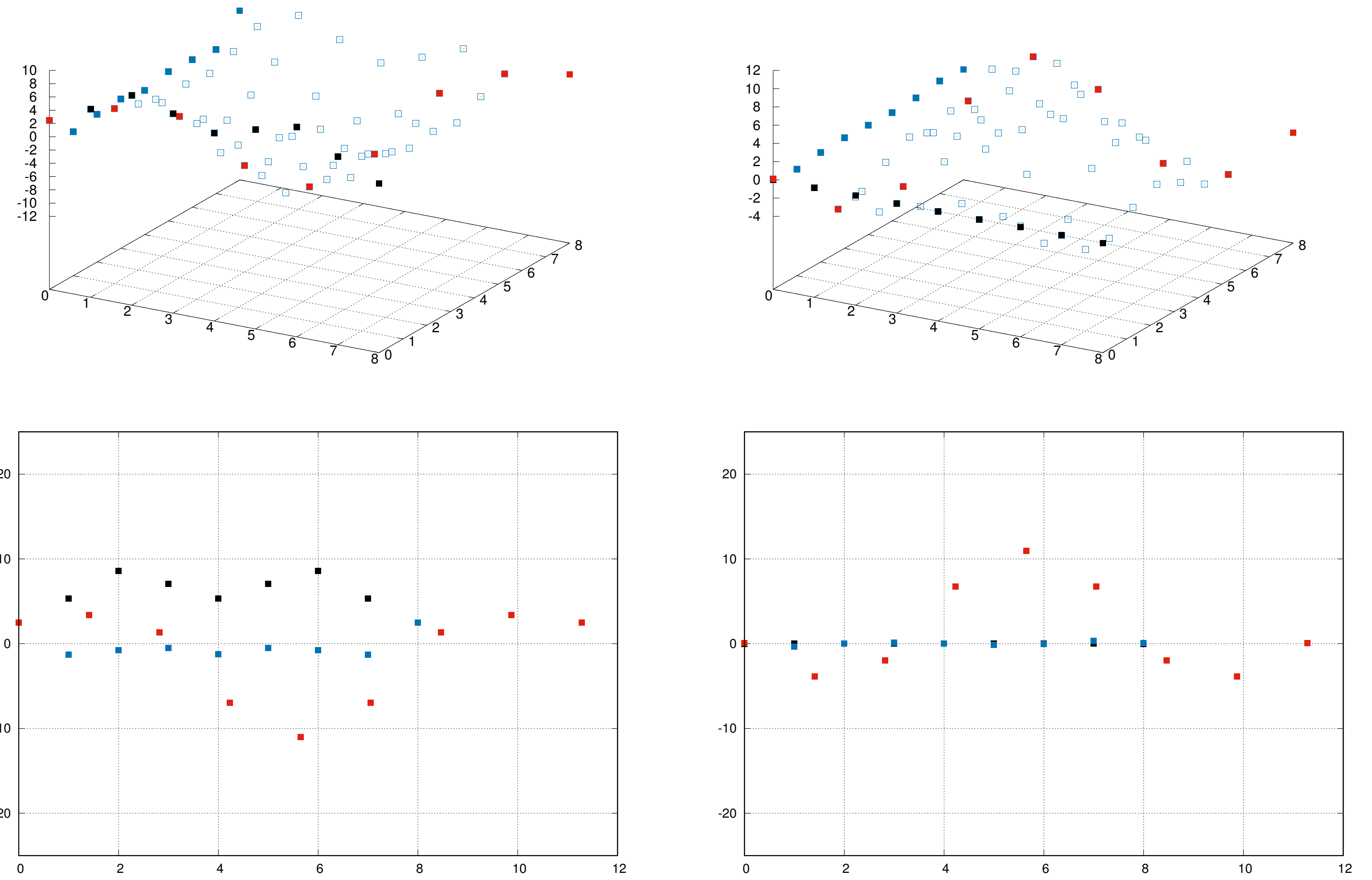


Figure 15. $L^2\,\tilde{\Pi}^{\hat{Q}\hat{Q}}_{00}(k)$ [left] and $L^2\,\tilde{\Pi}^{\hat{Q}\hat{Q}}_{01}(k)$ [right] vs. $|k|_2 \equiv \sqrt{k_0^2+k_1^2}$. The lattice size is $L=8$. The periodic boundary condition is assumed for the fermion fields. The black-, blue-, red-symbol plots are along the spacial momentum axis ($k_0=0$), the temporal momentum (energy) axis ($k_1=0$) and the diagonal momentum axis ($k_0=k_1$), respectively. 5,000 configurations are sampled with the interval of 20 trajectories. The errors are simple statistical ones.

Two approaches to non-abelian chiral gauge theory on the lattice with exact gauge invariance

- Overlap Weyl fermion / Neuberger's overlap Dirac operator / Ginsparg-Wilson rel.
- SO(10), SU(5) x$U(1)_Q$, SU(3)xSU(2)xU(1) x$U(1)_{B-L}$,…

  no local and global gauge anomalies

  $\Sigma_R Tr_R[T^a \{T^b,T^c\}]=0$, $\Omega^{spin}{}_5(BSO(10)) = 0$
- establish Luscher's integrability for chiral determinant of overlap Weyl fermions
  - Integrability condition with 5-, and 6-dimensional lattice Domain-wall fermions

    cf. bordism inv. / η-invariant, Dai-Freed theorem and APS index theorem
- decouple the mirror d.o.f. of overlap Dirac fermions

  by Eichten-Preskill-Wen / Kitaev-Fidkowski quartic interaction terms
  - 't Hooft vertex-type SO(10) / SO(7), SO(6) invariant quartic terms,

    which break explicitly the global symmetries with 't Hooft anomaly

    (16 x 16 x 16 x 16) = (10 x 10 x 10 x 5*) $_0$ **+ (10 x 5* x 5*)** $_{-5}$ **x 1** $_5$

    $$\lambda \left[\psi^T(x) i\gamma_5 c_d \mathrm{C}\Gamma^a \psi(x)\right]^2 \Leftrightarrow y \left[\psi^T(x) i\gamma_5 c_d \mathrm{C}\Gamma^a \psi(x)\right] E^a(x) \qquad E^a(x)E^a(x) = 1$$

    $$\pi_d(S^9) = 0 \quad (d = 0, \cdots, 9)$$

    cf. gapped boundary of interaction-reduced trivial phase of SPT

- Hamiltonian formalism
  - 4+1 DWF, 2+1 DWF TI/TSC model

    2dim. 3450 chiral Schwinger model [*Zeng, Zhu, Wang, You (2022)*]
  - Overlap Dirac fermion

    1+1 dim, overlap = Wilson, Exact chiral symmetry
    fine tuning problem

"Eventually, all things merge into one and a river runs through it"

What is the sound of one hand clapping?

隻手音声

両手の鳴る音は知る。
片手の鳴る音はいかに？
ー 禅の公案 ー

anomaly inflow / Callan-Harvey & Dai-Freed theorem on the lattice

[Aoyama-*YK (1999)*] [*YK (2002)*][Pedersen-*YK (2019)*]

$$\lim_{N\to\infty} \frac{\det X_{\rm w}^{(5)}|^{c}_{\rm Dir.}}{\sqrt{\det X_{\rm w}^{(5)}|^{c\cdot c^{-1}}_{\rm AP}}} = \det(\bar v D'_{\rm ov} v^1)\det(\bar v D'_{\rm ov} v^0)^* \frac{\det(v^{1\dagger}\prod_{t\in\tilde c} T_t v^0)}{\left|\det(v^{1\dagger}\prod_{t\in\tilde c} T_t v^0)\right|}$$

boundary contribution
(overlap Weyl fermions)

$$e^{\Gamma(\mathbb{X}_1\cup\bar{\mathbb{X}}_0)} = \det(\bar v D'_{\rm ov} v^1)\det(\bar v D'_{\rm ov} v^0)^*$$

bulk contribution
($DW|_{\rm Dir}$)

$$e^{-i2\pi\hat\eta_{\rm DF}(\mathbb{Y}|^{c}_{\rm Dir})} = \frac{\det(v^{0\dagger}\prod_{t\in\tilde c} T_t v^1)}{\left|\det(v^{0\dagger}\prod_{t\in\tilde c} T_t v^1)\right|}$$

- **same basis-vector dependence**

=> same U(1) bundle over $\mathfrak{U} = \left\{\{U(x,\mu)\}\,\middle|\, \| 1-U_{\mu\nu}(x)\| < \epsilon \ {}^\forall(x,\mu,\nu)\right\}$

- **connection of the U(1) bundle**

cf. [*Dai-Freed (1994)*]

[*Yonekura (2016)*]

$$\nabla_\eta\, e^{-i2\pi\hat\eta_{\rm DF}(\mathbb{Y}|_{\rm Dir})} = \left(\int_{\mathbb{Y}|^{c_1}_{\rm Dir}} q^{(5+1)}\right) e^{-i2\pi\hat\eta_{\rm DF}(\mathbb{Y}|_{\rm Dir})}$$

$$\nabla_\eta \equiv \delta_\eta + i\mathcal{L}_\eta$$

$$-i\mathcal{L}_\eta = \sum_i v_i(x)^\dagger \delta_\eta v_i(x)$$

$$\int_{\mathbb{Y}|^{c_1}_{\rm Dir}} q^{(5+1)} \equiv \sum_{x,t\in c_1} q^{(5+1)}(y,s)\big|_{s=1}$$

(integral of topological field on 5+1, 6 dim. lattices)

$$q^{(5+1)}(y,s) \equiv {\rm Im\,Tr}\{D_s(U_s)\,U_s^{-1}(y,\mu)\,J_{\rm w}^{(5)}(y,\mu)|_{U=U_s}\}$$

$$q^{(5+1)}(y,s)\big|_{s=1} = \lim_{a_6\to 0}\lim_{N_6\to\infty} q^{(6)}(z)\big|_{\mathbb{Y}^{c_1 c_2^{-1}}_{\rm AP}}$$

η-invariant on the lattice [Aoyama-*YK (1999)*]

$$e^{i\pi\hat{\eta}(\mathbb{Y}|_{\rm bc})} \equiv \lim_{N\to\infty} \frac{\det D_{\rm ov}^{(5)}|_{\rm bc}}{\left|\det D_{\rm ov}^{(5)}|_{\rm bc}\right|}$$

$$\mathrm{bc} = (\mathrm{Dir},\ \mathrm{AP},\ \mathrm{P})$$

$$i\not{D}^{(5)}\psi_\lambda = \lambda\psi_\lambda$$

$$\det i\not{D}^{(5)}|_{\rm reg.} = \prod_\lambda \frac{\lambda}{\lambda + iM}$$

$$= \left|\det i\not{D}^{(5)}|_{\rm reg.}\right| e^{-i\pi\eta_Y^{(5)}}$$

$$\eta_Y^{(5)} \equiv \frac{1}{2}\Big(\sum_{\lambda\neq 0}\mathrm{sign}(\lambda) + \ \dim\ \ker\ (i\not{D}^{(5)})\Big)$$

$$e^{i2\pi\hat{\eta}(\mathbb{Y}|_{\rm bc})} \equiv \lim_{N\to\infty}\left[\frac{\det D_{\rm ov}^{(5)}|_{\rm bc}}{\left|\det D_{\rm ov}^{(5)}|_{\rm bc}\right|}\right]^2 \equiv \lim_{N\to\infty}\frac{\det X_{\rm w}^{(5)}|_{\rm bc}}{\left|\det X_{\rm w}^{(5)}|_{\rm bc}\right|}.$$

$$e^{i2\pi\hat{\eta}_{\rm DF}(\mathbb{Y}|_{\rm Dir})} \equiv \frac{\det(v^{1\dagger}\prod_{t\in\tilde{c}}T_t v^0)}{\left|\det(v^{1\dagger}\prod_{t\in\tilde{c}}T_t v^0)\right|}$$

gluing property

$$\mathbb{Y}|_{\rm Dir}^{c_1}\cup\bar{\mathbb{Y}}|_{\rm Dir}^{c_2} = \mathbb{Y}|_{\rm AP}^{c_1c_2^{-1}}$$

$$e^{i2\pi\hat{\eta}_{\rm DF}(\mathbb{Y}|_{\rm Dir}^{c_1})}\ e^{-i2\pi\hat{\eta}_{\rm DF}(\mathbb{Y}|_{\rm Dir}^{c_2})}$$

$$= e^{i2\pi\hat{\eta}(\mathbb{Y}|_{\rm Dir}^{c_1})}\ e^{-i2\pi\hat{\eta}(\mathbb{Y}|_{\rm Dir}^{c_2})}$$

$$= e^{i2\pi\hat{\eta}(\mathbb{Y}|_{\rm AP}^{c_1c_2^{-1}})} \qquad (N\to\infty)$$

APS index on the lattice

[Fukaya, Kawai et al *(2019)*]

$$I(\mathbb{Z}|_{\rm Dir}) \equiv -\frac{1}{2}{\rm Tr}\left\{ \left.\frac{H_{\rm w}^{(6)}}{\sqrt{{H_{\rm w}^{(6)}}^2}}\right|_{\rm Dir} \right\}$$

cf. [Fukaya, Onogi, Yamaguchi *(2017)*]

cf. [Fukaya, Furuta et al *(2019)*]

$$(-1)^{I(\mathbb{Z}|_{\rm Dir})} = \frac{\det X_{\rm w}^{(6)}|_{\rm Dir}}{\left|\det X_{\rm w}^{(6)}|_{\rm Dir}\right|}$$

cf. [Witten *(2016)*]

$$I(\mathbb{Z}|_{\rm Dir}) = {\rm Index}\, D_{\rm ov}^{(6)}|_{\rm Dir}$$

cf. AS index theorem on the lattice

$$H_{\rm w}^{(6)}|_{\rm Dir} = \gamma_7 X_{\rm w}^{(6)}|_{\rm Dir}$$

Gap closed in the limit N_6 —> ∞

$$\frac{H_{\rm w}^{(6)}}{\sqrt{{H_{\rm w}^{(6)}}^2}}(x,x) \qquad D_{\rm ov}^{(6)}$$

Non-local

$${\rm Index}\, D_{\rm ov}^{(6)}|_{\rm Dir}$$

Index well-defined by exact chiral symmetry/GW rel.

Dir. b.c. is physically well-motivated, well-defined and local, Dirac operator is non-local,
but chiral symmetry/GW rel. is preserved and the index can be defined

APS b.c. is non-local (with 5dim. massless Dirac operator),
chiral symmetry is preserved and the index can be defined

In both cases, APS index is non-local

Exact APS index theorem on the lattice

[Pedersen-*YK* *(2019)*]

cf. [Aoyama-*YK* *(1999)*]

$$(-1)^{I(\mathbb{Z}|_{\mathrm{Dir}})} = \quad \mathrm{e}^{i\pi P(\mathbb{Z}|^{c}_{\mathrm{APS}})} \quad \mathrm{e}^{i\pi\left[\hat{\eta}'(\mathbb{Y}_1|_{\mathrm{AP}}) - \hat{\eta}'(\mathbb{Y}_0|_{\mathrm{AP}})\right]}$$

$$\mathrm{Index}\, D^{(6)}_{\mathrm{ov}}|_{\mathrm{Dir}} = P(\mathbb{Z}|^{c}_{\mathrm{APS}}) + \left[\hat{\eta}'(\mathbb{Y}_1|_{\mathrm{AP}}) - \hat{\eta}'(\mathbb{Y}_0|_{\mathrm{AP}})\right]$$

$$I(\mathbb{Z}|_{\mathrm{Dir}}) \equiv -\frac{1}{2}\mathrm{Tr}\left\{ \left.\frac{H^{(6)}_{\mathrm{w}}}{\sqrt{{H^{(6)}_{\mathrm{w}}}^2}}\right|_{\mathrm{Dir}} \right\}$$

$$P(\mathbb{Z}|^{c_1}_{\mathrm{APS}}) = \lim_{N\to\infty} \sum_{y,s\in c} q^{(6)}(z) \qquad q^{(6)}(z) \equiv -\frac{1}{2}\mathrm{tr}\left\{ \left.\frac{H^{(6)}_{\mathrm{w}}}{\sqrt{{H^{(6)}_{\mathrm{w}}}^2}}\right|_{\mathrm{AP}} \right\}(z,z)$$

(local, topological field in 6-dim. lattice AS index theorem)