格子フェルミオン再考

-グラフ理論と位相不変量の立場から-

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1. Reflection on Naive & Wilson fermions

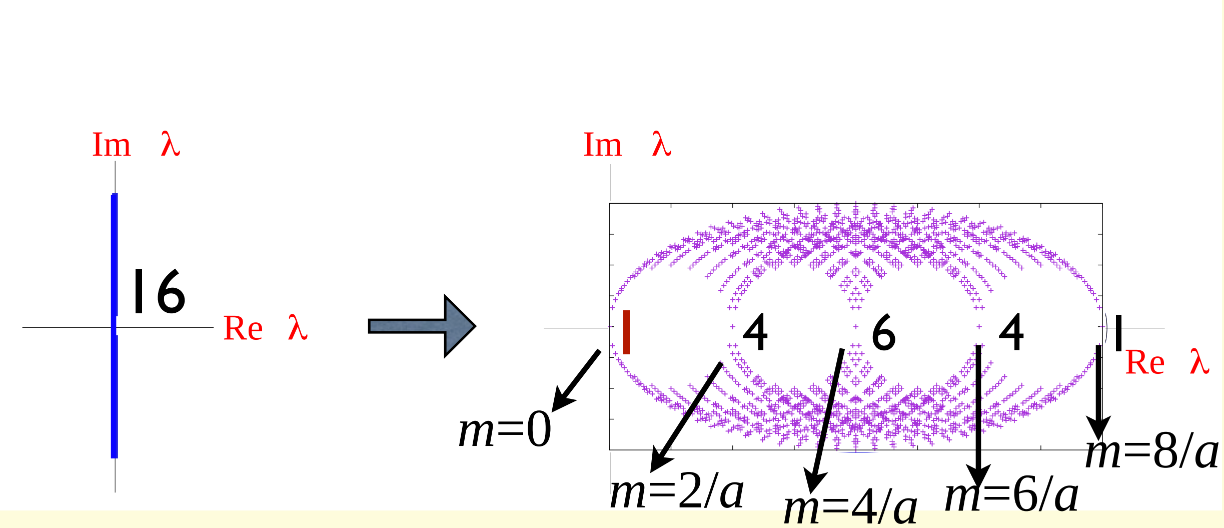
Wilson fermion : species-splitting mass fermion

Lattice fermion action with species-splitting term $\sum_{n,\mu} \frac{a^5}{2}\bar{\psi}_n(2\psi_n - \psi_{n+\mu} - \psi_{n-\mu})$

$$\Rightarrow \quad D_W(p) = \frac{1}{a}\sum_{\mu}[i\gamma_\mu \sin ap_\mu \, + \, \underline{(1-\cos ap_\mu)}]$$

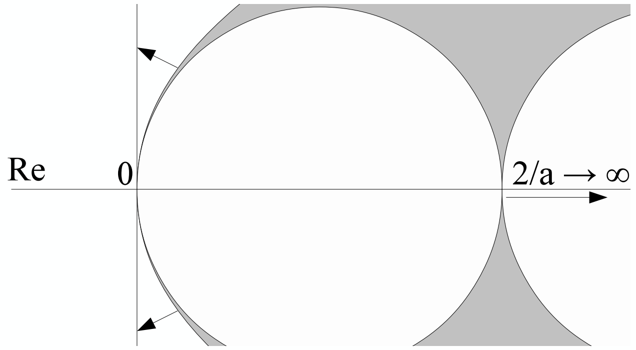
Physical (0,0,0,0) : $D_W(p) = i\gamma_\mu p_\mu \, + \, O(a)$

Doubler(π/a,0,0,0) : $D_W(p) = i\gamma_\mu p_\mu \, + \, \frac{2}{a} \, + \, O(a)$



Only one flavor is massless, while others have $O(1/a)$ mass.

- 15 species are decoupled → doubler-less
- $1/a$ additive mass renormalization → Fine-tune
- Domain-wall & Overlap fermions → costs



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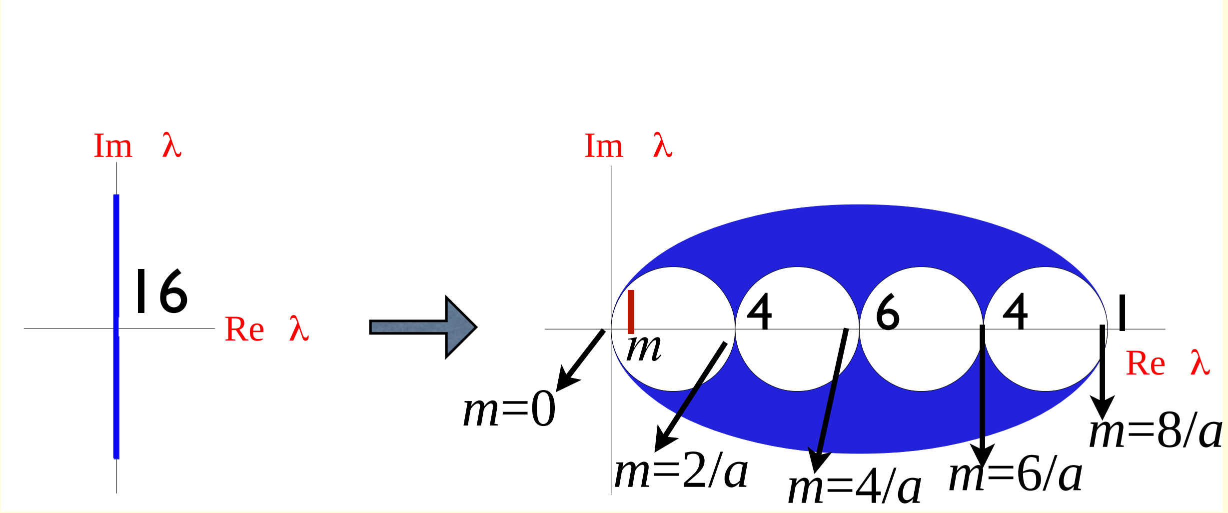

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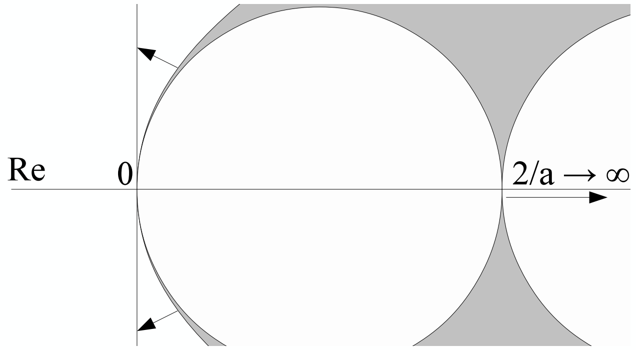
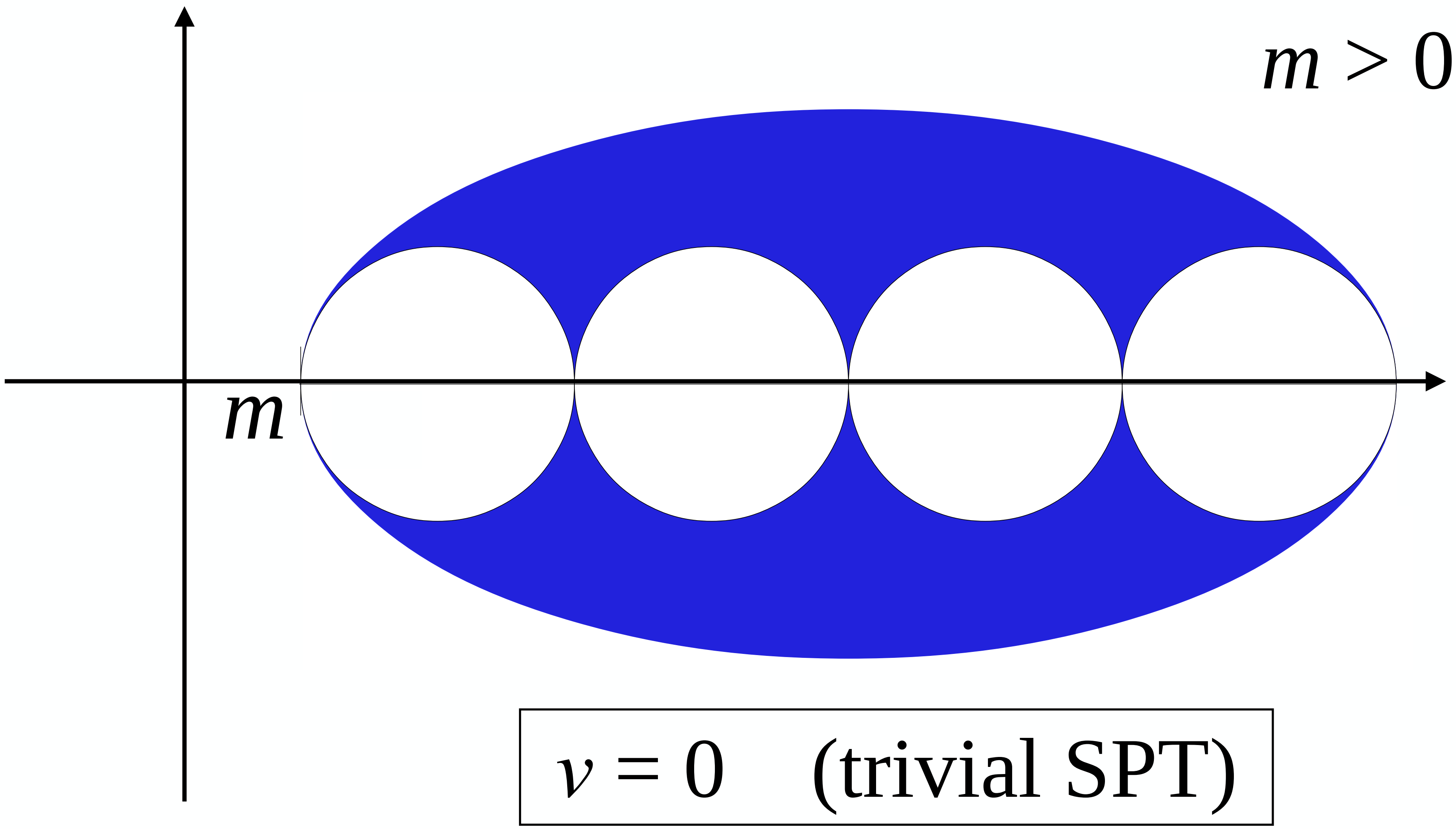


Wilson fermion as U(1) SPT phases

Topological # of SPT ~ index of modes with negative mass

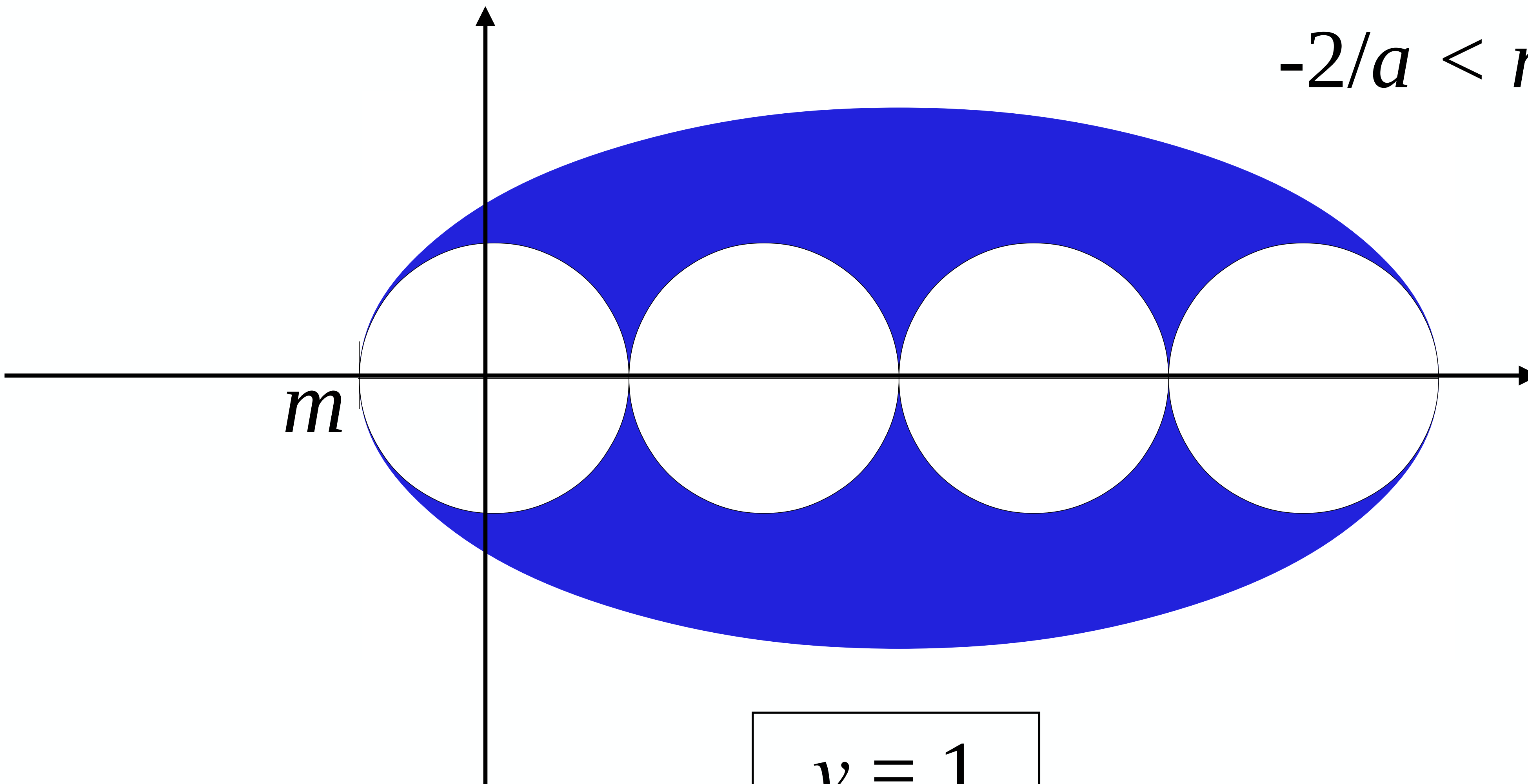


These indices reflect topology of Berry connection for free fermion, while gauge field topology plays the same role in gauged theory.

Wilson fermion as U(1) SPT phases

Topological # of SPT ~ index of modes with negative mass

$-2/a < m < 0$

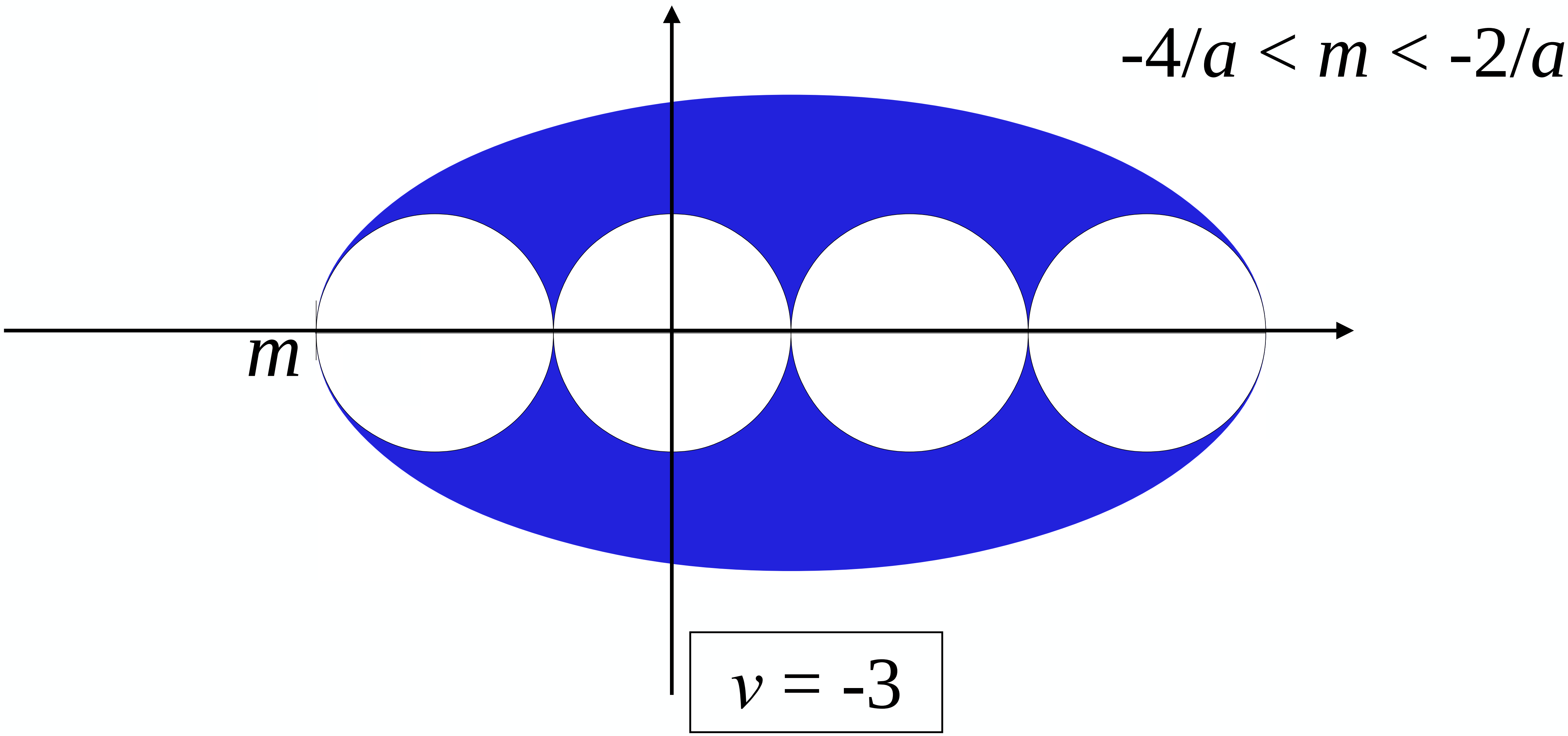


$\nu = 1$

Domain-wall fermion : gapless mode emerging at boundary between ν=0 and ν=1 SPTs, where 't Hooft anomaly cancels.

Wilson fermion as U(1) SPT phases

Topological # of SPT ~ index of modes with negative mass



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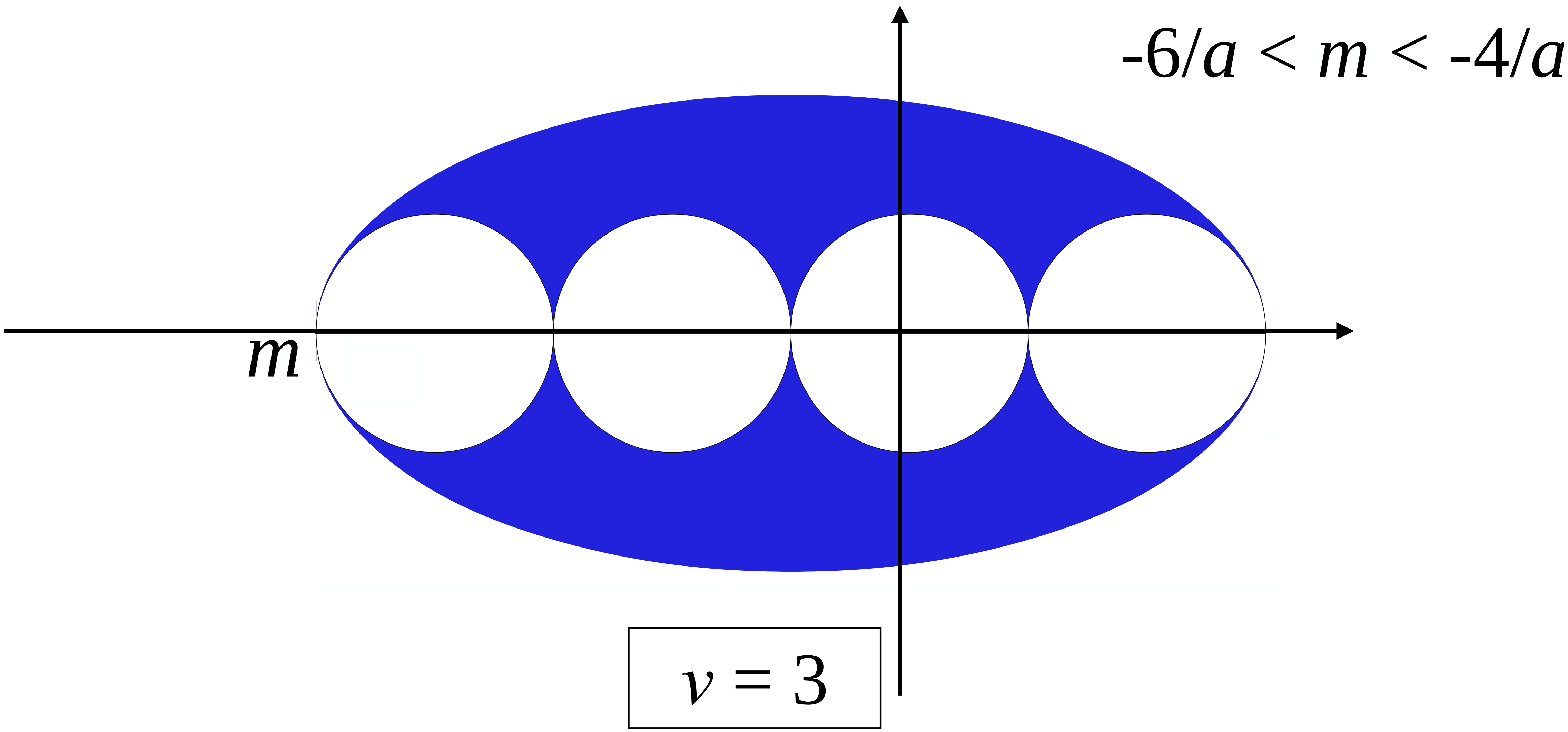

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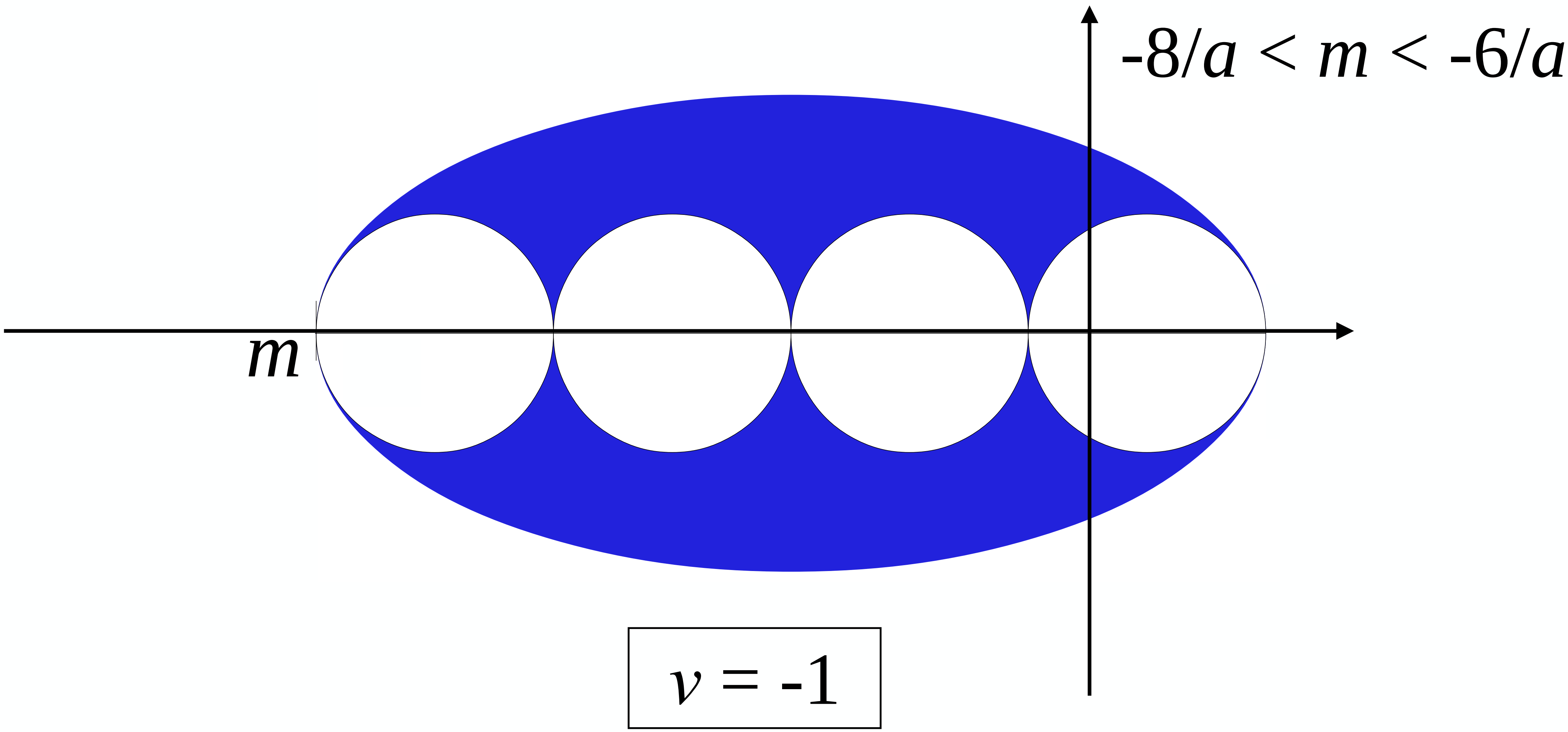

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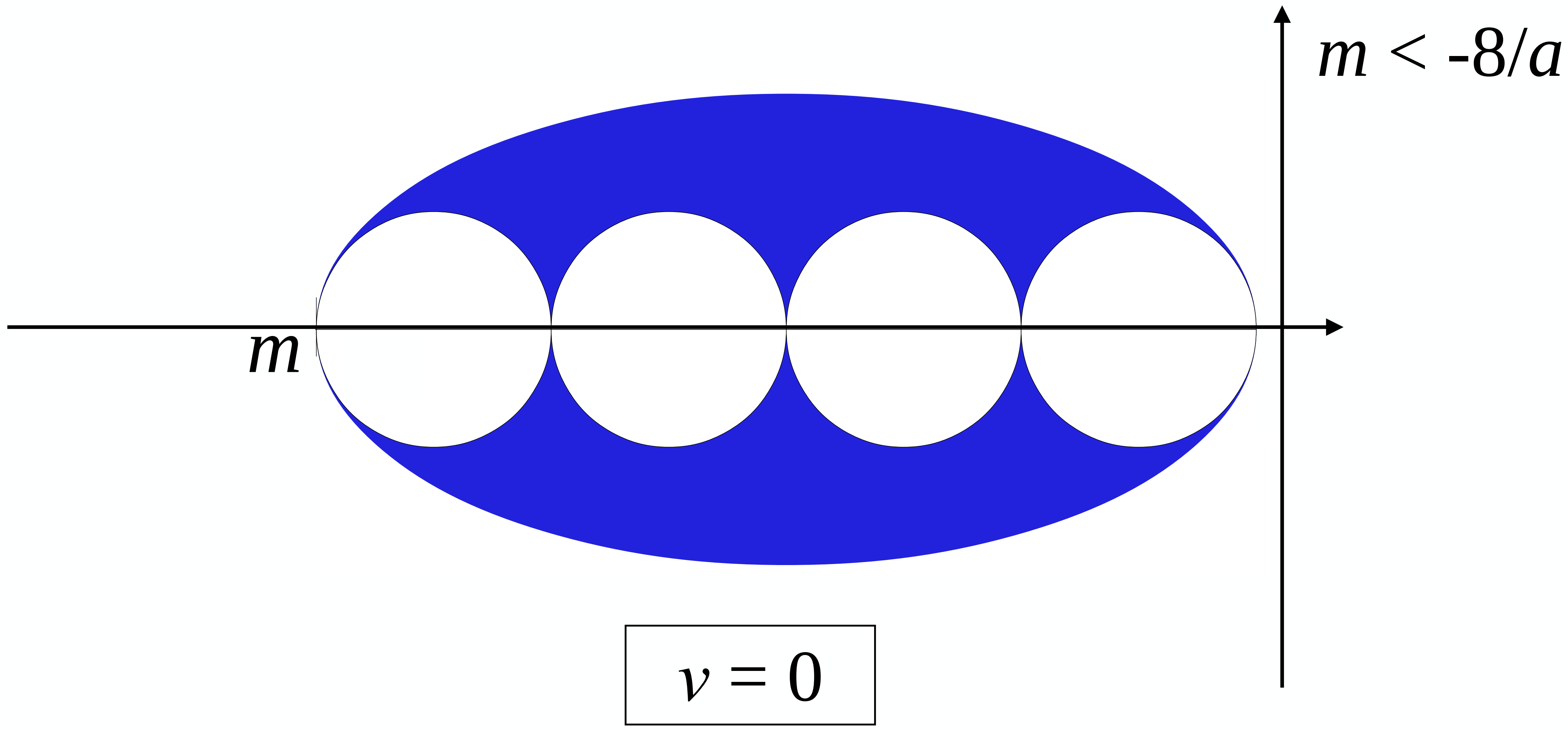
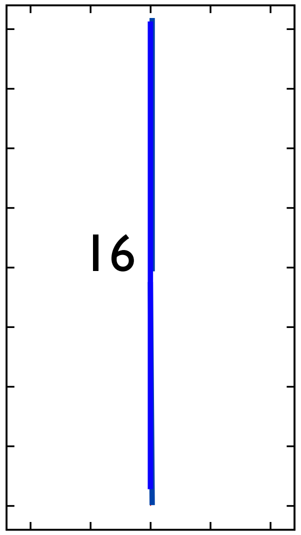
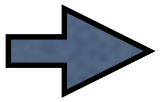


Flavor-chiral symmetries of Wilson fermion

Naive



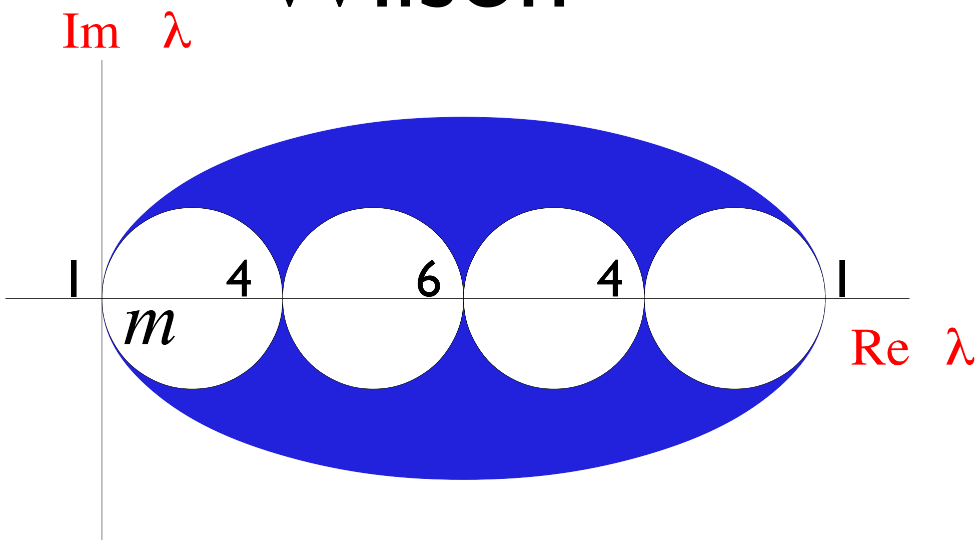
$$\sum_\mu C_\mu$$



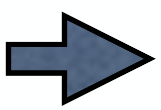
$C_\mu = (T_{+\mu} + T_{-\mu})/2$

$T_{\pm\mu}\psi_n = U_{n,\pm\mu}\psi_{n\pm\mu}$

Wilson



U(4)×U(4)



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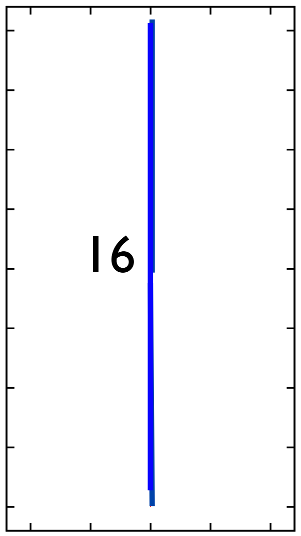

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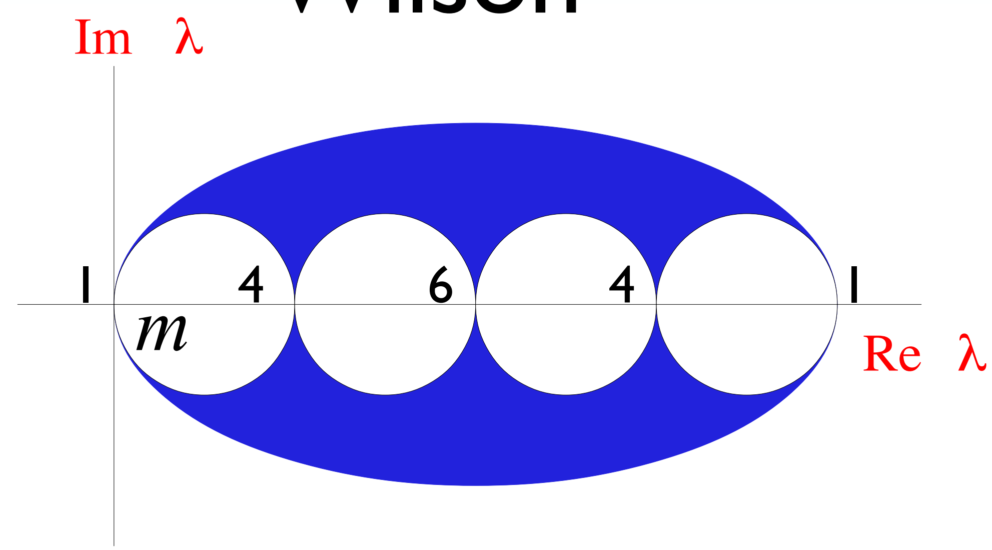
Wilson



U(4)×U(4) →

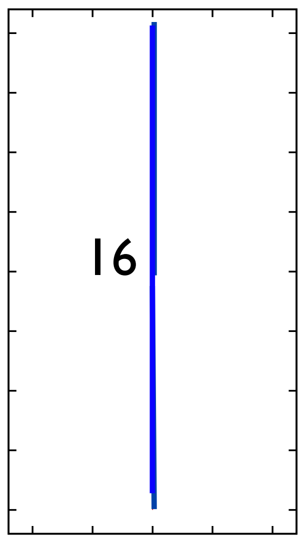
$$\Gamma_X^{(+)} \in \left\{ \mathbf{1}_4\,,\ (-1)^{n_1+\ldots+n_4}\gamma_5\,,\ (-1)^{\check{n}_\mu}\gamma_\mu\,,\ (-1)^{n_\mu} i\gamma_\mu\gamma_5\,,\ (-1)^{n_{\mu,\nu}}\frac{[\gamma_\mu\,,\gamma_\nu]}{2} \right\}$$

$$\Gamma_X^{(-)} \in \left\{ (-1)^{n_1+\ldots+n_4}\mathbf{1}_4\,,\ \gamma_5\,,\ (-1)^{n_\mu}\gamma_\mu\,,\ (-1)^{\check{n}_\mu}\gamma_\mu\gamma_5\,,\ (-1)^{\check{n}_{\mu,\nu}}\frac{[\gamma_\mu\,,\gamma_\nu]}{2} \right\}$$

$$\psi_n \to \psi'_n = \exp\left[\mathrm{i}\sum_X \left(\theta_X^{(+)}\Gamma_X^{(+)} + \theta_X^{(-)}\Gamma_X^{(-)}\right)\right]\psi_n\,, \quad \bar{\psi}_n \to \bar{\psi}'_n = \bar{\psi}_n \exp\left[\mathrm{i}\sum_X \left(-\theta_X^{(+)}\Gamma_X^{(+)} + \theta_X^{(-)}\Gamma_X^{(-)}\right)\right]$$

Flavor-chiral symmetries of Wilson fermion

Naive



$$\sum_\mu C_\mu$$

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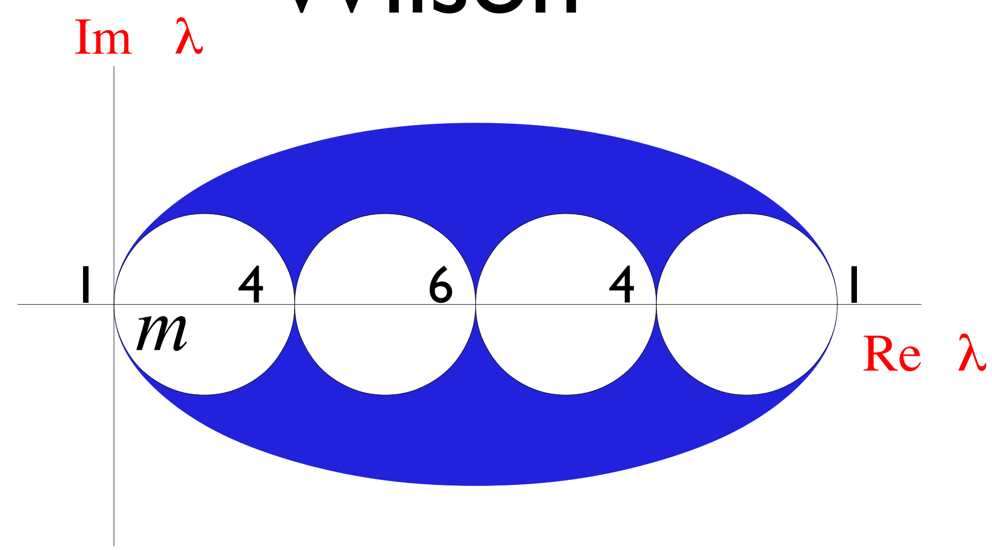
Wilson

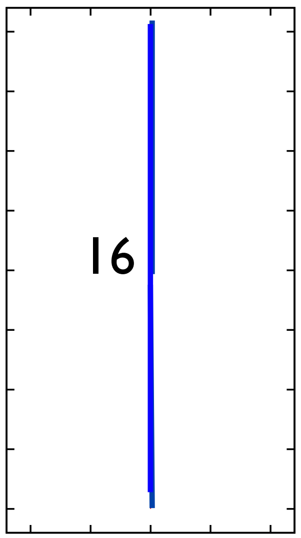


U(4)×U(4) → U(1)

$$\Gamma_X^{(+)} \in \left\{ \mathbf{1}_4 \, , \qquad \right\}$$
$$\Gamma_X^{(-)} \in \left\{ \qquad \right\}$$

Flavor-chiral symmetries of Wilson fermion

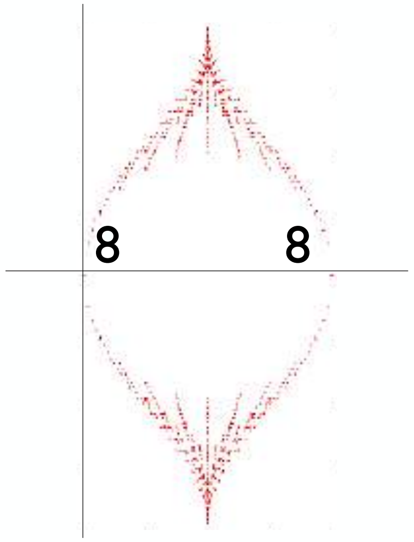
Naive



$$\sum_{sym.} C_1 C_2 C_3 C_4$$

$$C_\mu = (T_{+\mu} + T_{-\mu})/2$$
$$T_{\pm\mu}\psi_n = U_{n,\pm\mu}\psi_{n\pm\mu}$$

Wilson'



U(4)×U(4) → U(2)×U(2)

$$\Gamma_X^{(+)} \in \left\{ \mathbf{1}_4 \,,\ (-1)^{n_1+\ldots+n_4}\gamma_5 \,, \qquad (-1)^{n_{\mu,\nu}} \frac{[\gamma_\mu \,, \gamma_\nu]}{2} \right\}$$
$$\Gamma_X^{(-)} \in \left\{ \qquad \right\}$$

Flavor-chiral symmetries of Wilson fermion

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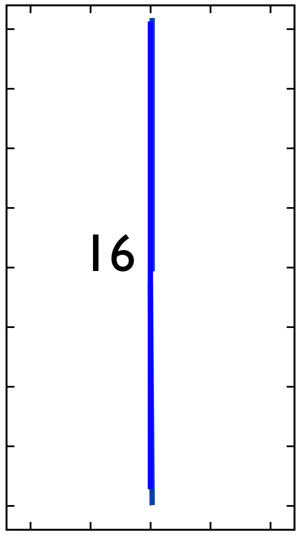

$$C_{12} + C_{34}$$

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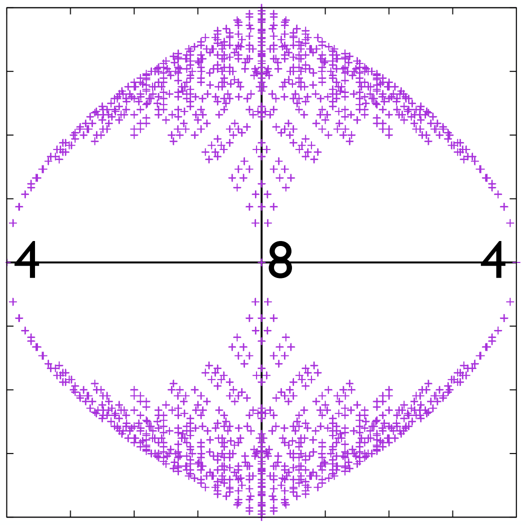
Wilson'



U(4)×U(4) → U(2)

$$\Gamma_X^{(+)} \in \left\{ \mathbf{1}_4\,,\ (-1)^{n_1+\ldots+n_4}\gamma_5\,,\qquad (-1)^{n_{1,2}}\frac{i\,[\gamma_1\,,\gamma_2]}{2}\,,\ (-1)^{n_{3,4}}\frac{i\,[\gamma_3\,,\gamma_4]}{2} \right\}$$

$$\Gamma_X^{(-)} \in \left\{ \qquad \right\}$$

Flavored-mass and generalized Wilson

Adams (09)
Hoelbling (10)
Creutz, Kimura, TM (10)

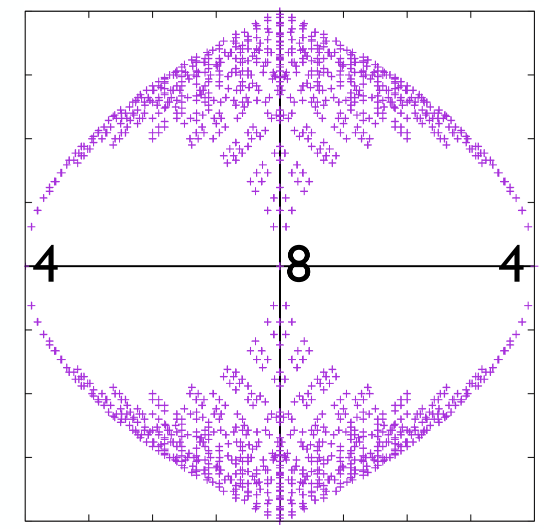
$$M_{\rm V} = \sum_{\mu} C_\mu, \quad \textit{Vector (1-link)}$$

$$M_{\rm T} = \sum_{perm.} \sum_{sym.} C_\mu C_\nu, \quad \textit{Tensor (2-link)}$$

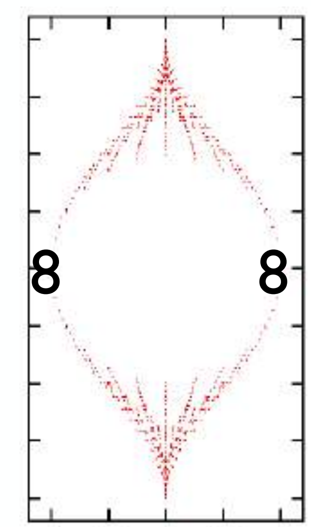
$$M_{\rm A} = \sum_{perm.} \sum_{sym.} \prod_{\nu} C_\nu, \quad \textit{Axial-V (3-link)}$$

$$M_{\rm P} = \sum_{sym.} \prod_{\mu=1}^{4} C_\mu, \quad \textit{Pseudo-S (4-link)}$$

Dirac eigenvalues



$M_{\rm T}$



$M_{\rm P}$

- *Hypercubic sym, gamma5-hermiticity, C, P, T...*
- *Lattice laplacian* $\sum_n \bar{\psi}_n (M_P - 1)\psi_n \;\rightarrow\; -a \int d^4x \bar{\psi}(x) D_\mu^2 \psi(x) + O(a^2)$
- *Generalized Wilson fermions*

Central-branch Wilson

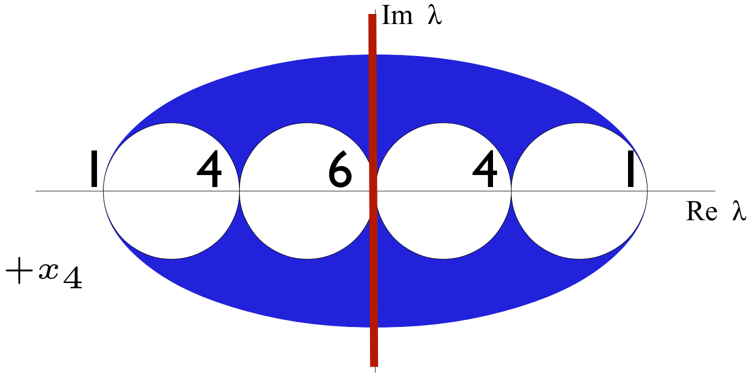
Creutz, Kimura, TM (11)
Kimura, Komatsu, TM, Noumi, Torii, Aoki (11)

◆ Wilson without onsite terms $M_W \equiv m + 4r = 0$

$$S = \frac{1}{2}\sum_{x,\mu} \bar{\psi}_x[\gamma_\mu(U_{x,\mu}\psi_{x+\mu} - U_{x,-\mu}\psi_{x-\mu}) - (U_{x,\mu}\psi_{x+\mu} + U_{x,-\mu}\psi_{x-\mu})]$$

⇨ Extra U(1) for 6 flavors!

$$\psi_x \to e^{i\theta(-1)^{x_1+x_2+x_3+x_4}}\psi_x,\ \ \bar{\psi}_x \to \bar{\psi}_x e^{i\theta(-1)^{x_1+x_2+x_3+x_4}}$$



- <u>Flavor-chiral symmetry</u>

$$\Gamma_X^{(+)} \in \left\{\mathbf{1}_4\,,\ (-1)^{n_1+\ldots+n_4}\gamma_5\,,\ (-1)^{\check{n}_\mu}\gamma_\mu\,,\ (-1)^{n_\mu}i\gamma_\mu\gamma_5\,,\ (-1)^{n_{\mu,\nu}}\frac{[\gamma_\mu\,,\gamma_\nu]}{2}\right\}$$

$$\Gamma_X^{(-)} \in \left\{(-1)^{n_1+\ldots+n_4}\mathbf{1}_4\,,\ \gamma_5\,,\ (-1)^{n_\mu}\gamma_\mu\,,\ (-1)^{\check{n}_\mu}\gamma_\mu\gamma_5\,,\ (-1)^{\check{n}_{\mu,\nu}}\frac{[\gamma_\mu\,,\gamma_\nu]}{2}\right\}$$

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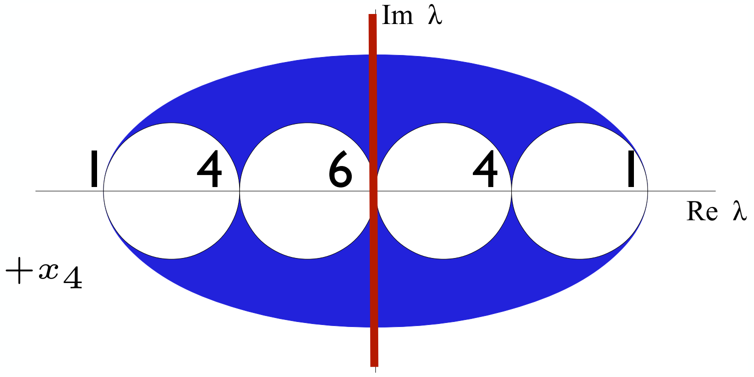

- Flavor-chiral symmetry

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$$\Gamma_X^{(-)} \in \left\{(-1)^{n_1+\ldots+n_4}\mathbf{1}_4\,, \qquad\qquad\qquad\qquad\qquad\qquad\right\}$$

↓

Prohibits additive mass renormalization !
No fine-tuning !

◆ Lattice perturbation

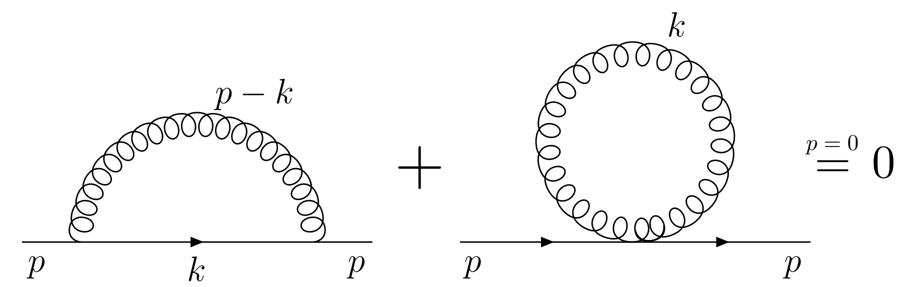
TM (12) Chowdhury, et.al. (13) TM, Yumoto (20)

- quark self-energy

$$\frac{g_0^2}{16\pi^2}\left(\frac{\Sigma_0}{a} + i\gamma_\mu p_\mu \Sigma_1 + m_0 \Sigma_2\right)$$

↓

$$\Sigma_0^{(\alpha)} = \Sigma_0^{(\alpha)}(\mathrm{sun}) + \Sigma_0^{(\alpha)}(\mathrm{tad}) = 0$$



No additive mass renormalization

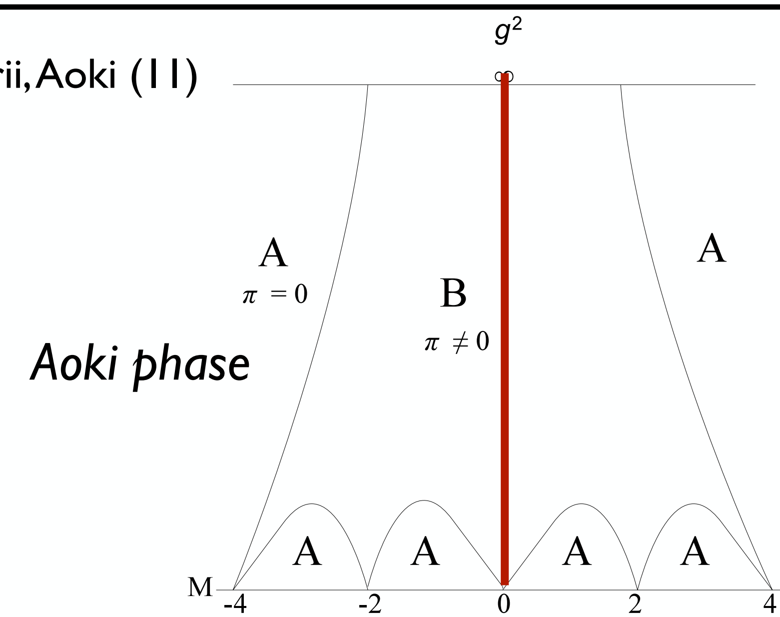
◆ Strong-coupling QCD

Kimura, Komatsu, TM, Noumi, Torii, Aoki (11)
TM, Yumoto (20)

- Massless meson a/w extra U(1)

$$\pi \sim \bar{\psi}(-1)^{\sum n}\psi$$

$$\cosh(m_\pi) = 1 + \frac{2M_W^2(16 + M_W^2)}{16 - 15M_W^2} \;\rightarrow\; m_\pi = 0$$



- Pseudo-scalar condensate

$$\langle\bar{\psi}\gamma_5\psi\rangle \neq 0 \qquad \langle\bar{\psi}\psi\rangle = 0$$

Parity is also broken!

➡

NG boson emerges due to SSB of the extra U(1)

◆ CB for other flavored masses

de Forcrand et.al. (12) TM (12) TM,Yumoto (20)

M_T : U(2) restored M_H : C_T' restored for staggered

$$\Gamma_X^{(+)} \in \left\{ \mathbf{1}_4 \,,\; (-1)^{n_1+\ldots+n_4}\gamma_5 \,,\; (-1)^{n_{1,2}}\frac{i\,[\gamma_1\,,\gamma_2]}{2} \,,\; (-1)^{n_{3,4}}\frac{i\,[\gamma_3\,,\gamma_4]}{2} \right\},$$

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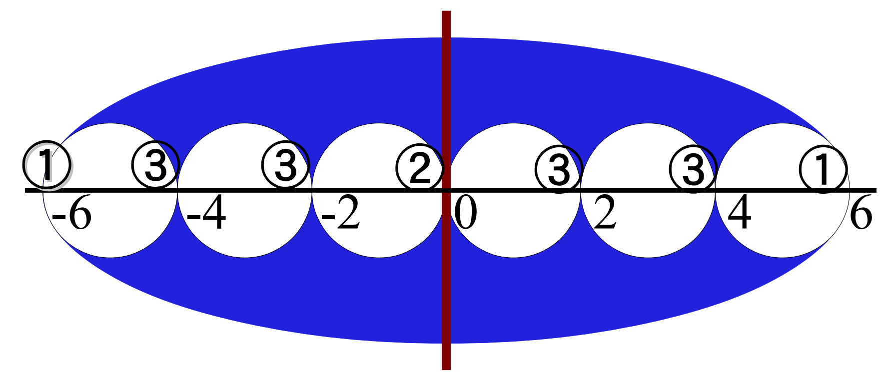
◆ 2-flavor CB

TM (12) TM,Yumoto (20)

§ 4D $\sum_{\mu=1}^{4} C_\mu \;\rightarrow\; \sum_{j=1}^{3} C_j + 3C_4.$



(1,3,3,2,3,3,1) splitting

- 2-flavor fermion
- Hypercubic symmetry → Cubic symmetry

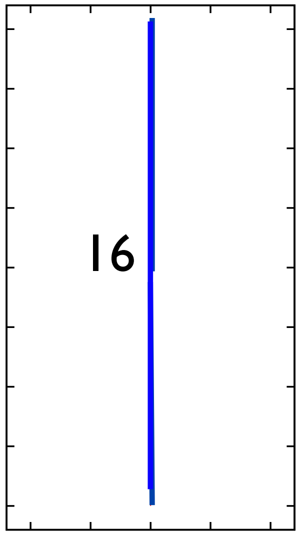
§ 5D $\sum_{\mu=1}^{5} C_\mu \;\rightarrow\; \sum_{j=1}^{4} C_j + 4C_5$

(1,4,6,4,2,4,6,4,1) splitting

- 5D 2-flavor fermion
- 5D hypercubic → 4D hyperubic

Flavor-chiral symmetries of Wilson fermion

Naive



$$C_{12}+C_{34}$$

$$C_\mu=(T_{+\mu}+T_{-\mu})/2$$
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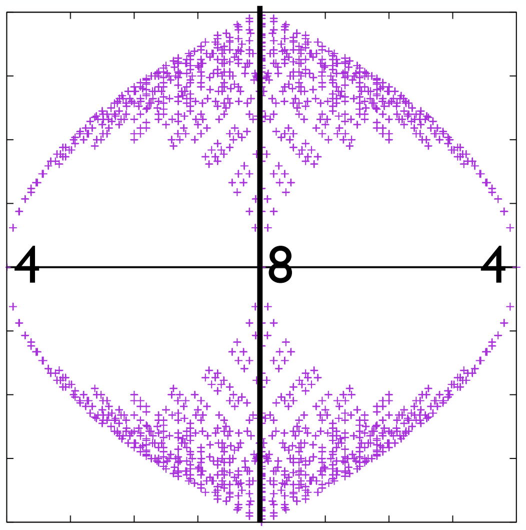
CB Wilson'



U(4)×U(4) → U(2)×U(2)

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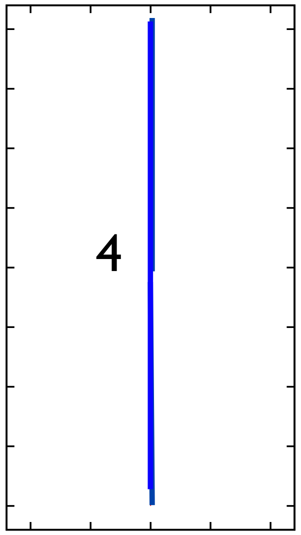
Flavor-chiral symmetries of Wilson fermion

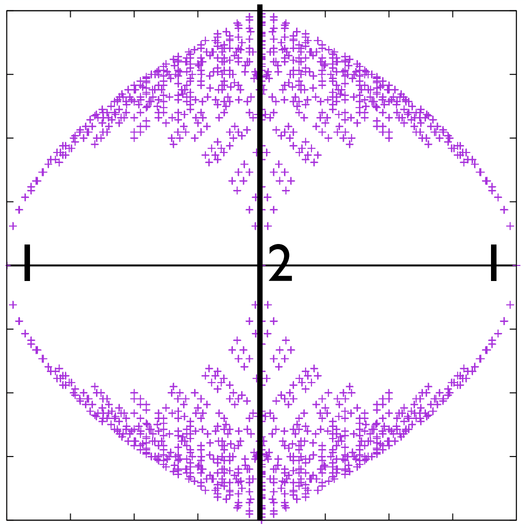
Staggered



$$i(\eta_{12}C_{12} + \eta_{34}C_{34})$$

$$(\eta_{\mu\nu})_{xy} = \epsilon_{\mu\nu}\eta_\mu\eta_\nu\delta_{x,y},\ (\epsilon_{\mu\nu})_{xy} = (-1)^{x_\mu + x_\nu}\delta_{x,y},$$

CB staggered-Wilson



$$\{C_0,\, \Xi_\mu,\, I_s,\, R_{\mu\nu}\} \times \{U^\epsilon(1)\}_{m=0} \quad \Rightarrow \quad \{C_T,\ C_T',\ \Xi_\mu',\ R_{12},\ R_{34},\ R_{24}R_{31}\}$$

$$C_T' :\ \chi_x \rightarrow \bar{\chi}_x^T,\ \bar{\chi}_x \rightarrow \chi_x^T,\ U_{x,\mu} \rightarrow U_{x,\mu}^*$$

◆ CB for other flavored masses

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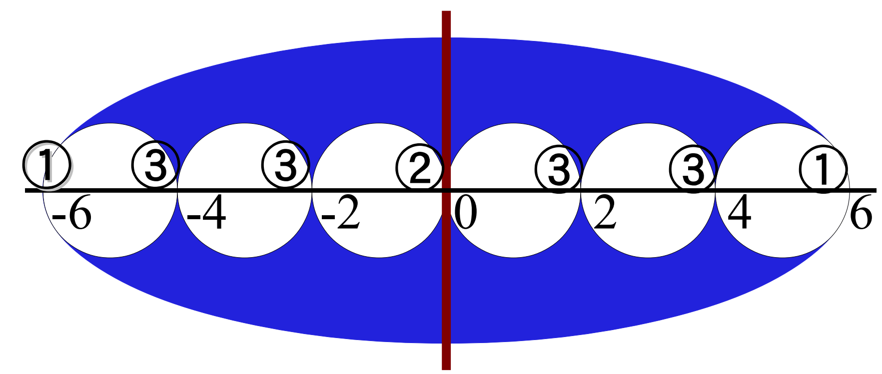
TM (12) TM,Yumoto (20)

§ 4D $\displaystyle\sum_{\mu=1}^{4} C_\mu \;\rightarrow\; \sum_{j=1}^{3} C_j + 3C_4.$



- 2-flavor fermion
- Hypercubic symmetry → Cubic symmetry

(1,3,3,2,3,3,1) splitting

§ 5D $\displaystyle\sum_{\mu=1}^{5} C_\mu \;\rightarrow\; \sum_{j=1}^{4} C_j + 4C_5$

(1,4,6,4,2,4,6,4,1) splitting

- 5D 2-flavor fermion
- 5D hypercubic → 4D hyperubic

Semi-positivity of det(D)

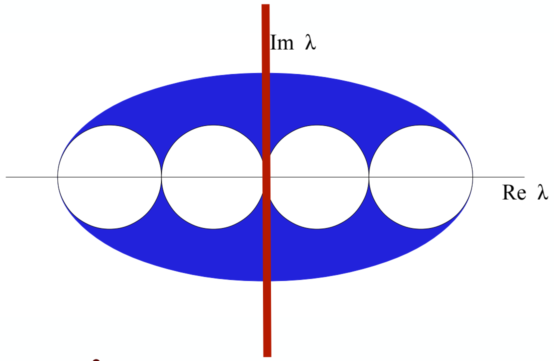
Tanizaki, TM (19)

◆Use of extra U(1) symmetry $\mathsf{D}(-1)^{\sum_\mu n_\mu} = -(-1)^{\sum_\mu n_\mu}\mathsf{D}$

$$\mathsf{D}|R_\lambda\rangle = \lambda|R_\lambda\rangle \qquad \langle L_\lambda|\mathsf{D} = \lambda\langle L_\lambda|$$

⇒

$$\mathsf{D}(-)^{\sum_\mu n_\mu}|R_\lambda\rangle = -\lambda(-)^{\sum_\mu n_\mu}|R_\lambda\rangle, \qquad \langle L_\lambda|(-)^{\sum_\mu n_\mu}\mathsf{D} = -\lambda\langle L_\lambda|(-)^{\sum_\mu n_\mu}.$$



It shows that +λ and -λ make a pair.

◆Use of hermitian Dirac operator $H = \gamma_5\mathsf{D}$

$$H(-1)^{\sum_\mu n_\mu} = -(-1)^{\sum_\mu n_\mu}H \Rightarrow \{\pm\varepsilon_i\}_{i=1,\dots,N}$$

Pair of +ε and -ε

$$\Rightarrow \det(\mathsf{D}) = \det(H) = \prod_{i=1}^{N}\varepsilon_i(-\varepsilon_i) = (-1)^N\prod_{i=1}^{N}\varepsilon_i^2 > 0$$

$N = N_1N_2N_3N_4$: even

This procedure shows semi-positivity $\det(\mathsf{D}) \geq 0$

2D Central-branch Wilson

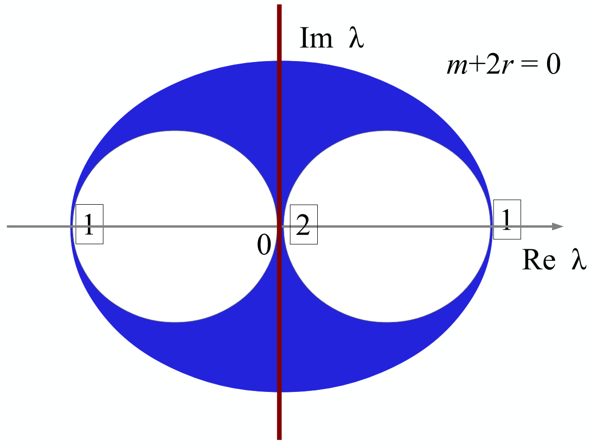
Tanizaki, TM (19)

◆Wilson without onsite terms $m+2r=0$

$$S_{\rm CB}=\sum_{n,\mu}\left(\bar{\psi}_n\gamma_\mu D_\mu\psi_n - r\bar{\psi}_n C_\mu\psi_n\right)$$

➡ Extra U(1) for 2 flavors!

$$U(1)_{\overline{V}}:\psi_n\mapsto \mathrm{e}^{\mathrm{i}(-1)^{n_1+n_2}\beta}\psi_n,\ \overline{\psi}_n\mapsto\overline{\psi}_n\mathrm{e}^{\mathrm{i}(-1)^{n_1+n_2}\beta}$$



Dirac eigenvalue distribution

- Flavor-chiral symmetry for central-branch Wilson fermion

$$\Gamma_X^{(+)}\in\{\mathbf{1}_2 \qquad\qquad\qquad\qquad \}$$

$$\Gamma_X^{(-)}\in\{(-1)^{n_1+n_2}\mathbf{1}_2 \qquad\qquad\qquad \}$$

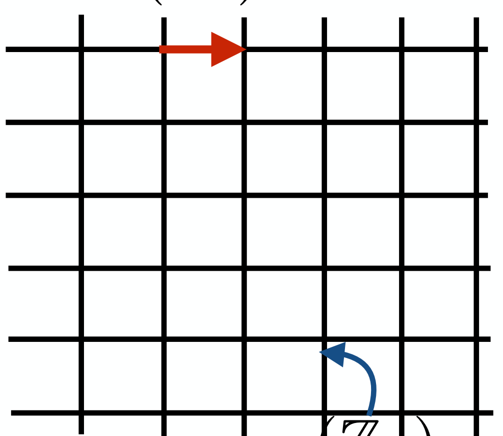
↓

This extra U(1) symmetry to prohibit mass term

Symmetries of CB Wilson

Tanizaki, TM (19)

◆Relevant symmetries



$$U(1)_V : \psi_n \mapsto \mathrm{e}^{\mathrm{i}\alpha}\psi_n, \quad \overline{\psi}_n \mapsto \overline{\psi}_n \mathrm{e}^{-\mathrm{i}\alpha}$$

$$U(1)_{\overline{V}} : \psi_n \mapsto \mathrm{e}^{\mathrm{i}(-1)^{n_1+n_2}\beta}\psi_n, \; \overline{\psi}_n \mapsto \overline{\psi}_n \mathrm{e}^{\mathrm{i}(-1)^{n_1+n_2}\beta}$$

$$(\mathbb{Z}_2)_{\rm lat.trans.} : \quad \psi(x,y) \mapsto \psi(x+1,y) \quad \psi(x,y) \mapsto \psi(x,y+1)$$

$$(\mathbb{Z}_2)_\chi : \quad \psi(x,y) \mapsto \mathrm{e}^{\mathrm{i}\frac{\pi}{4}\gamma_3}\psi(y,-x), \; \overline{\psi}(x,y) \mapsto \overline{\psi}(y,-x)\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}\gamma_3}$$

$$G_{\rm CB\ fermion} = \frac{U(1)_V \times [U(1)_{\overline{V}} \rtimes (\mathbb{Z}_2)_{\rm lat.\ trans.}]}{(\mathbb{Z}_2)_F} \times (\mathbb{Z}_2)_\chi$$

't Hooft anomaly matching tells CB-Wilson Schwinger model is gapless or has SSB of Z_2

Anomaly matching for QED with CB

◆ XXZ spin chain

$$\hat{H} = -\sum_{\ell}(J_x \hat{X}_\ell \hat{X}_{\ell+1} + J_y \hat{Y}_\ell \hat{Y}_{\ell+1} + J_z \hat{Z}_\ell \hat{Z}_{\ell+1}) \qquad J \equiv J_x = J_y \neq J_z$$

Symmetries : $SO(2) \rtimes \mathbb{Z}_2 \times (\mathbb{Z}_2)_{\mathrm{lat.trans.}}$

same symmetry structure
as CB-Wilson Schwinger model

◆ Known facts on the system

$|J_z/J| < 1$ gapless phase with spin-wave or spinon

$|J_z/J| > 1$ Z_2 SSB : ferromagnetic or anti-ferromagnetic

⇒ consistent to our mixed anomaly !

CB-Wilson Schwinger enables us to simulate Heisenberg spin chain

2. Lattice fermions as spectral graphs

Lattice fermion as spectral graph

Yumoto, TM (21)
cf.) Ohta, Sakai (20)
Ohta, Matsuura (21)

Definition 1. ***A graph G is a pair G = (V, E).*** *V is a set of vertices and E is a set of edges.*

Definition 2. ***A directed graph is a pair (V, E) of sets of vertices and edges together with two maps* init *: E → V and* ter *: E → V.*** *The two maps are assigned to every edge e_{ij} with an initial vertex* $\mathrm{init}(e_{ij}) = v_i \in V$ *and a terminal vertex* $\mathrm{ter}(e_{ij}) = v_j \in V$. *If* $\mathrm{init}(e_{ij}) = \mathrm{ter}(e_{ij})$, *the edge e_{ij} is called a loop.*

Definition 3. ***A weighted graph has a value (weight) for each edge in a graph.***

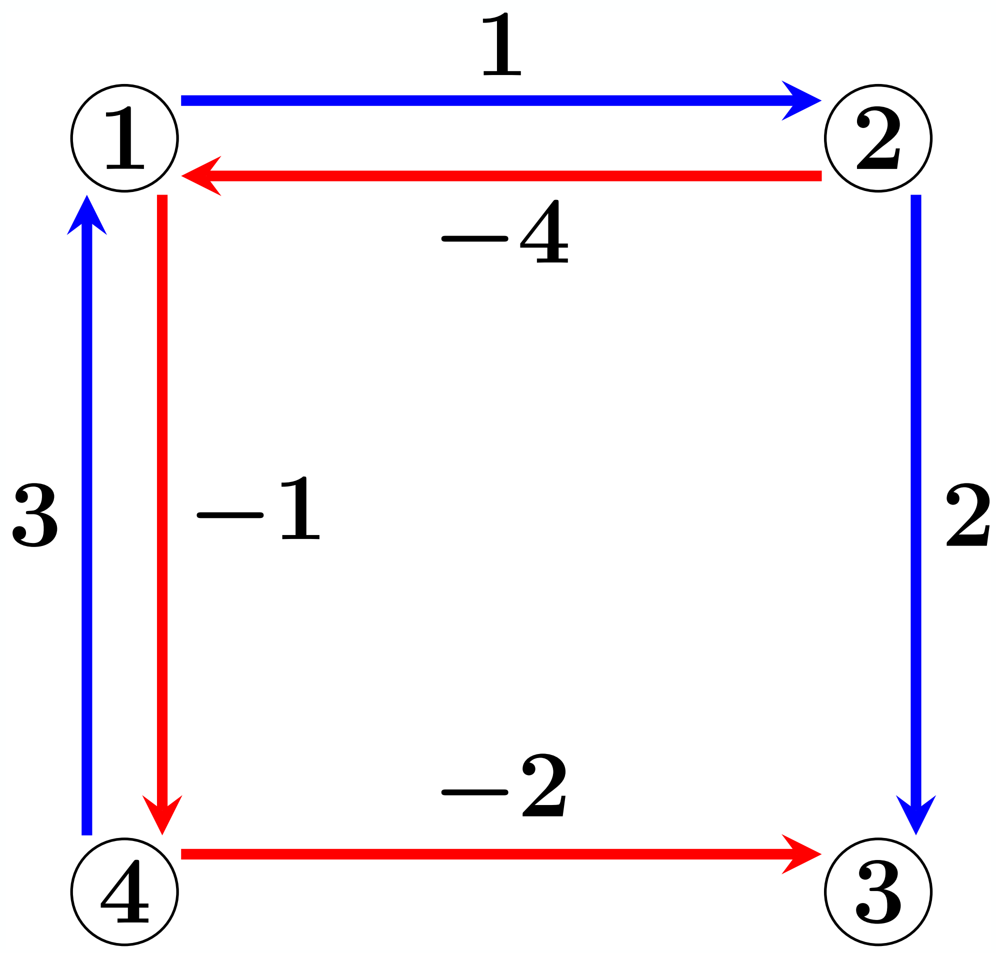
Definition 4. ***A adjacency matrix A of a graph is the |V| × |V| matrix given by***

$$A_{ij} = \begin{cases} w_{ij} & \textit{if there is a edge from } i \textit{ to } j \\ 0 & \textit{otherwise} \end{cases},$$

where w_{ij} is the weight of an edge from i to j.

Lattice fermion as spectral graph

Yumoto, TM (21)



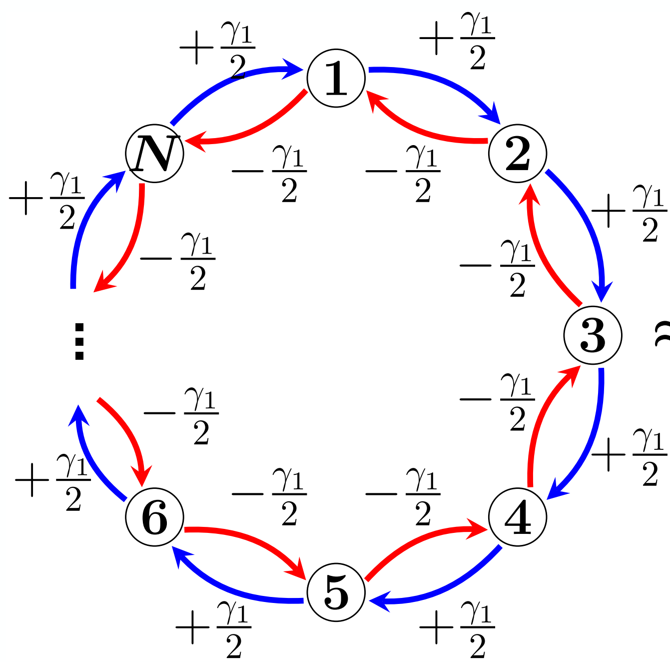
$$A = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -4 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & -2 & 0 \end{pmatrix}$$

Lattice fermion as spectral graph

Yumoto, TM (21)

- Naive fermion

1D



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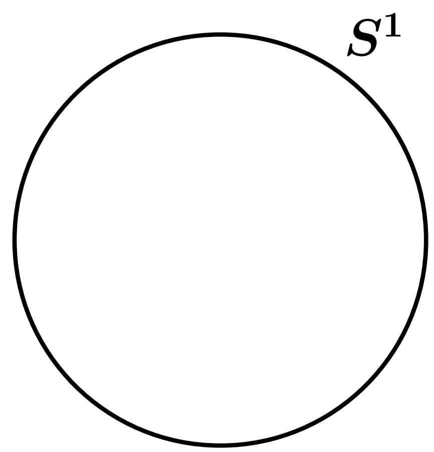
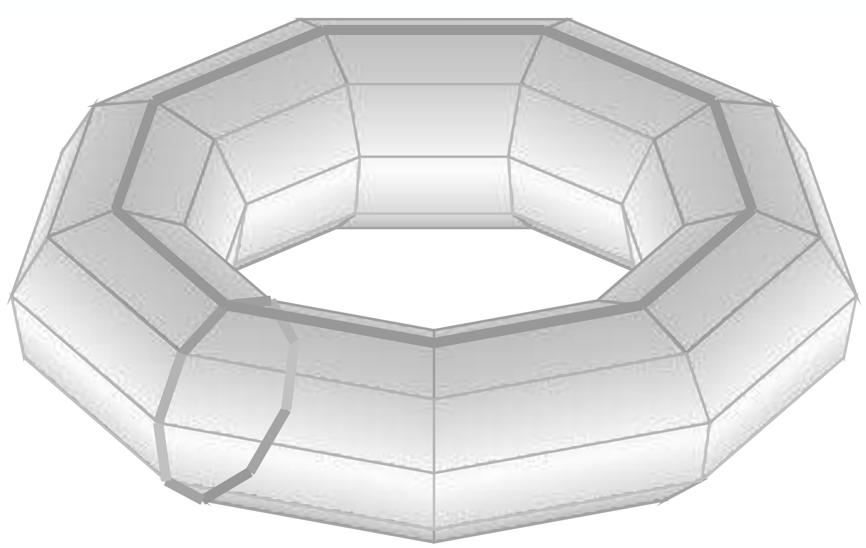


$$\mathcal{D}^{1\mathrm{d}} = P_N \otimes \gamma_1.$$

2D

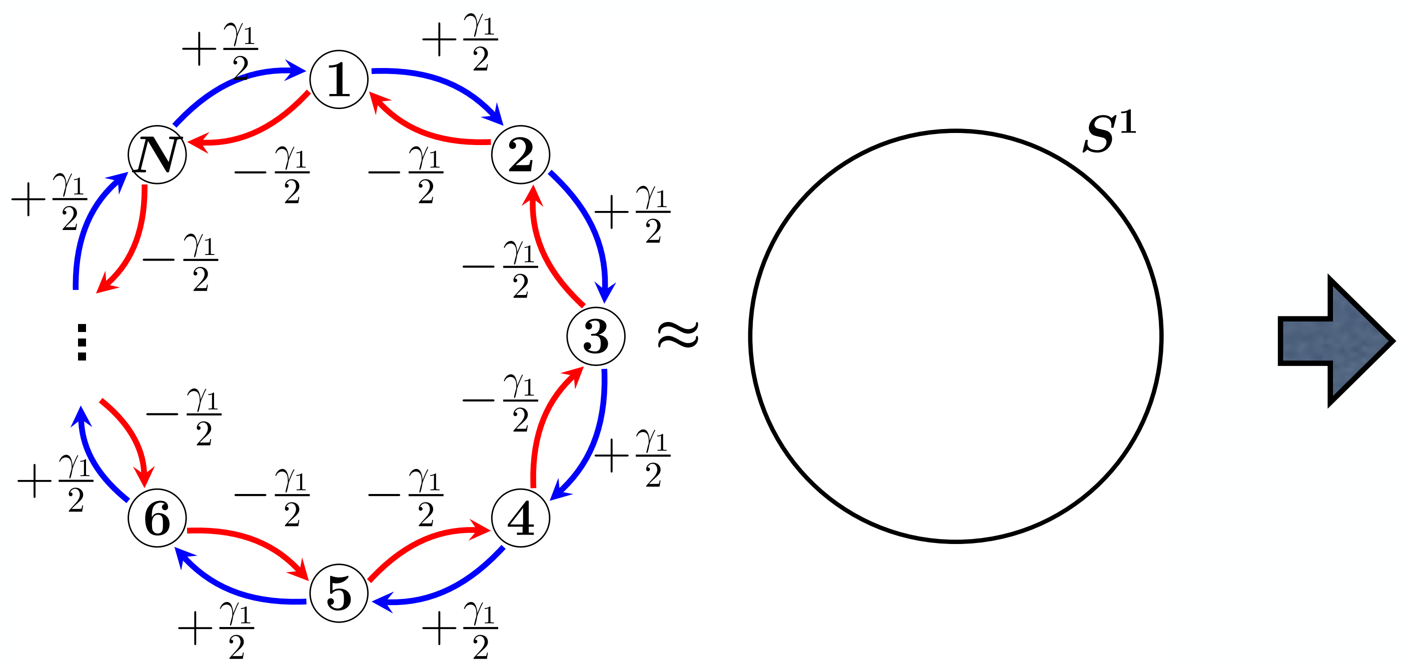


$$D^{2\mathrm{d}} = \mathbf{1}_N \otimes P_N \otimes \gamma_1 + P_N \otimes \mathbf{1}_N \otimes \gamma_2$$

Lattice fermion as spectral graph

Yumoto, TM (21)

- 4D Naive fermion



$\times\, 4$

$$
\begin{aligned}
\mathcal{D} = & \ \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes P_N \otimes \gamma_1 \\
& + \mathbf{1}_N \otimes \mathbf{1}_N \otimes P_N \otimes \mathbf{1}_N \otimes \gamma_2 \\
& + \mathbf{1}_N \otimes P_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \gamma_3 \\
& + P_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \gamma_4
\end{aligned}
$$

$$
P_N = \frac{1}{2}\begin{pmatrix}
0 & 1 & 0 & & 0 & 0 & -1 \\
-1 & 0 & 1 & \cdots & 0 & 0 & 0 \\
0 & -1 & 0 & & 0 & 0 & 0 \\
 & \vdots & & \ddots & & \vdots & \\
0 & 0 & 0 & & 0 & 1 & 0 \\
0 & 0 & 0 & \cdots & -1 & 0 & 1 \\
1 & 0 & 0 & & 0 & -1 & 0
\end{pmatrix}
$$

Lattice fermion as spectral graph

Yumoto, TM (21)

- Diagonalization

$$P_N X = i\,\mathrm{Diag}\left[0,\ \sin\frac{2\pi}{N},\ \sin\frac{4\pi}{N},\ \cdots,\ \sin\frac{2(N-1)\pi}{N}\right] X \equiv \Lambda_{P_N} X.$$

$$\begin{aligned}\mathcal{U}^\dagger \mathcal{D} \mathcal{U} = &\ \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \Lambda_{P_N} \otimes \gamma_1 \\ &+ \mathbf{1}_N \otimes \mathbf{1}_N \otimes \Lambda_{P_N} \otimes \mathbf{1}_N \otimes \gamma_2 \\ &+ \mathbf{1}_N \otimes \Lambda_{P_N} \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \gamma_3 \\ &+ \Lambda_{P_N} \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \gamma_4\end{aligned}$$

$$\mathcal{U} = \bigotimes_{\mu=1}^{4} X \otimes \mathbf{1}_4$$

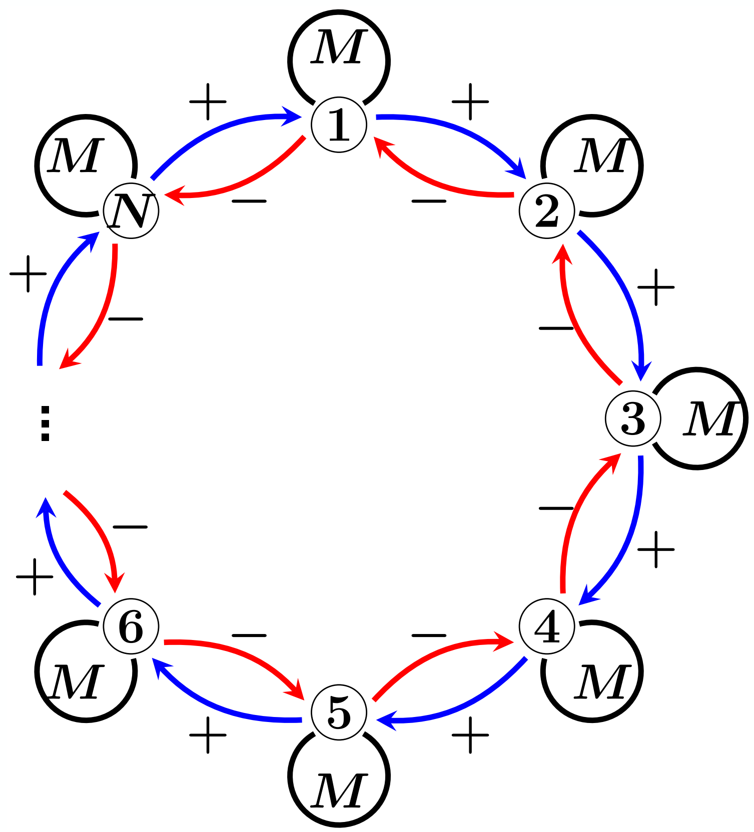
➡ $$\sum_{\mu=1}^{4} \sin\left(\frac{2\pi(k_\mu - 1)}{N}\right)\gamma_\mu = \mathbf{0}.$$ ➡ 16 species

Lattice fermion as spectral graph

Yumoto, TM (21)

◆ Wilson fermion

1D



$\pm(\gamma_1 \mp r\mathbf{1}_4)/2$

$M = m + r$

$$D_W^{1\mathrm{d}} = P_N \otimes \gamma_1 + m\mathbf{1}_N \otimes \mathbf{1}_4 + rM_W \otimes \mathbf{1}_4$$

$$M_W = \mathbf{1}_N - (E + E^\dagger)/2$$

$$D_W^{2\mathrm{d}} = \mathbf{1}_N \otimes P_N \otimes \gamma_1 + P_N \otimes \mathbf{1}_N \otimes \gamma_2$$

$$+ m\,(\mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_4) + r\,(\mathbf{1}_N \otimes M_W + M_W \otimes \mathbf{1}_N) \otimes \mathbf{1}_4$$

⇨

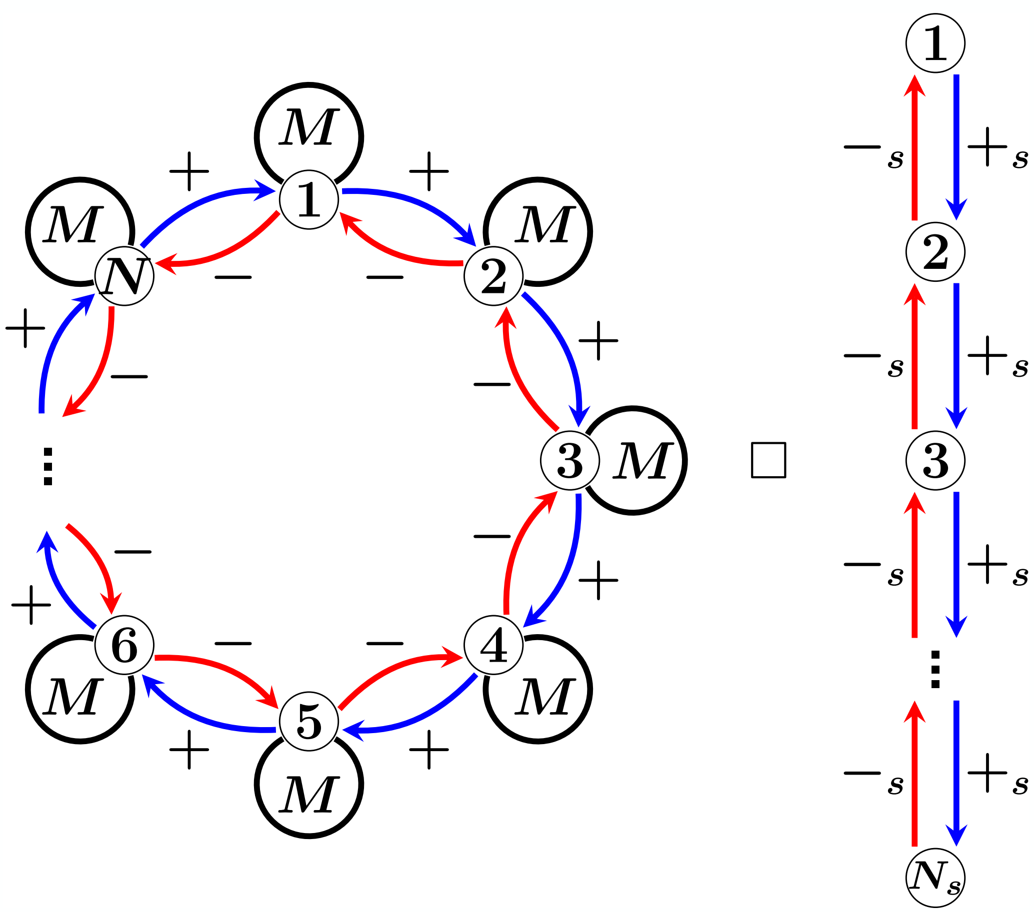
$$\sin\left(\frac{2\pi(k_\mu - 1)}{N}\right) = 0$$

$$m + 4r - r\sum_\mu \cos\left(\frac{2\pi(k_\mu - 1)}{N}\right) = 0$$

Lattice fermion as spectral graph

Yumoto, TM (21)

- Domain-wall fermion



$$S_{DW} = \sum_{s,t} \bar{\psi}_s \left[\mathcal{D}_{DW} + \mathcal{M}_{DW}\right]_{st} \psi_t$$

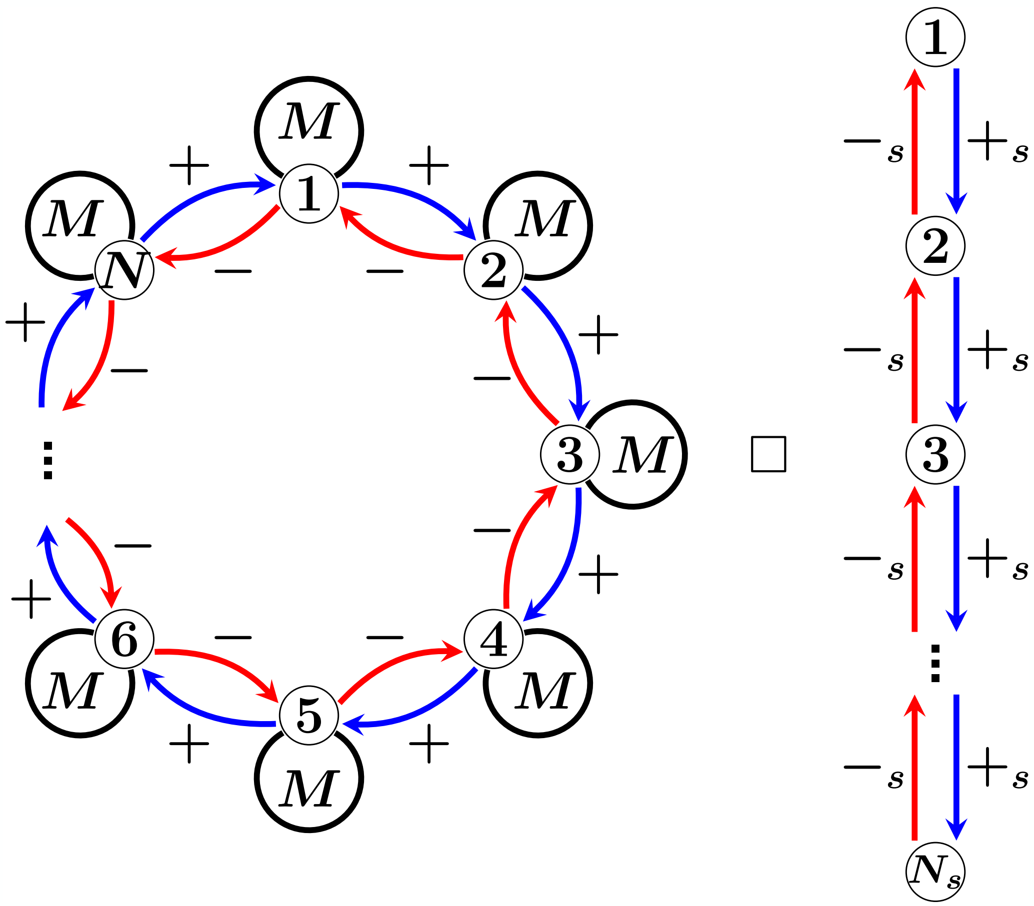
$$\begin{aligned}[\mathcal{D}_{DW}]_{st} =& \delta_{st} \cdot \big(\mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes P_N \otimes \gamma_1 + \mathbf{1}_N \otimes \mathbf{1}_N \otimes P_N \otimes \mathbf{1}_N \otimes \gamma_2 \\ & + \mathbf{1}_N \otimes P_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \gamma_3 + P_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \gamma_4 \big)\end{aligned}$$

$$\begin{aligned}[\mathcal{M}_{DW}]_{st} =& \frac{1}{2}\Delta^{(-)}_{st} \cdot \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \gamma_5 \\ & + \delta_{st}\big(-M_0 \cdot \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_4 \\ & + \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes M_W \otimes \mathbf{1}_4 + \mathbf{1}_N \otimes \mathbf{1}_N \otimes M_W \otimes \mathbf{1}_N \otimes \mathbf{1}_4 \\ & + \mathbf{1}_N \otimes M_W \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_4 + M_W \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_4 \big) \\ & + \frac{1}{2}\left(2\delta_{st} - \Delta^{(+)}_{st}\right) \cdot \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_4\end{aligned}$$

Lattice fermion as spectral graph

Yumoto, TM (21)

- Domain-wall fermion



$$S_{DW} = \sum_{s,t} \bar{\psi}_s \left[\mathcal{D}_{DW} + \mathcal{M}_{DW}\right]_{st} \psi_t$$

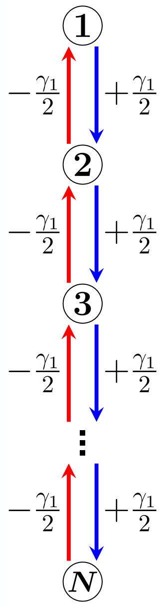
the solutions of k_μ	the range of M_0	the number of zero eigenvalues
any $k_\mu = 1$	$0 < M_0 < 2$	1
one $k_\mu = 1 + N/2$ otherwise $k_\mu = 1$	$2 < M_0 < 4$	4
two $k_\mu = 1 + N/2$ otherwise $k_\mu = 1$	$4 < M_0 < 6$	6
three $k_\mu = 1 + N/2$ otherwise $k_\mu = 1$	$6 < M_0 < 8$	4
any $k_\mu = 1 + N/2$	$8 < M_0 < 10$	1

Lattice fermion as spectral graph

Yumoto, TM (21)

- Naive fermion on hyperball

1D



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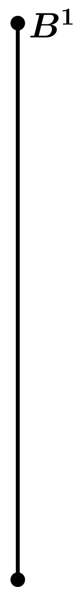
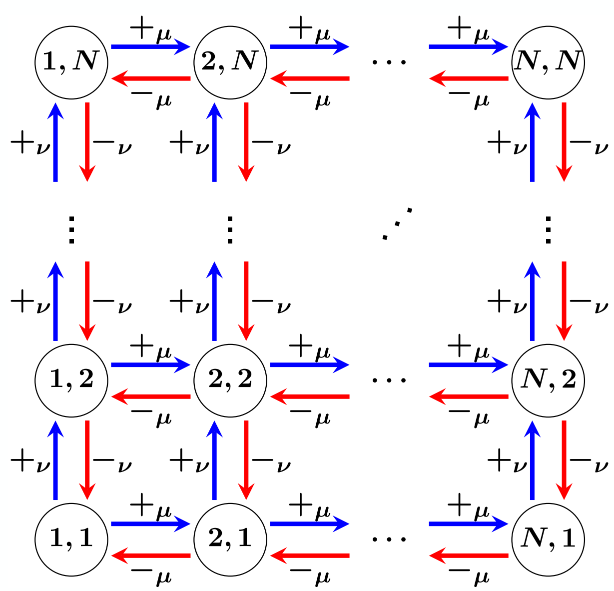


2D



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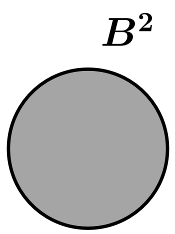


$$\mathcal{D}_{B^1} = Q_N \otimes \gamma_1$$

$$\mathcal{D}_{B^2} = \mathbf{1}_N \otimes Q_N \otimes \gamma_1 + Q_N \otimes \mathbf{1}_N \otimes \gamma_2$$

$$\begin{aligned}\mathcal{D}_{B^4} &= \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes Q_N \otimes \gamma_1 \\ &+ \mathbf{1}_N \otimes \mathbf{1}_N \otimes Q_N \otimes \mathbf{1}_N \otimes \gamma_2 \\ &+ \mathbf{1}_N \otimes Q_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \gamma_3 \\ &+ Q_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \gamma_4.\end{aligned}$$

$$Q_N = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & & 0 & 0 & 0 \\ -1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 1 \\ 0 & 0 & 0 & & 0 & -1 & 0 \end{pmatrix}$$

Lattice fermion as spectral graph

Yumoto, TM (21)

- Naive fermion on 4D hyperball

$$Q_N Y = i\mathrm{Diag}\left[\cos\left(\frac{\pi}{N+1}\right),\, \cos\left(\frac{2\pi}{N+1}\right),\, \cdots,\, \cos\left(\frac{N\pi}{N+1}\right)\right] \equiv \Lambda_{Q_N} X$$

$$\begin{aligned}
\mathcal{V}^\dagger \mathcal{D}_{B^4} \mathcal{V} = &\mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \Lambda_{Q_N} \otimes \gamma_1 \\
&+ \mathbf{1}_N \otimes \mathbf{1}_N \otimes \Lambda_{Q_N} \otimes \mathbf{1}_N \otimes \gamma_2 \\
&+ \mathbf{1}_N \otimes \Lambda_{Q_N} \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \gamma_3 \\
&+ \Lambda_{Q_N} \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \gamma_4
\end{aligned}$$

$$\mathcal{V} \equiv \bigotimes_{\mu=1}^{4} Y \otimes \mathbf{1}_4$$

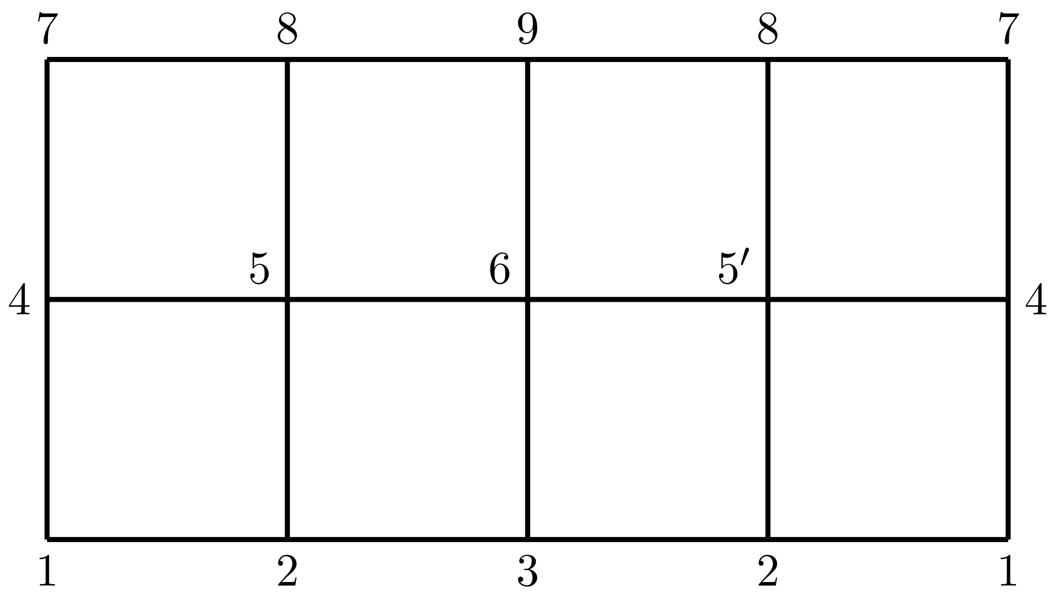
$$\Rightarrow \sum_\mu \cos\left(\frac{\lambda_\mu \pi}{N+1}\right)\gamma_\mu = 0$$

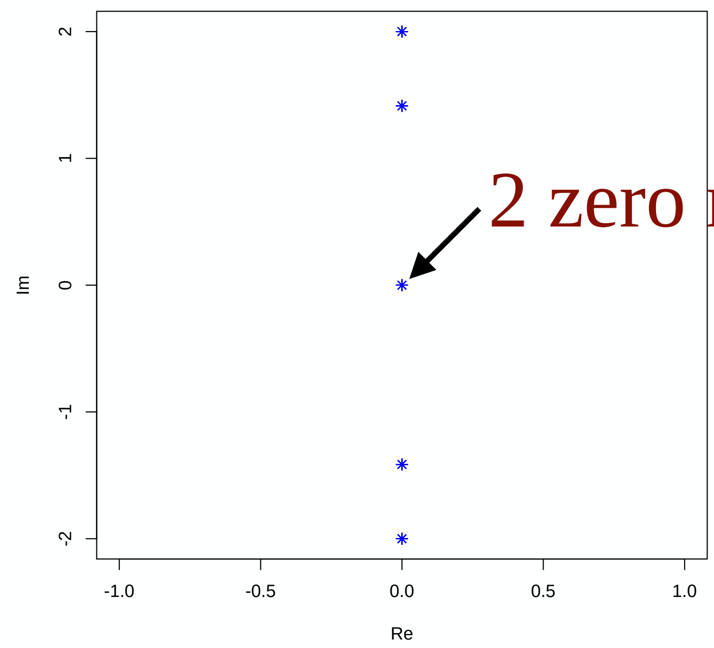
⇒ 1 species in bulk !

Lattice fermion as spectral graph

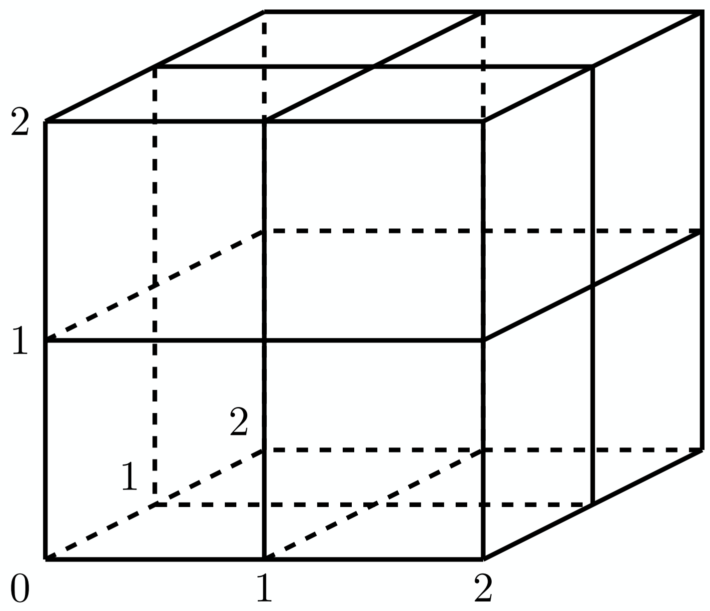
Yumoto, TM (21)

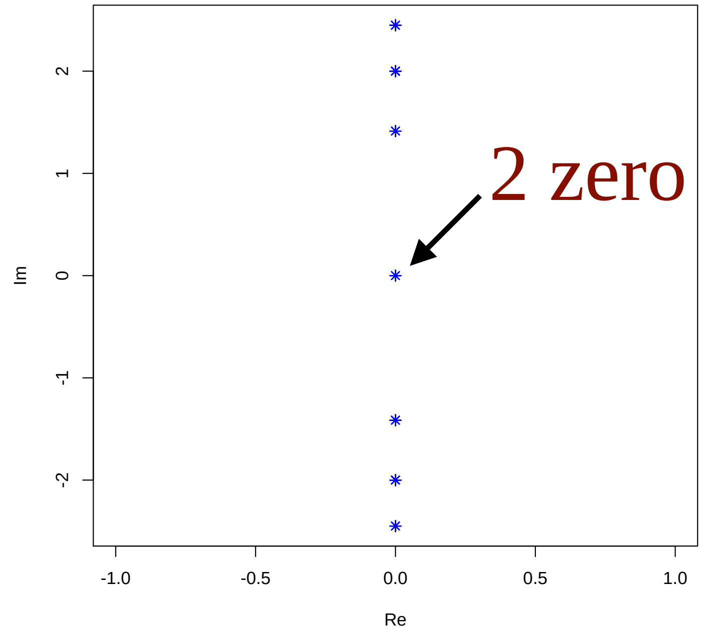
◆ Naive fermion on sphere





Empirically
2 species !





Lattice fermion as spectral graph

Yumoto, TM (21)

Lattice field theory	➡	Spectral graph theory
Lattice fermion	➡	Directed and Weighted spectral graph
# of Fermion species	➡	Nullity of spectral matrix

We can use known theorems to study lattice field theory.

3. New conjecture on fermion doubling

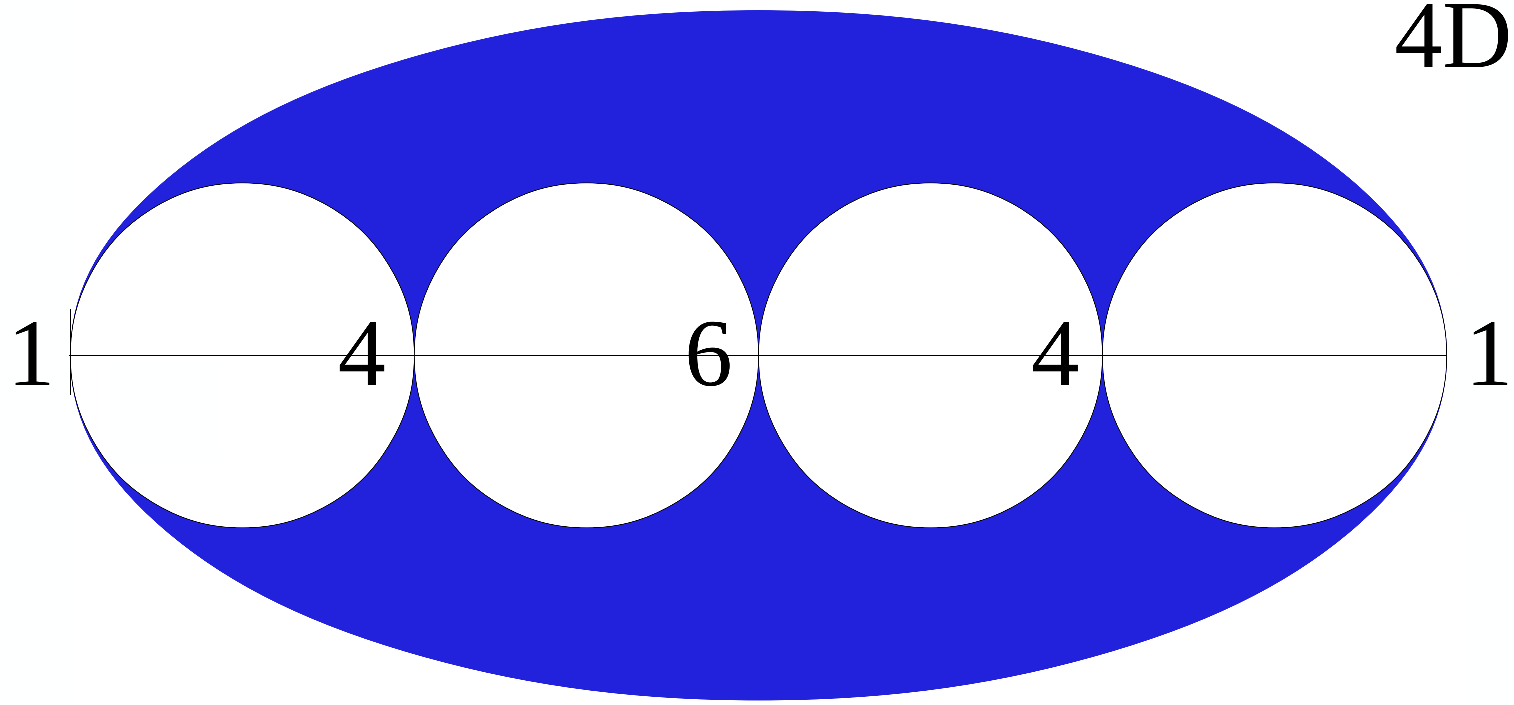
Nielsen-Ninomiya's no-go theorem is just no-go theorem.
It never tells us how many fermion species emerge
given a lattice fermion formulation.

Is there a theorem which informs us of # of species?

Reconsider Naive and Wilson

Yumoto, TM (22)

4D Wilson



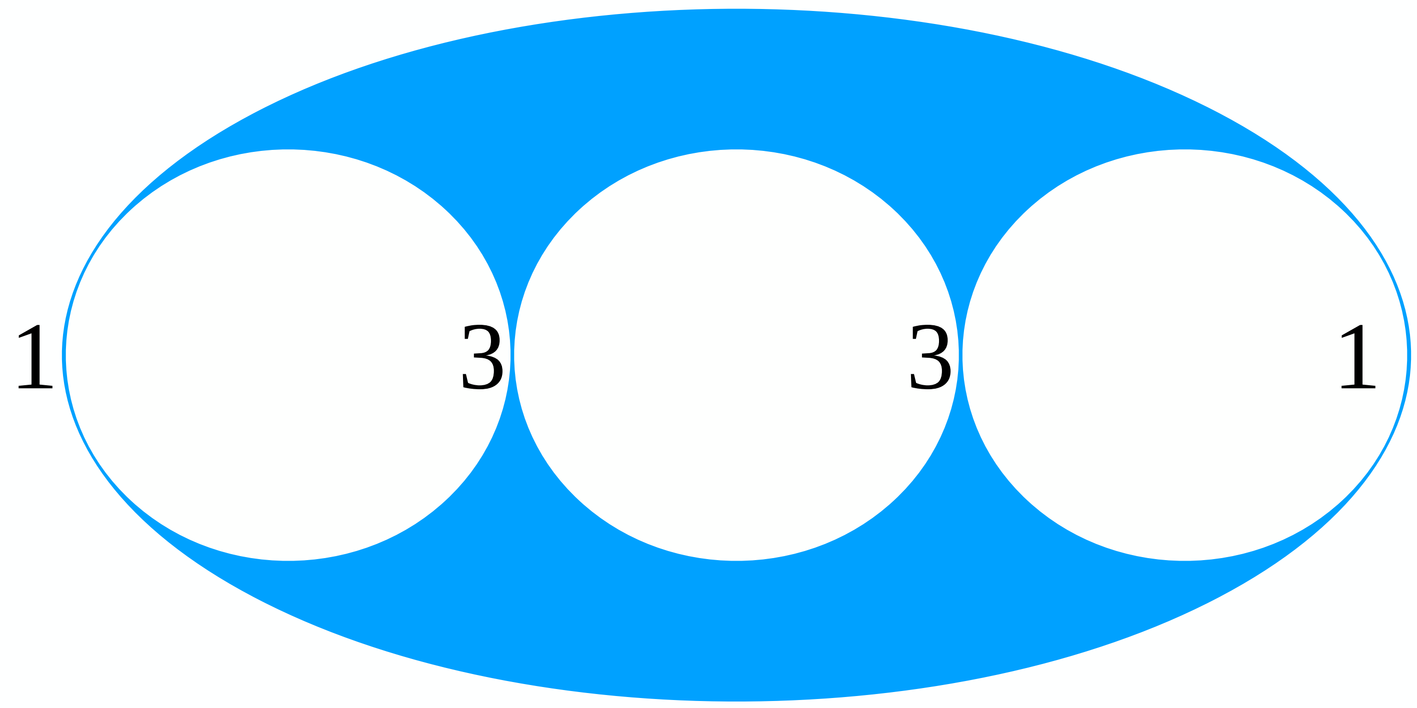
$(0,0,0,0)$	$(\pi,0,0,0)$	$(\pi,\pi,0,0)$	$(\pi,\pi,\pi,0)$	(π,π,π,π)
	$(0,\pi,0,0)$	$(\pi,0,\pi,0)$	$(\pi,\pi,0,\pi)$	
	$(0,0,\pi,0)$	$(\pi,0,0,\pi)$	$(\pi,0,\pi,\pi)$	
	$(0,0,0,\pi)$	$(0,\pi,0,\pi)$	$(0,\pi,\pi,\pi)$	
		$(0,0,\pi,\pi)$		
		$(0,\pi,\pi,0)$		

What is the meaning of the numbers?

Reconsider Naive and Wilson

Yumoto, TM (22)

3D Wilson



$(0,0,0)$

$(\pi,0,0)$
$(0,\pi,0)$
$(0,0,\pi)$

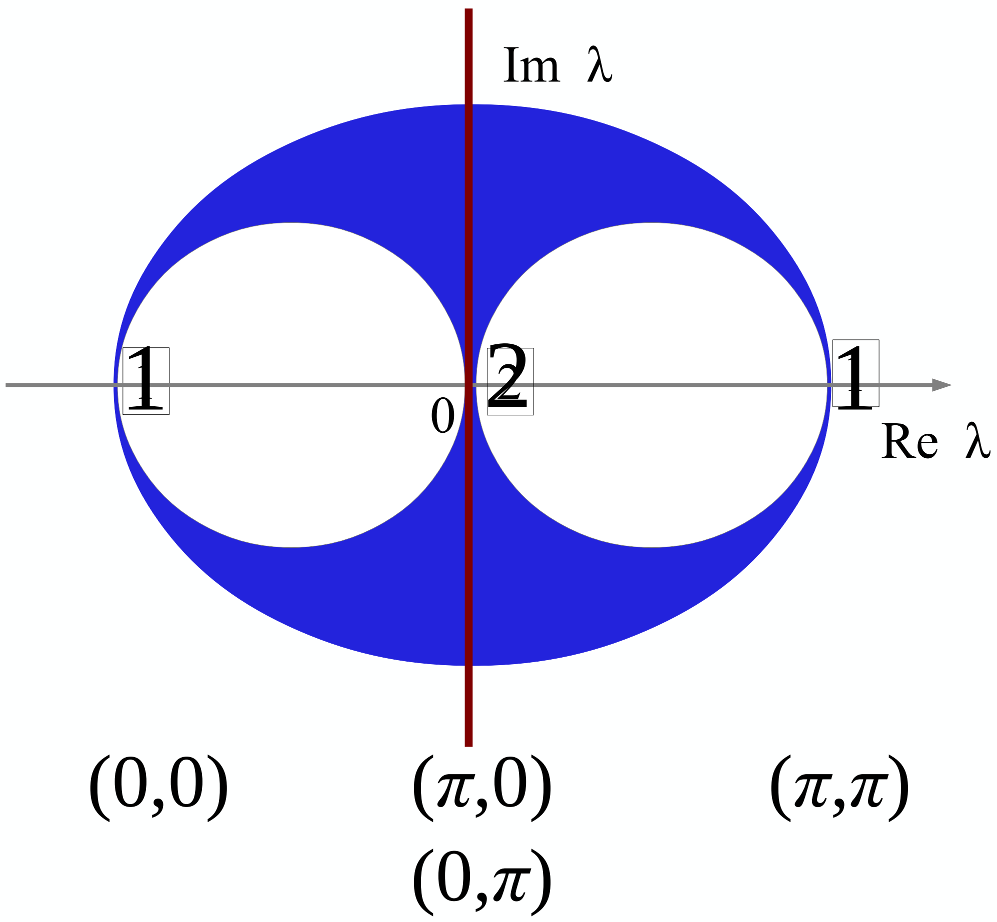
$(\pi,\pi,0)$
$(\pi,0,\pi)$
$(0,\pi,\pi)$

(π,π,π)

What is the meaning of the numbers?

Reconsider Naive and Wilson

Yumoto, TM (22)

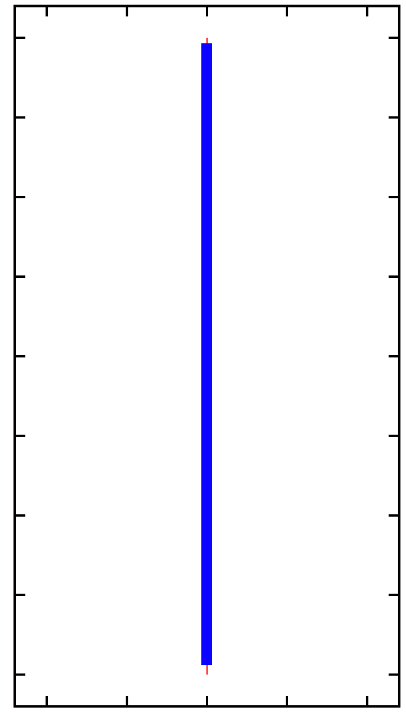


2D Wilson

What is the meaning of the numbers?

Reconsider Naive and Wilson

Yumoto, TM (22)



4D Naive

16

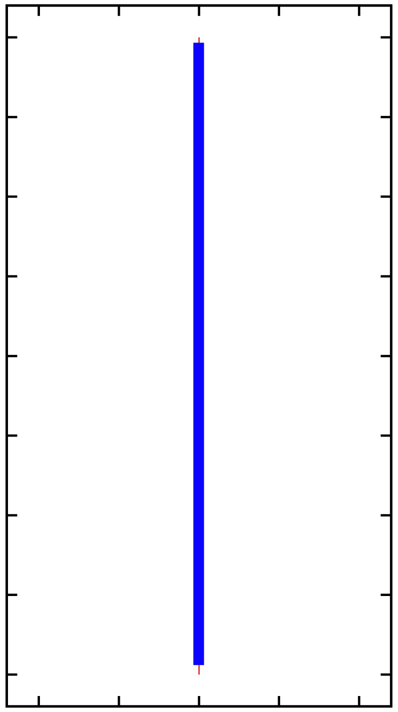
1	4	6	4	1
(0,0,0,0)	(π,0,0,0)	(π,π,0,0)	(π,π,π,0)	(π,π,π,π)
	⋮	⋮	⋮	

What is the meaning of the numbers?

The reason why $p=\pi$ becomes zero of Dirac operator is "periodicity"

Reconsider Naive and Wilson

Yumoto, TM (22)



4D Naive

$$16$$

1	4	6	4	1
$(0,0,0,0)$	$(\pi,0,0,0)$	$(\pi,\pi,0,0)$	$(\pi,\pi,\pi,0)$	(π,π,π,π)
	$\vdots$	$\vdots$	$\vdots$	

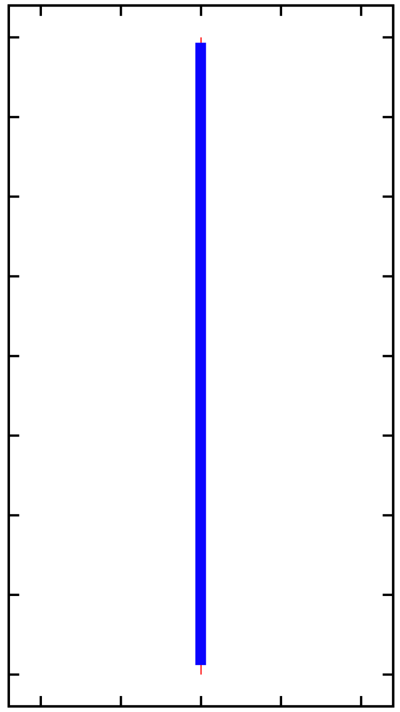
What is the meaning of the numbers?

It means these numbers are related to certain topological invariants

Reconsider Naive and Wilson

Yumoto, TM (22)

4D Naive



16

1	4	6	4	1
$(0,0,0,0)$	$(\pi,0,0,0)$	$(\pi,\pi,0,0)$	$(\pi,\pi,\pi,0)$	(π,π,π,π)
# of 0d holes	# of 1d holes	# of 2d holes	# of 3d holes	# of 4d holes

Topological invariants

Yumoto, TM (22)

- Topological invariant

Betti number is an indicator how many *n*-dimensional holes the space has.

$$\beta_n(M) \;=\; \text{rank of } H_n(M)=\text{Ker}\partial_n/\text{Im}\partial_{n+1}$$

n-th Betti number is a rank of *n*-th homology group

- **4D torus**

$\beta_0(M) = 1$ $\beta_1(M) = 4$ $\beta_2(M) = 6$ $\beta_3(M) = 4$ $\beta_4(M) = 1$

↓

Sum of Betti numbers is 16 $\longrightarrow$ # of naive fermion species !

Topological invariants

Yumoto, TM (22)

- Topological invariant

Betti number is an indicator how many n-dimensional holes the space has.

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n-th Betti number is a rank of n-th homology group

- **3D torus**

$$\beta_0(M) = 1 \quad \beta_1(M) = 3 \quad \beta_2(M) = 3 \quad \beta_3(M) = 1$$

Sum of Betti numbers is 8 $\rightarrow$ # of naive fermion species !

Topological invariants

Yumoto, TM (22)

- Topological invariant

Betti number is an indicator how many *n*-dimensional holes the space has.

$$\beta_n(M) \;=\; \text{rank of } H_n(M)=\text{Ker}\partial_n/\text{Im}\partial_{n+1}$$

n-th Betti number is a rank of *n*-th homology group

- **2D torus**

$$\beta_0(M) = 1 \qquad \beta_1(M) = 2 \qquad \beta_2(M) = 1$$

↓

Sum of Betti numbers is 4 → # of naive fermion species !

Topological invariants

Yumoto, TM (22)

- Topological invariant

Betti number is an indicator how many n-dimensional holes the space has.

$$\beta_n(M) \ = \ \text{rank of } H_n(M) = \mathrm{Ker}\partial_n / \mathrm{Im}\partial_{n+1}$$

n-th Betti number is a rank of n-th homology group

- **D-dim hyperball**

$$\beta_0(M) = 1 \quad \beta_1(M) = 0 \quad \beta_2(M) = 0 \ \ .....$$

Sum of Betti numbers is 1 $\rightarrow$ # of bulk fermion species !

Topological invariants

Yumoto, TM (22)

- Topological invariant

Betti number is an indicator how many n-dimensional holes the space has.

$$\beta_n(M) \;=\; \text{rank of } H_n(M) = \mathrm{Ker}\partial_n / \mathrm{Im}\partial_{n+1}$$

n-th Betti number is a rank of n-th homology group

- **D-dim hyperball**

$$\beta_0(M) = 1$$

$$Q_N = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & & 0 & 0 & 0 \\ -1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 1 \\ 0 & 0 & 0 & & 0 & -1 & 0 \end{pmatrix}$$

Sum of Betti numbers is 1 $\longrightarrow$ # of bulk fermion species !

Topological invariants

Yumoto, TM (22)

- Topological invariant

Betti number is an indicator how many n-dimensional holes the space has.

$$\beta_n(M) \ = \ \text{rank of } H_n(M)=\text{Ker}\partial_n/\text{Im}\partial_{n+1}$$

n-th Betti number is a rank of n-th homology group

- **$T^4 \times R^1$**

$\beta_0(M) = 1$ $\beta_1(M) = 4$ $\beta_2(M) = 6$ $\beta_3(M) = 4$ $\beta_4(M) = 1$ $\beta_5(M) = 0$

Sum of Betti numbers is 16 → maximal # of species !

Topological invariants

Yumoto, TM (22)

- Topological invariant

Betti number is an indicator how many n-dimensional holes the space has.

$$\beta_n(M) \;=\; \text{rank of } H_n(M) = \mathrm{Ker}\,\partial_n / \mathrm{Im}\,\partial_{n+1}$$

n-th Betti number is a rank of n-th homology group

- $\mathbf{T^2 \times R^2}$

$\beta_0(M) = 1 \quad \beta_1(M) = 2 \quad \beta_2(M) = 1 \quad \beta_3(M) = 0 \quad \beta_4(M) = 0$

↓

Sum of Betti numbers is 4 $\rightarrow$ maximal # of species !

Topological invariants

Yumoto, TM (22)

- Topological invariant

Betti number is an indicator how many *n*-dimensional holes the space has.

$$\beta_n(M) \;=\; \text{rank of } H_n(M) = \text{Ker}\,\partial_n / \text{Im}\,\partial_{n+1}$$

n-th Betti number is a rank of *n*-th homology group

- **2D Spheres**

$$\beta_0(M) = 1 \quad \beta_1(M) = 0 \quad \beta_2(M) = 1$$

↓

Kamata, Matsuura, TM, Ohta (16)
Yumoto, TM (21)

Sum of Betti numbers is 2 $\longrightarrow$ # of fermion species !

Topological invariants

Yumoto, TM (22)

	sum of $\beta_n(M)$	maximum # of species
1D torus	1+1	2
2D torus	1+2+1	4
3D torus	1+3+3+1	8
4D torus	1+4+6+4+1	16
T^D	$(1+1)^D$	2^D
Hyperball	1+0+0+….	1 for bulk
Sphere	1+0+0+…+1	2
$T^D \times R^d$	$2^D + 0$	2^D

Conjecture on fermion species

Yumoto, TM (22)

- **Conjecture**

A sum of Betti numbers of background space is
a maximal number of fermion species
when the fermion is defined on the discretized space.

How can we prove it?

Sketch of proof

Yumoto, TM (22)

Prove each of Betti numbers (β_0=1 and β_1=1) is equivalent to each of nullity of the Dirac matrix on 1D torus or 1D ball by regarding lattice fermion as chain complex.

↓

By use of Künneth theorem, elevate the above argument to higher dimensional space such as 4D Torus and Hyperball.

$$H_n(C_* \otimes C'_*) \cong \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*)$$

↓

Classify necessary conditions and complete proof.

Sketch of proof

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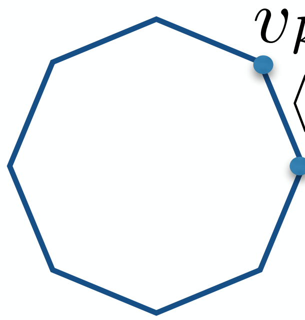
Prove each of Betti numbers (β_0=1 and β_1=1) is equivalent to each of nullity of the Dirac matrix on 1D torus or 1D ball by regarding lattice fermion as chain complex.

- 1D torus lattice fermion as Chain complex



$$\mathcal{C}^1(L_N^{(\mathrm{p})}) \equiv \left\{ \sum_{k=1}^{N} a_{k,k+1} \langle v_k, v_{k+1}\rangle \;\middle|\; a_{k,k+1} \in \mathbb{Z} \right\}$$

$$= \{a_{1,2}\langle v_1, v_2\rangle + a_{2,3}\langle v_2, v_3\rangle + \cdots + a_{N,1}\langle v_N, v_1\rangle \mid a_{1,2}, a_{2,3}, \cdots, a_{N,1} \in \mathbb{Z}\}$$

$L_N^{(\mathrm{p})}$: simplical complex $\rightarrow$ graph (1D lattice)

$\langle v_k, v_{k+1}\rangle$: 1-simplices of complex $L_N^{(\mathrm{p})}$ $\rightarrow$ edges (links)

v_k : boundaries of simplices $\rightarrow$ vertices (lattice points)

$a_{k,k+1}$: coefficients of simplices (should be abelian ring)

Sketch of proof

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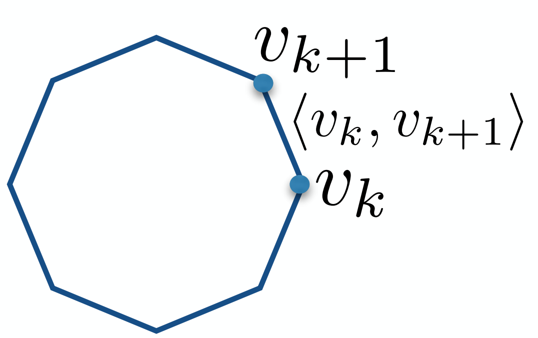

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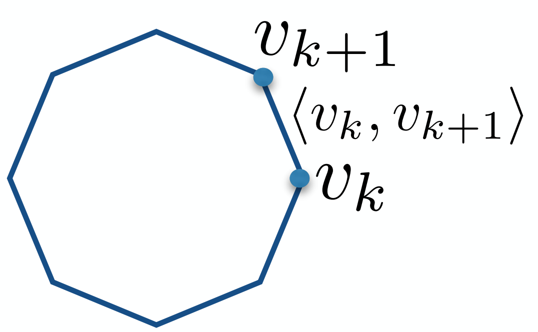

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$$= \{ a_{1,2} \langle v_1, v_2 \rangle + a_{2,3} \langle v_2, v_3 \rangle + \cdots + a_{N,1} \langle v_N, v_1 \rangle \mid a_{1,2}, a_{2,3}, \cdots, a_{N,1} \in \mathbb{Z} \}$$

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ . \\ . \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ . \\ . \\ 0 \end{pmatrix} \quad \cdots\cdots\cdots \quad v_N = \begin{pmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ 1 \end{pmatrix}$$

represented as linearly independent vectors

Sketch of proof

Yumoto, TM (22)

Prove each of Betti numbers (β_0=1 and β_1=1) is equivalent to each of nullity of the Dirac matrix on 1D torus or 1D ball by regarding lattice fermion as chain complex.

- prove β_1=1 is equivalent to degeneracy of Dirac matrix

$$\beta_1(M) \;=\; \text{rank of } H_1(M)=\mathrm{Ker}\partial_1$$

Sketch of proof

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Cycle group Ker∂_1 $\quad c_1 \in \mathcal{C}^1(L_N^{(\mathrm{p})})$

$$\partial_1 c_1 = \sum_{k=1}^{N} a_{k,k+1} \partial_1 \langle v_k, v_{k+1} \rangle = \sum_{k=1}^{N} a_{k,k+1} (v_k - v_{k+1})$$

$$= a_{1,2}(v_1 - v_2) + a_{2,3}(v_2 - v_3) + \cdots + a_{N,1}(v_N - v_1)$$

$$= (a_{12} - a_{N1}) v_1 + (a_{23} - a_{12}) v_2 + \cdots + (a_{N,1} - a_{N-1,N}) v_N = 0$$

$$a = a_{1,2} = a_{2,3} = \ldots = a_{N-1,N}$$

$$c_1' = a \sum_{k=1}^{N} \langle v_k, v_{k+1} \rangle \qquad c_1' \in \mathrm{Ker}\, \partial_1$$

Sketch of proof

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$$a = a_{1,2} = a_{2,3} = \ldots = a_{N-1,N}$$

$$H_1(L_N^{(\mathrm{p})}) = \mathrm{Ker}\, \partial_1 = \{ a \left(\langle v_1, v_2 \rangle + \langle v_2, v_3 \rangle + \cdots + \langle v_N, v_1 \rangle \right) \mid a \in \mathbb{Z} \} \cong \mathbb{Z}$$

Sketch of proof

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$$\beta_1(L_N^{(\mathrm{p})}) = 1$$

Sketch of proof

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$$c_1' = a\sum_{k=1}^{N} \langle v_k, v_{k+1}\rangle \qquad c_1' \in \mathrm{Ker}\,\partial_1$$

$$\Rightarrow \quad \partial_1 c_1' = a\sum_{k=1}^{N} \partial_1 \langle v_k, v_{k+1}\rangle = a\sum_{k=1}^{N} (v_k - v_{k+1}) = 0$$

$$\Rightarrow \quad \begin{aligned} \partial_1 c_1' &= a\,(v_2 - v_1 + v_2 - v_3 + \cdots + v_N - v_1) \\ &= a\,(-v_2 + v_N + v_1 - v_3 + \cdots + v_{N-1} - v_1) \end{aligned}$$

Sketch of proof

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$$\Rightarrow \quad \partial_1 c_1' = a\,(v_2 - v_1 + v_2 - v_3 + \cdots + v_N - v_1)$$
$$= a\,(\underline{-v_2 + v_N} + \underline{v_1 - v_3} + \cdots + \underline{v_{N-1} - v_1}) = 0$$

Sketch of proof

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$$\begin{pmatrix} 0 \\ -1 \\ 0 \\ . \\ . \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \\ -1 \\ . \\ . \\ 0 \end{pmatrix} \qquad \cdots\cdots \qquad \begin{pmatrix} -1 \\ 0 \\ . \\ . \\ 1 \\ 0 \end{pmatrix}$$

Sketch of proof

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$$\begin{aligned}\partial_1 c_1' &= a\left(v_2 - v_1 + v_2 - v_3 + \cdots + v_N - v_1\right)\\ &= a\left(\underline{-v_2 + v_N} + \underline{v_1 - v_3} + \cdots + \underline{v_{N-1} - v_1}\right) = 0\end{aligned}$$

$$\underbrace{\begin{pmatrix} 0 \\ -1 \\ 0 \\ \cdot \\ \cdot \\ 1 \end{pmatrix}}_{w_1} \quad \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}}_{w_2} \quad \cdots\cdots \quad \underbrace{\begin{pmatrix} -1 \\ 0 \\ \cdot \\ \cdot \\ 1 \\ 0 \end{pmatrix}}_{w_N}$$

$$\mathcal{D}_{1\mathrm{D}} = \frac{1}{2}\Big(w_1, w_2 \cdots w_N\Big)$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & & 0 & 0 & -1 \\ -1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 1 \\ 1 & 0 & 0 & & 0 & -1 & 0 \end{pmatrix}$$

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⇨ degeneracy (nullity) of Dirac matrix

↓

zero mode (fermion species)

Sketch of proof

Yumoto, TM (22)

Prove each of Betti numbers (β_0=1 and β_1=1) is equivalent to each of nullity of the Dirac matrix on 1D torus or 1D ball by regarding lattice fermion as chain complex.

- prove β_0=1 is equivalent to degeneracy of Dirac matrix

$$\beta_0(M) \;=\; \text{rank of } H_0(M) = \ker\partial_0/\mathrm{Im}\partial_1$$

↓

$$w_1 - w_2 + w_3 - w_4 \cdots + w_{N-1} - w_N = 0$$

⇒

degeneracy (nullity) of Dirac matrix

↓

zero mode (fermion species)

Sketch of proof

Yumoto, TM (22)

Prove each of Betti numbers (β_0=1 and β_1=1) is equivalent to each of nullity of the Dirac matrix on 1D torus or 1D ball by regarding lattice fermion as chain complex.

↓

By use of Künneth theorem, elevate the above argument to higher dimensional space such as 4D Torus and Hyperball.

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↓

Classify necessary conditions and complete proof.

Summary

- Wilson fermion has an unknown aspect charming in both theoretical and practical terms.
- CB Wilson Schwinger model is in the same universality class of XXZ spin chain.
- Lattice fermions are interpreted as spectral graphs. It means we can study them in terms of topology of graphs.
- New conjecture on fermion doubling is proposed:
  The maximal number of doublers is the sum of Betti numbers.

Summary

- Wilson fermion has an unknown aspect charming in both theoretical and practical terms.
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- Lattice fermions are interpreted as spectral graphs. It means we can study them in terms of topology of graphs.
- New conjecture on fermion doubling is proposed:
  The maximal number of doublers is the sum of Betti numbers

I hope you join Yumoto's talk too!

Symmetry-protected topological phase

- G-Symmetry Protected Topological phase (SPT) Wen, et.al., (13)

1. Unique ground state with trivial gap as long as G is unbroken
2. Gap should be closed when moving to another SPT
3. Massless modes at boundary btwn two different SPTs
4. 't Hooft anomaly cancelled btwn bulk & boundary with gauged G

All 't Hooft anomalies are (expected to be) classified by SPTs.

Kapustin (14), Witten (15), Yonekura (16), Yonekura, Witten (19)

Symmetry-protected topological phase

ex.) (2+1)-dim free massive Dirac fermion = U(1) SPT = IQHS

m < 0	m > 0
$Z = e^{-2\pi i \eta}$	Z = 1
(η: APS η-invariant $\equiv \sum_i \mathrm{sgn}[\lambda_i]$)	

2-dim chiral fermions Z_{bndry}

Symmetry-protected topological phase

ex.) (2+1)-dim free massive Dirac fermion = U(1) SPT = IQHS

m < 0	m > 0
$Z = e^{-2\pi i \eta}$	Z = 1
APS index theorem cf.)Fukaya, Onogi, Yamaguchi,et.al.(17-19) $= e^{\frac{i}{4\pi}\int AdA}$	

2-dim chiral fermions $Z_{\rm bndry}$

$$Z_{\rm total} = Z_{\rm bulk} \cdot Z_{\rm bndry} \longrightarrow Z_{\rm bulk} e^{\frac{i}{4\pi}\int F} \cdot Z_{\rm bndry} e^{-\frac{i}{4\pi}\int F} = Z_{\rm total}$$

’t Hooft anomaly is cancelled between bulk and boundary

Staggered symmetry

$$C_0: \ \chi_x \to \epsilon_x \bar{\chi}_x^T, \quad \bar{\chi}_x \to -\epsilon_x \chi_x^T, \quad U_{x,\nu} \to U_{x,\nu}^* \, .$$

$$\Xi_\mu: \ \chi_x \to \zeta_\mu(x) \chi_{x+\hat{\mu}}, \quad \bar{\chi}_x \to \zeta_\mu(x) \bar{\chi}_{x+\hat{\mu}}, \quad U_{x,\nu} \to U_{x+\hat{\mu},\nu} \, ,$$

$\zeta_1(x) = (-1)^{x_2+x_3+x_4}$, $\zeta_2(x) = (-1)^{x_3+x_4}$, $\zeta_3(x) = (-1)^{x_4}$ and $\zeta_4(x) = 1$.

$$I_\mu: \ \chi_x \to (-1)^{x_\mu} \chi_{x'}, \quad \bar{\chi}_x \to (-1)^{x_\mu} \bar{\chi}_{x'}, \quad U_{x,\nu} \to U_{x',\nu} \, ,$$

$$R_{\rho\sigma}: \ \chi_x \to S_R(\tilde{R}^{-1}x) \chi_{\tilde{R}^{-1}x}, \quad \bar{\chi}_x \to S_R(\tilde{R}^{-1}x) \bar{\chi}_{\tilde{R}^{-1}x}, \quad U_{x,\nu} \to U_{\tilde{R}x,\nu}$$