Bridging 3d coupled-wire models and cellular topological states

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Collaborator: Akira Furusaki (RIKEN)

arXiv:2112.07926

格子上の場の理論と連続空間上の場の理論@YITP, Jul 20, 2022

Outline

1. Introduction

    --- Topological order and fracton order

    --- Coupled-wire construction, Cellular topological state/Topological defect network

2. General recipe for 3d coupled-wire models

    --- K-matrix formalism for 2d Abelian topological orders and gapped interfaces

    --- Coupled-edge and coupled-wire Hamiltonians

3. Applications

    --- 3d topological order, fracton order, and their hybrid

4. Summary and outlooks

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Gapped phases of matter

Chen, Gu, & Wen, PRB **82**, 155138 (2010).

Quantum phases: A family of ground states that can be smoothly connected with each other by continuous deformations of the Hamiltonian.

$$H\,|\psi_n\rangle = E_n\,|\psi_n\rangle \qquad (E_0 \leq E_1 \leq E_2 \leq \cdots)$$

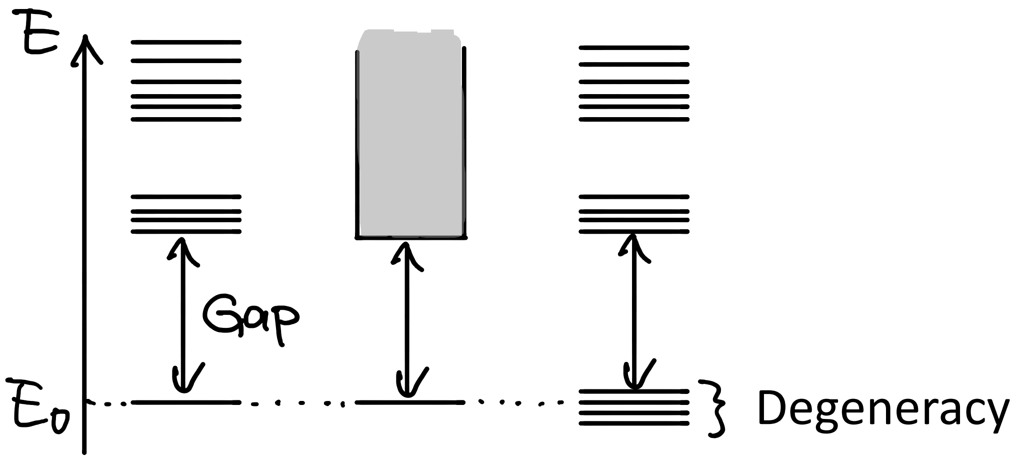
Energy eigenvalues

$|\psi_0\rangle$: Ground state of Hamiltonian H

1. We focus on Hamiltonians with short-range interactions = local Hamiltonians.

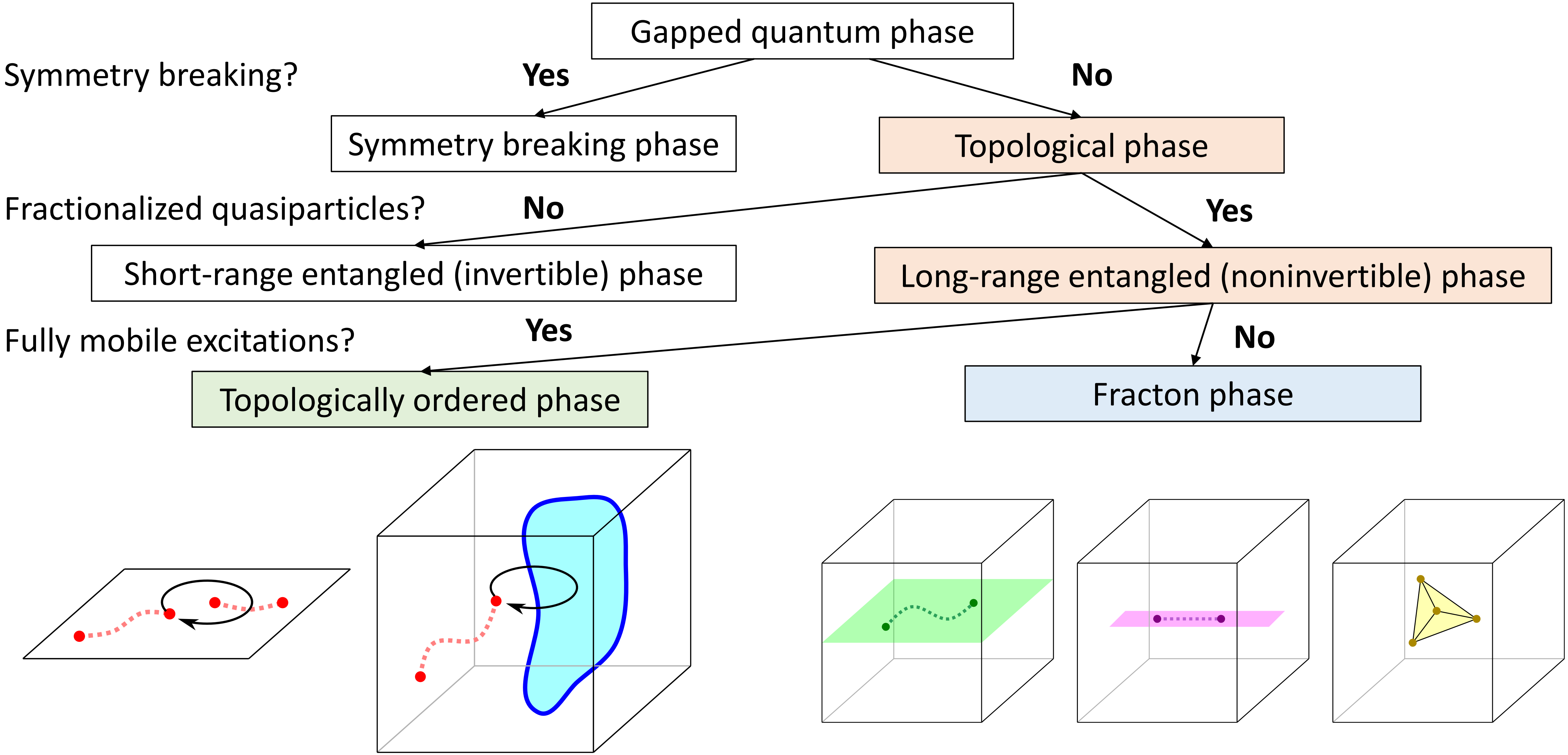
$H = \sum_i h_i$: Sum of Hermitian operators on finite supports

2. We assume a spectral gap above the ground state.



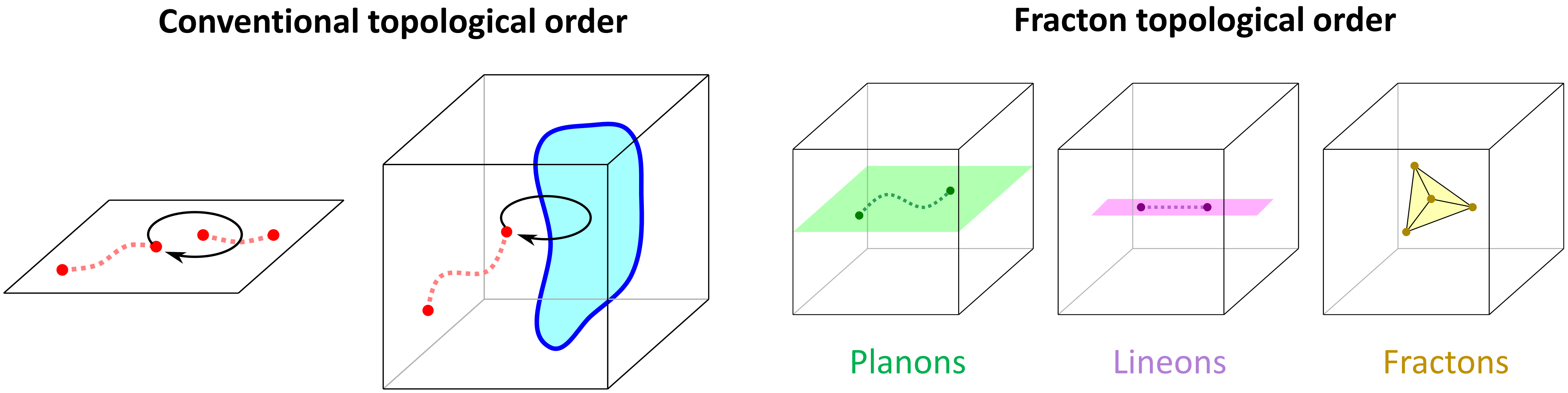
Gapped phases of matter

Wen, Rev. Mod. Phys. **89**, 041004 (2017).



Fracton topological order

Nandkishore & Hermele, Ann. Rev. Cond. Mat. Phys. **10**, 295 (2019).
Pretko, Chen, & You, Int. J. Mot. Phys. A **35**, 2030003 (2020).



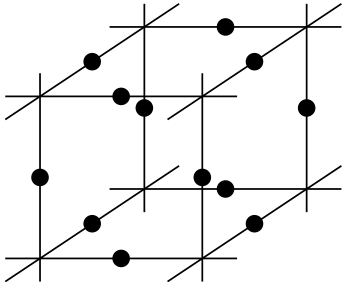
Properties	Conventional TO	Fracton TO
Shape of quasiparticles (3d)	Particle or loop	Particle
Mobility of quasiparticles	Full space	Lower-dimensional subspaces
Degeneracy on torus	Constant	Roughly exp(L) with linear size L
Emergent symmetry	Higher-order symmetries	Subsystem or fractal symmetries
Quantum field theory	Topological QFT (TQFT)	Something beyond TQFT

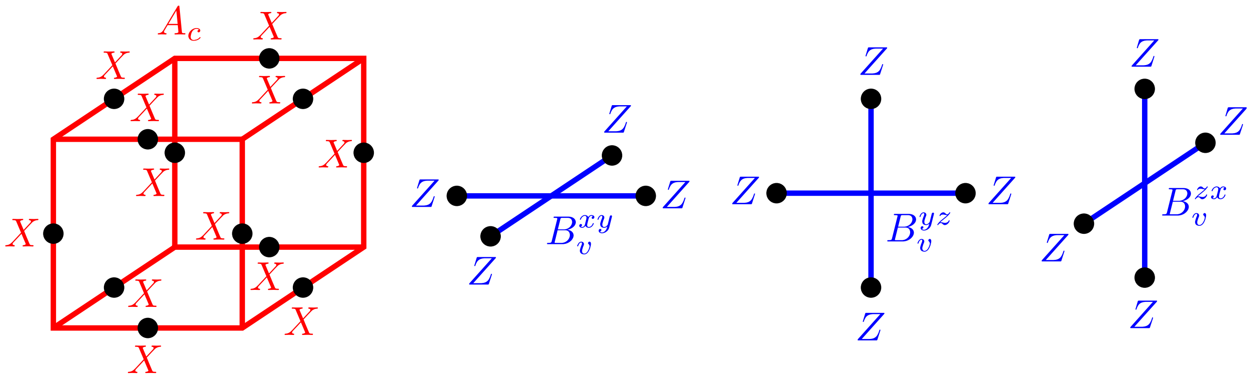
E.g., Seiberg & Shao, SciPost Phys. **10**, 003 (2021).

Example: X-cube model

Vijay, Haah, & Fu, PRB **94**, 235157 (2016).

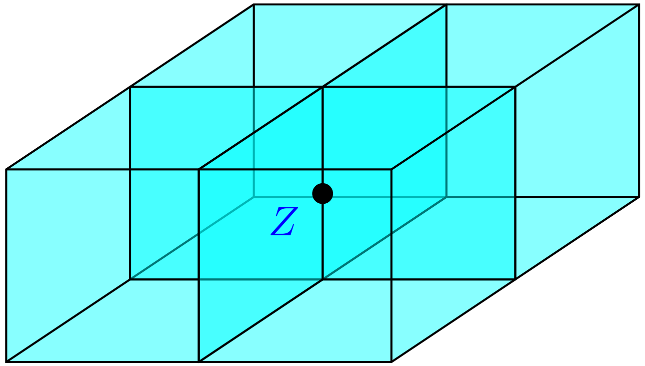
X-cube model is defined on the 3d cubic lattice: $H = -\sum (A_c + B_v^{xy} + B_v^{yz} + B_v^{zx})$

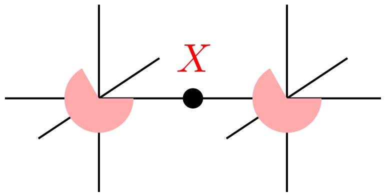




--- Ground state satisfies $A_c |\Phi\rangle = B_v |\Phi\rangle = |\Phi\rangle$

--- Quasiparticles mobile only in 0d (fraction), 1d (lineon), or 2d (planon) subspaces.





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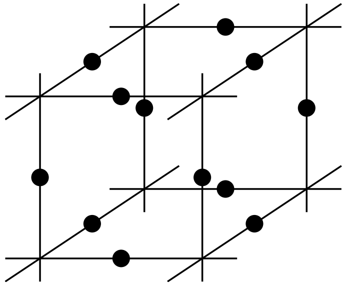

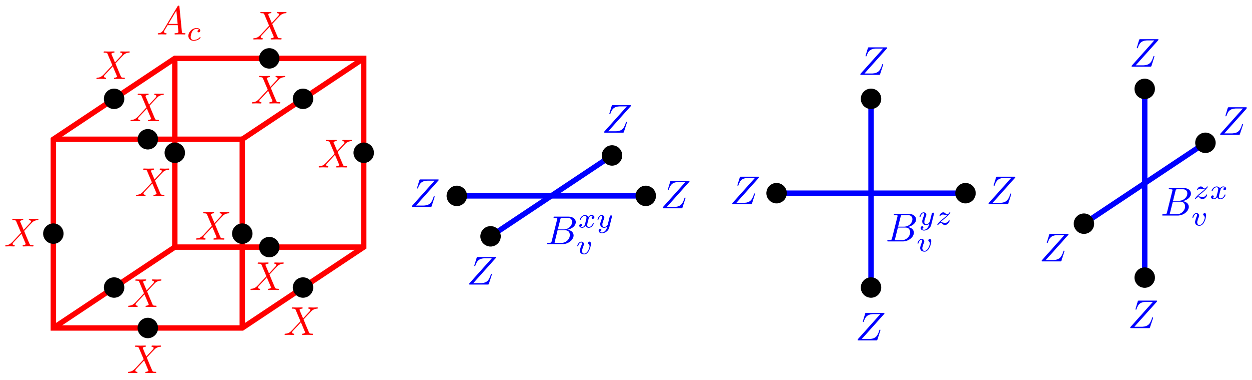

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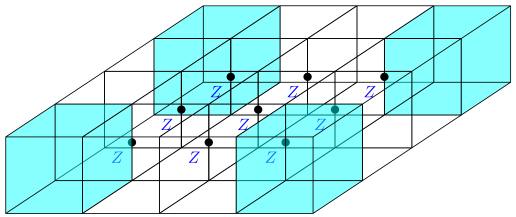
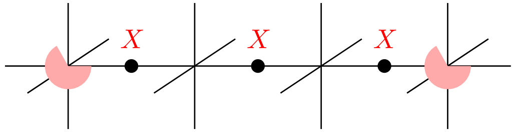
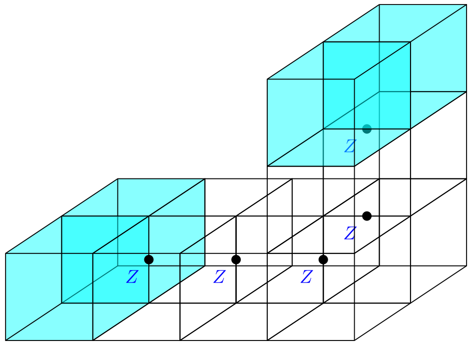
Monopole: fracton



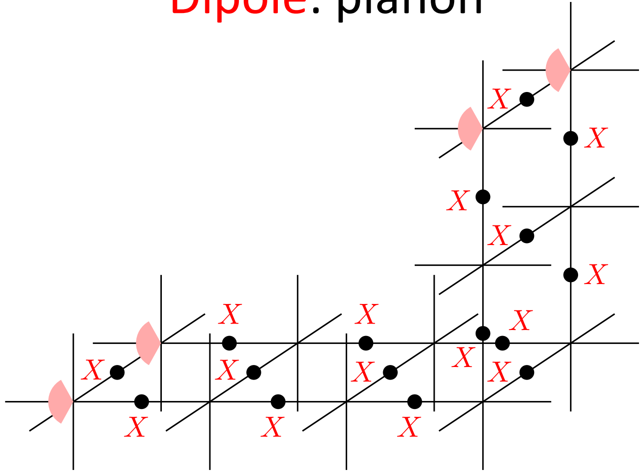
Monopole: lineon



Dipole: planon



Dipole: planon



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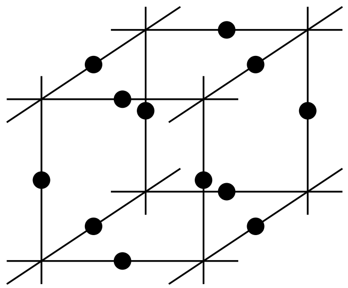
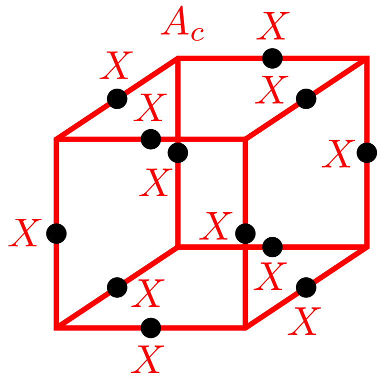
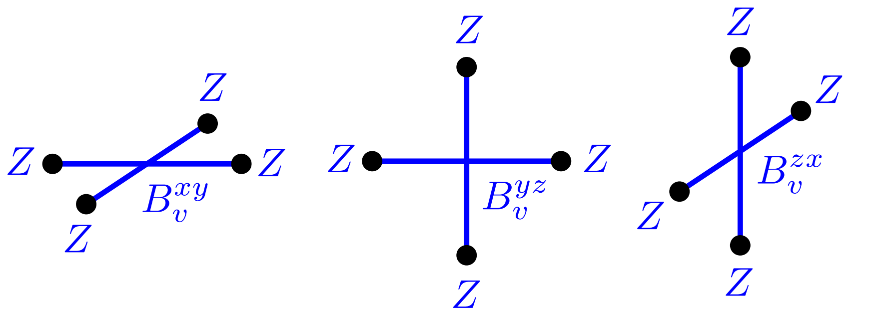




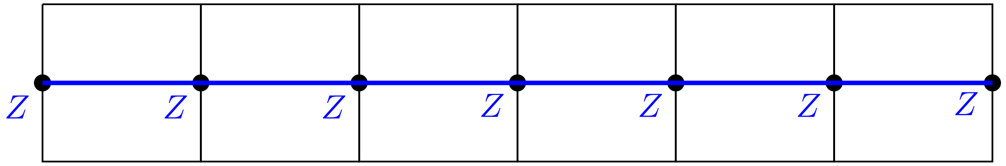


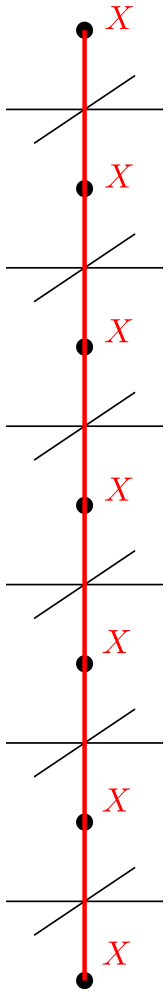
--- Ground state satisfies $A_c |\Phi\rangle = B_v |\Phi\rangle = |\Phi\rangle$

--- Quasiparticles mobile only in 0d (fraction), 1d (lineon), or 2d (planon) subspaces.

--- Nonlocal operators commuting with Hamiltonian (subsystem symmetry generators)

→ Subextensive GS degeneracy $2^{2(L_x+L_y+L_z)-3}$ on T^3 robust against local perturbations.





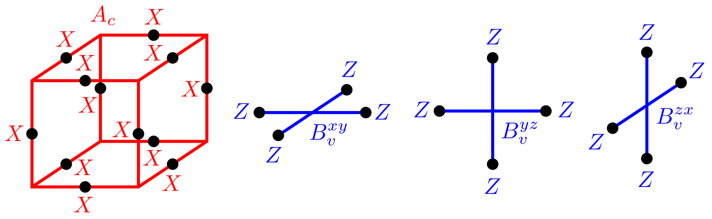
Lattice models for topological or fracton order

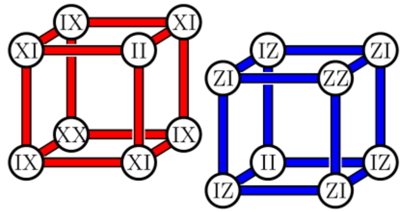
Why are lattice Hamiltonians important?

--- To understand basic properties of topological or fraction orders

--- To seek experimentally feasible models

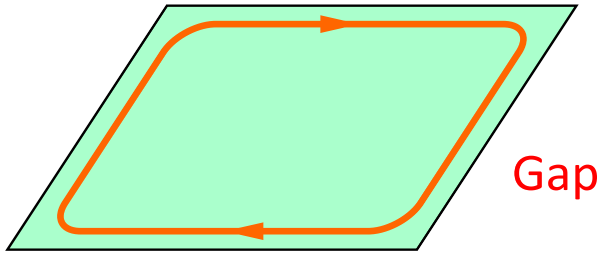
--- To derive effective QFT via appropriate continuum limit.





There are some fundamental limitations:

--- No commuting-projector or frustration-free Hamiltonians for 2d chiral topological orders.



Gapless chiral edge mode

Kapustin & Fidkowski, Commun. Math. Phys. **373**, 763 (2020)
Kapustin & Spodyneiko, PRB **101**, 045137 (2020).
Lemm & Mozgunov, J. Math. Phys. **60**, 051901 (2019).

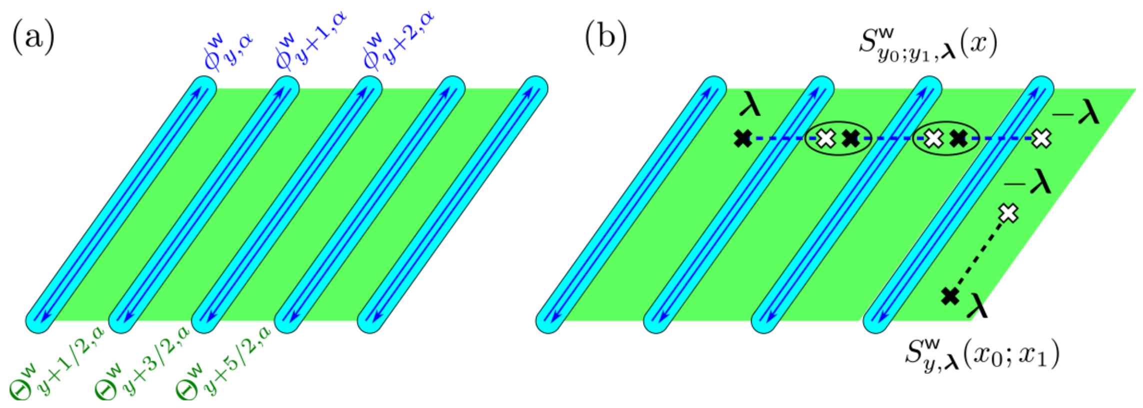
--- There might be similar limitations for 3d topological orders with gapless surface states.

Coupled-wire construction

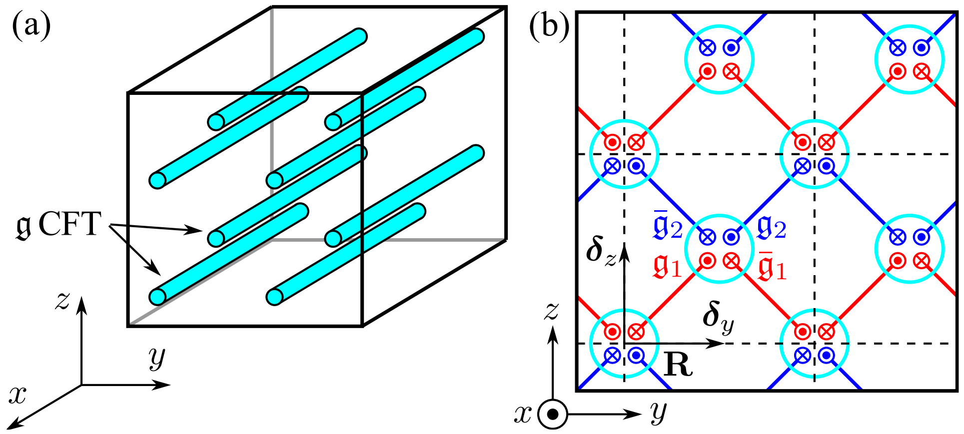
Kane, Mukhopadhyay, & Lubensky, PRL **88**, 036401 (2002).
Meng, Eur. Phys. J. Special Topics **229**, 527 (2020).

Hamiltonians from arrays of quantum wires: **hybrid of lattice and continuum**

2d coupled-wire model



3d coupled-wire model



--- Exactly solvable models for various 2d topological phases including chiral ones

--- Applications to some 3d topological phases

Iadecola, Neupert, Chamon, & Mudry, PRB **93**, 195136 (2016).
Fuji & Furusaki, PRB **99**, 241107 (2019).
Sullivan, Iadecola, & Cheng, PRB **99**, 245138 (2021)...

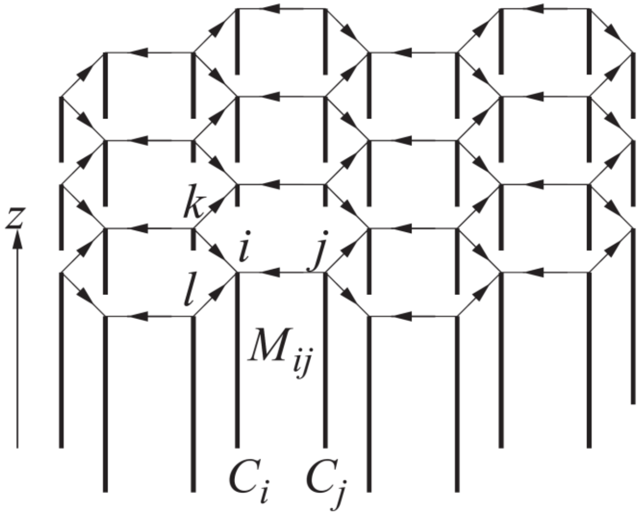
Models related to infinite-layer Chern-Simons theory:
Sullivan, Dua, & Cheng, Phys. Rev. Res. **3**, 02323 (2021).
Sullivan, Dua, & Cheng, arXiv:2109.13267.

--- Investigation for microscopic lattice models

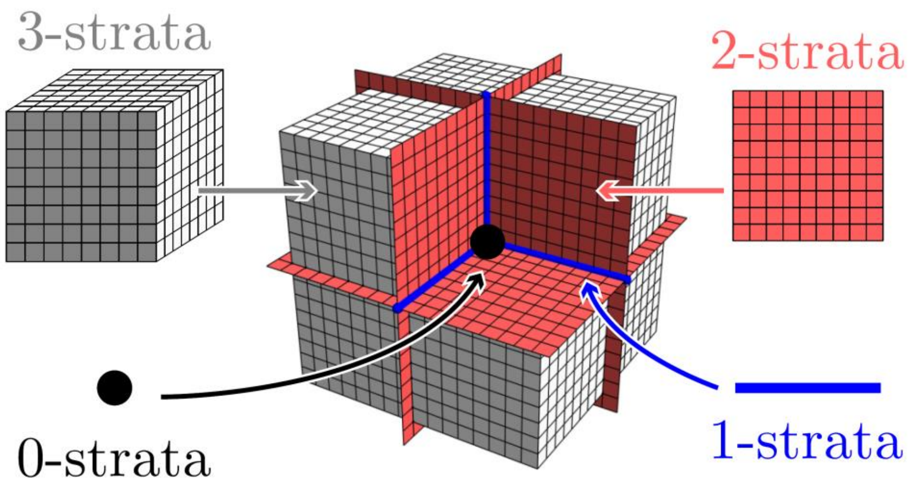
Cellular construction

Wen, Phys. Rev. Res. **2**, 033300 (2020).
Aasen, Bulmash, Prem, Slagle, & Williamson, Phys. Rev. Res. **2**, 043165 (2020).
Wang, Phys. Rev. Res. **4**, 023258 (2022).

Cellular topological state



Topological defect network



--- Decompose the 3d space into 0d,1d, 2d, and 3d cells.

--- Place topological orders on each cell.

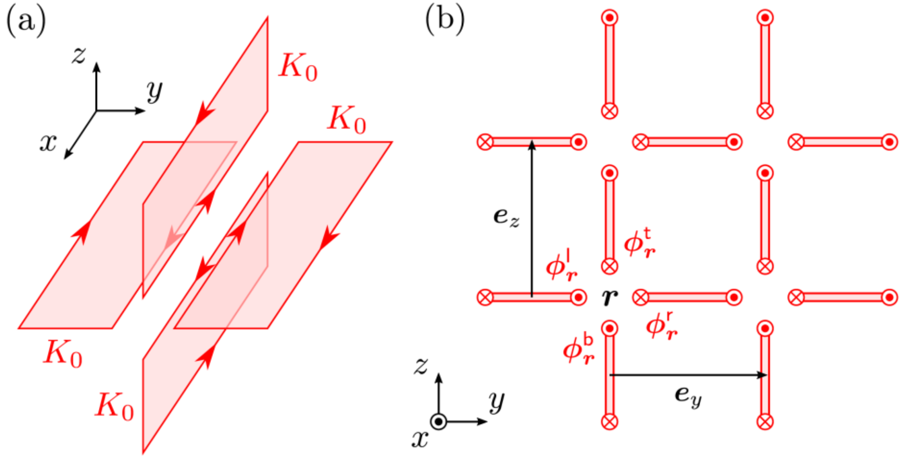
--- Couple cells via gapped interfaces/topological defects.

→ Nontrivial quasiparticle dynamics leads to various 3d topological order or fracton order.

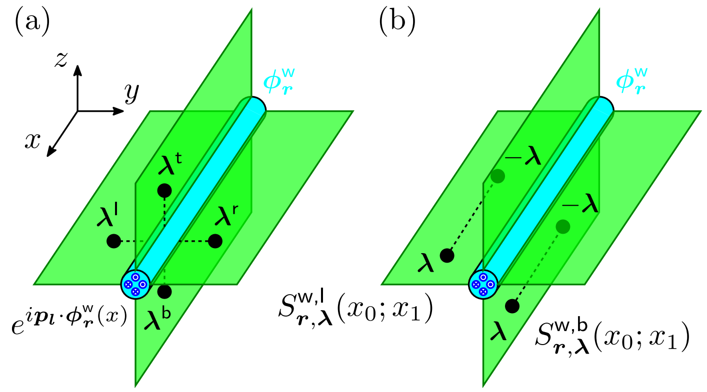
--- Not immediately provide Hamiltonians.

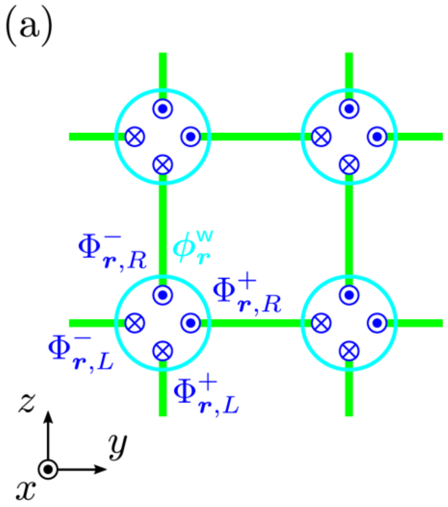
Bridge between cellular and coupled-wire constructions

Cellular topological states



Coupled-wire models





Input:

Cellular topological states built from 2d Abelian topological orders and their 1d gapped interfaces.

Output:

Coupled-wire Hamiltonian for 3d topological order or fracton order.

--- They can have fully chiral gapless surface states.

--- They can (in principle) be written in terms of lattice degrees of freedom such as spins or electrons.

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    --- Topological order and fracton order

    --- Coupled-wire construction, Cellular topological state/Topological defect network

2. General recipe for 3d coupled-wire models

    --- K-matrix formalism for 2d Abelian topological orders and gapped interfaces

    --- Coupled-edge and coupled-wire Hamiltonians

3. Applications

    --- 3d topological order, fracton order, and their hybrid

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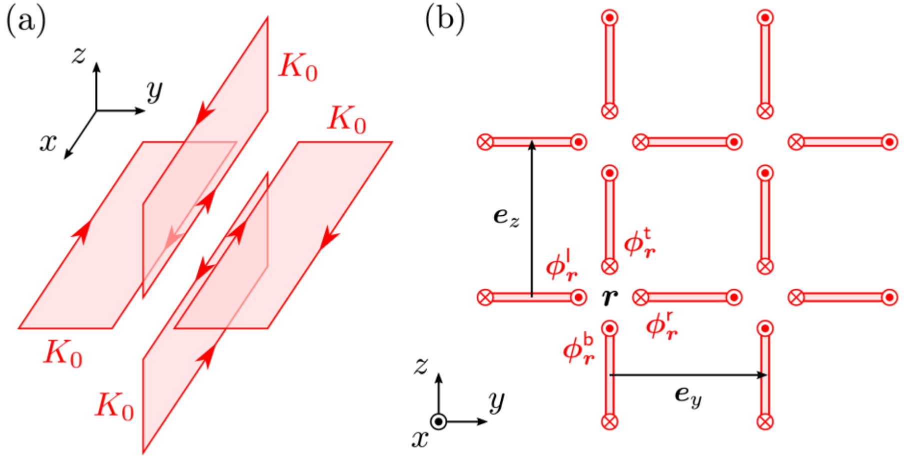
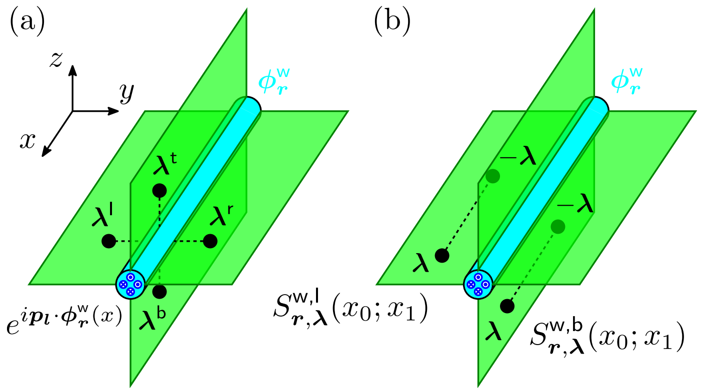
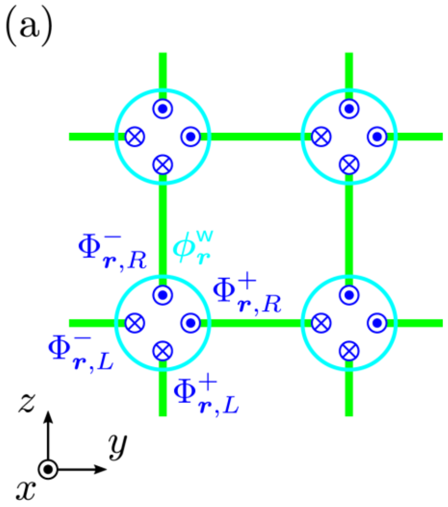
Bridge between cellular and coupled-wire constructions

Cellular topological states



Coupled-wire models





Input:

Cellular topological states built from 2d Abelian topological orders and their 1d gapped interfaces.

To be more specific, we need:

1. Cellular structure → Cells of thin 2d strips extended along the x axis
2. 2d Abelian topological orders → K matrix
3. 1d gapped interfaces → Lagrangian subgroup

K matrix formalism for 2d Abelian topological order

Let K an $N \times N$ symmetric integer matrix, which is called the **K matrix**.

Wen, Adv. Phys. **44**, 405 (1995).

Bulk theory: Chern-Simon theory

$$\mathcal{L}_{\mathrm{CS}} = -\sum_{I,J=1}^{N} \frac{K_{IJ}}{4\pi} \epsilon^{\mu\nu\lambda} a_{\mu}^{I} \partial_{\nu} a_{\lambda}^{J}$$

Edge theory: free boson theory

$$\mathcal{H}_{\mathrm{edge}} = \frac{1}{4\pi} \int dx \sum_{I,J=1}^{N} v_{IJ} \partial_x \phi_I \partial_x \phi_J$$

$$[\partial_x \phi_I(x), \phi_J(x')] = 2\pi i K_{IJ}^{-1} \delta(x - x')$$

All topological properties are encoded in the K matrix:

Local boson or fermion excitations

Quasiparticles: Integer vectors $\boldsymbol{m} \in \mathbb{Z}^N$

Identification: $\boldsymbol{m} \sim \boldsymbol{m} + K\mathbb{Z}^N$

Vacuum: $\boldsymbol{m}_{\mathbf{1}} = \mathbf{0}$

Topological angle: $\theta_a = 2\pi \boldsymbol{m}_a^T K^{-1} \boldsymbol{m}_a$

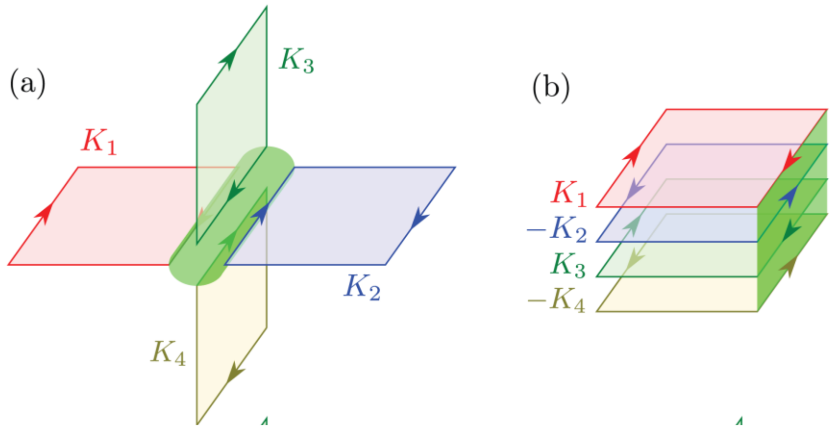
Mutual statistics: $\theta_{ab} = \pi \boldsymbol{m}_a^T K^{-1} \boldsymbol{m}_b$

Chiral central charge: $c_- =$ (# of positive eigenvalues of K) – (# of negative eigenvalues of K)

Theory of gapped interfaces

Levin, PRX **3**, 021009 (2013).
Barkeshli, Jian, & Qi, PRB **88**, 235103 (2013).

Folding trick: Gapped interfaces between TOs → Gapped boundary of a stack of TOs



$$K_{\mathrm{e}} = \begin{pmatrix} K_1 & & & \\ & -K_2 & & \\ & & K_3 & \\ & & & -K_4 \end{pmatrix}$$

Gapped interface is described by a subset of quasiparticles **(Lagrangian subgroup)** $L = \{\boldsymbol{l}\} \subseteq \mathbb{Z}^N$

1. All quasiparticles in L have bosonic or fermionic self statistics $\boldsymbol{l}_a^T K_{\mathrm{e}}^{-1} \boldsymbol{l}_a \in \mathbb{Z}$
2. Any two quasiparticles in L have trivial mutual statistics $\boldsymbol{l}_a^T K_{\mathrm{e}}^{-1} \boldsymbol{l}_b \in \mathbb{Z}$
3. Quasiparticles not in L have nontrivial mutual statistics with at least one quasiparticle in L

$$\boldsymbol{n}^T K_{\mathrm{e}}^{-1} \boldsymbol{l}_a \notin \mathbb{Z} \text{ for } \boldsymbol{n} \notin L$$

→ Quasiparticles in L are condensed at the interface.

Theory of gapped interfaces

Levin, PRX **3**, 021009 (2013).
Barkeshli, Jian, & Qi, PRB **88**, 235103 (2013).

Lagrangian subgroup $\boldsymbol{l} \in L$ → Gapping potential (Sine-Gordon Hamiltonian)

In general, we need to add $2N$ extra quantum wires.

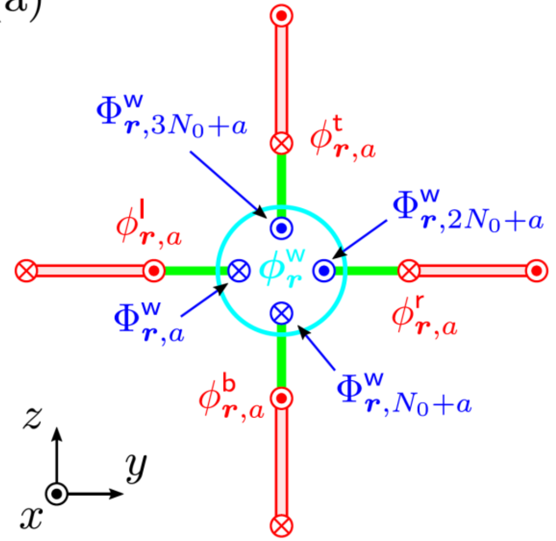
$\widetilde{\boldsymbol{\phi}}_{\boldsymbol{r}} = (\boldsymbol{\phi}^{\mathsf{l}}_{\boldsymbol{r}}, \boldsymbol{\phi}^{\mathsf{b}}_{\boldsymbol{r}}, \boldsymbol{\phi}^{\mathsf{r}}_{\boldsymbol{r}}, \boldsymbol{\phi}^{\mathsf{t}}_{\boldsymbol{r}}, \boldsymbol{\phi}^{\mathsf{w}}_{\boldsymbol{r}})^T$ --- Edge modes from the four strips and additional wires.

$\widetilde{K} = \begin{pmatrix} K_{\mathsf{e}} & \\ & K_{\mathsf{w}} \end{pmatrix}$ --- Extended $2N \times 2N$ K matrix

--- If L contains only bosons. → $K_{\mathsf{w}} = X^{\otimes N} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

--- If L contains at least one fermion. → $K_{\mathsf{w}} = Z^{\otimes N} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(a)



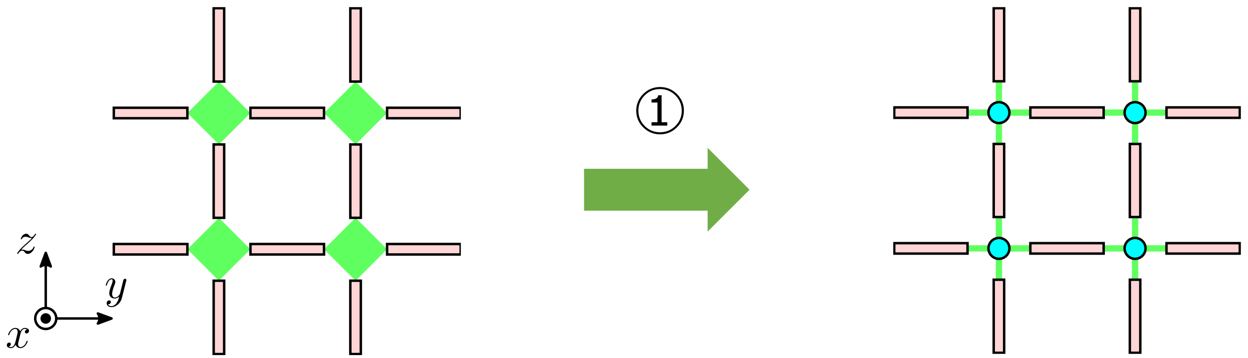
Gapping potential is specified by a set of integer vectors $\{\widetilde{\boldsymbol{\Lambda}}_I\}$

$$\widetilde{\mathcal{V}} = -g \int dx \sum_{I=1}^{4N} \cos(\widetilde{\boldsymbol{\Lambda}}_I^T \widetilde{K} \widetilde{\boldsymbol{\phi}})$$

General algorithm can be found in Barkeshli, Jian, & Qi, PRB **88**, 235103 (2013).

Derivation of coupled-wire models

Cellular topological state | Coupled-edge model



① Given 2d Abelian TOs K and gapped interfaces L → Integer vectors $\{\widetilde{\mathbf{\Lambda}}_I\}$ for a gapping potential

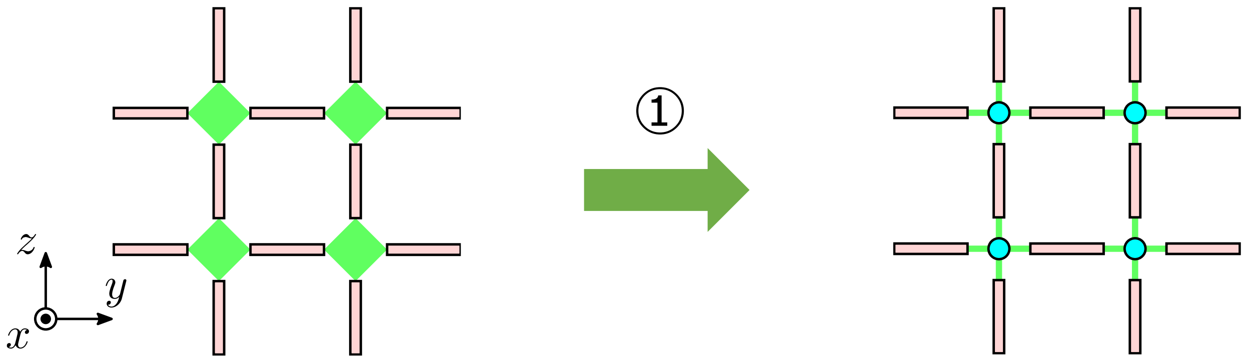
We obtain a "coupled-edge" Hamiltonian, but this is not a "coupled-wire" Hamiltonian that we want.

--- Microscopic origin of edge modes is not immediately clear.

→ We need one more step.

Derivation of coupled-wire models

Cellular topological state   Coupled-edge model



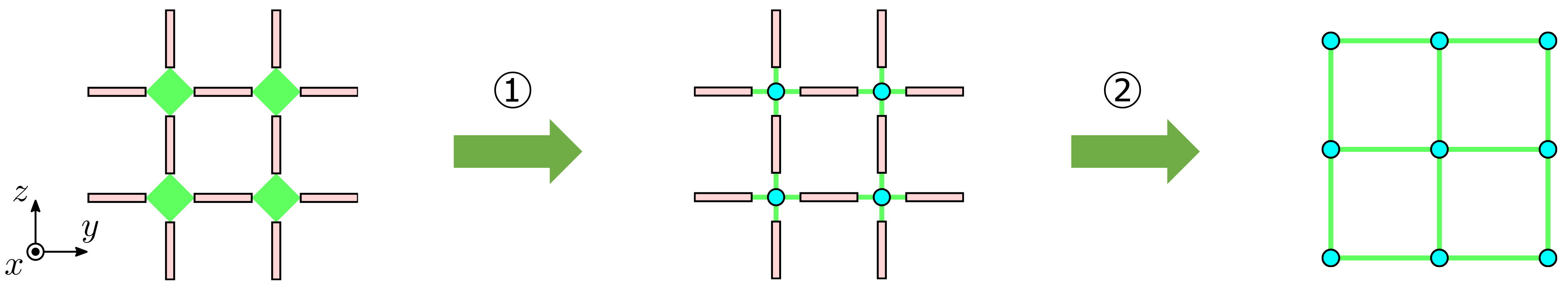
① Given 2d Abelian TOs K and gapped interfaces L → Integer vectors $\{\widetilde{\mathbf{\Lambda}}_I\}$ for a gapping potential

② We can find integer vectors $\{\widetilde{\mathbf{\Lambda}}_I\}$ of a special form $\begin{pmatrix} \widetilde{\mathbf{\Lambda}}_1 & \cdots & \widetilde{\mathbf{\Lambda}}_{2N} \end{pmatrix} = \begin{pmatrix} I_{2N} \\ \Lambda_{\mathrm{w}} \end{pmatrix}$

→ Edge modes are coupled only through additional quantum wires.

Derivation of coupled-wire models

Cellular topological state — Coupled-edge model — Coupled-wire model



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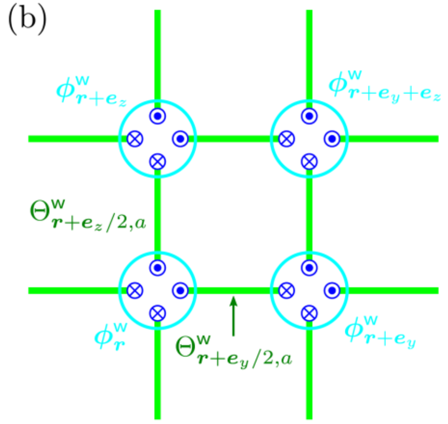
→ Edge modes are coupled only through additional quantum wires.

--- In the thin strip limit, quasiparticle tunnelings across the strip become dominant.

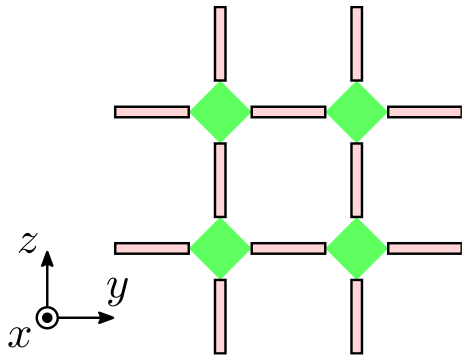
→ A coupled-wire Hamiltonian is derived from 2nd-order perturbation theory.

→ Shrinking and removing the strips yields **coupled-wire model**!



3d coupled-wire models

Input: 2d Abelian topological orders $K_{\mathrm{e}} = \begin{pmatrix} K_0 & & & \\ & K_0 & & \\ & & -K_0 & \\ & & & -K_0 \end{pmatrix}$ and gapped interfaces L



Output: Coupled-wire Hamiltonian

$$\mathcal{H}_{\mathrm{w}} = \frac{1}{4\pi}\int dx \sum_{\boldsymbol{r}\in\mathbb{Z}^2}\sum_{\alpha,\beta=1}^{4N} v^{\mathrm{w}}_{\alpha\beta}\partial_x\phi^{\mathrm{w}}_{\boldsymbol{r},\alpha}\partial_x\phi^{\mathrm{w}}_{\boldsymbol{r},\beta},$$

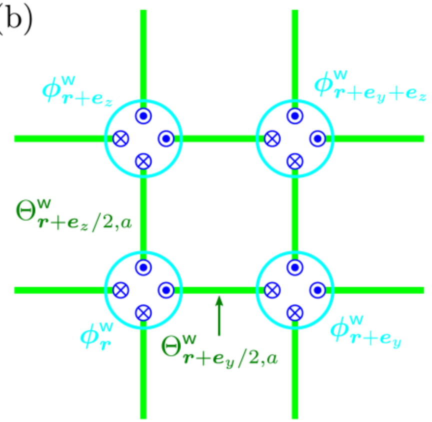
$$\mathcal{V}_{\mathrm{w}} = -g\int dx \sum_{\boldsymbol{r}\in\mathbb{Z}^2}\sum_{a=1}^{N}\Big(\cos\Theta^{\mathrm{w}}_{\boldsymbol{r}+\boldsymbol{e}_y/2,a} + \cos\Theta^{\mathrm{w}}_{\boldsymbol{r}+\boldsymbol{e}_z/2,a}\Big),$$

$$\Theta^{\mathrm{w}}_{\boldsymbol{r}+\boldsymbol{e}_y/2,a} = \Phi^{\mathrm{w}}_{\boldsymbol{r},2N+a} + \Phi^{\mathrm{w}}_{\boldsymbol{r}+\boldsymbol{e}_y,a},$$

$$\Theta^{\mathrm{w}}_{\boldsymbol{r}+\boldsymbol{e}_z/2,a} = \Phi^{\mathrm{w}}_{\boldsymbol{r},3N+a} + \Phi^{\mathrm{w}}_{\boldsymbol{r}+\boldsymbol{e}_z,N+a},$$

$$\Phi^{\mathrm{w}}_{\boldsymbol{r},\alpha} = \boldsymbol{\Lambda}^{T}_{\mathrm{w},\alpha}K_{\mathrm{w}}\boldsymbol{\phi}^{\mathrm{w}}_{\boldsymbol{r}}.$$

(b)



--- Quasiparticles are always mobile along the x axis.

--- Mobility of quasiparticles in the yz plane is dictated by L.

We classify gapped interfaces between $U(1)_k$ topological orders (Laughlin states) for small $K_0 = k$.

→ Construct 3d coupled-wire models for topological orders and (type-I) fracton orders.

Outline

1. Introduction

--- Topological order and fracton order

--- Coupled-wire construction, Cellular topological state/Topological defect network

2. General recipe for 3d coupled-wire models

--- K-matrix formalism for 2d Abelian topological orders and gapped interfaces

--- Coupled-edge and coupled-wire Hamiltonians

3. Applications

--- 3d topological order, fracton order, and their hybrid

4. Summary and outlooks

Sorting fracton orders

Dua, Kim, Cheng, & Williamson, PRB **100**, 155137 (2019).

	TQFT [D 5]	Foliated type I [D 4]	Fractal type I [D 2]	Type II [D 1]
Mobilities of particles	3	0, 1, 2	0, 1, 2	0
Scaling of number of qubits	constant	subextensive	subextensive + fluctuations or fluctuating with subextensive envelope	fluctuating with subextensive envelope
Examples	3D toric code (with bosonic or fermionic charge)	checkerboard model X-cube model Chamon's model	Sierpinski FSL model cubic codes 0,5,6,9,11-17	cubic codes 1–4,7,8,10 Hsieh-Halsz-II model[a]

[a]Our results are consistent with fractal type I or type II.

Our models always have quasiparticles moving along strips. → Presence of lineons

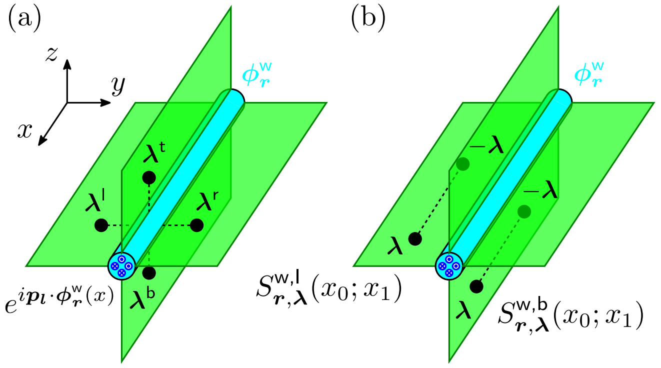
--- 3d TQFT-type topological order with point- and loop-like excitations

--- Foliated type-I fracton order with only planons

--- Foliated type-I fracton order with lineons and planons

--- Hybrid of TQFT-type topological order and foliated type-I fracton order

--- Fractal type-I fracton order



Foliated type-I fracton order with lineons and planons

Input: $K_\mathrm{e} = \begin{pmatrix} 4 & & & \\ & 4 & & \\ & & -4 & \\ & & & -4 \end{pmatrix}$ $\boldsymbol{m}_1 = (1,0,3,2)^T,$ $\boldsymbol{m}_2 = (0,1,2,3)^T,$

Output:

$$\begin{aligned} \boldsymbol{\Lambda}_{\mathrm{w},1} &= (-1,1,0,2)^T, \\ \boldsymbol{\Lambda}_{\mathrm{w},2} &= (0,-2,-1,1)^T, \\ \boldsymbol{\Lambda}_{\mathrm{w},3} &= (1,-1,-2,0)^T, \\ \boldsymbol{\Lambda}_{\mathrm{w},4} &= (2,0,1,-1)^T. \end{aligned}$$

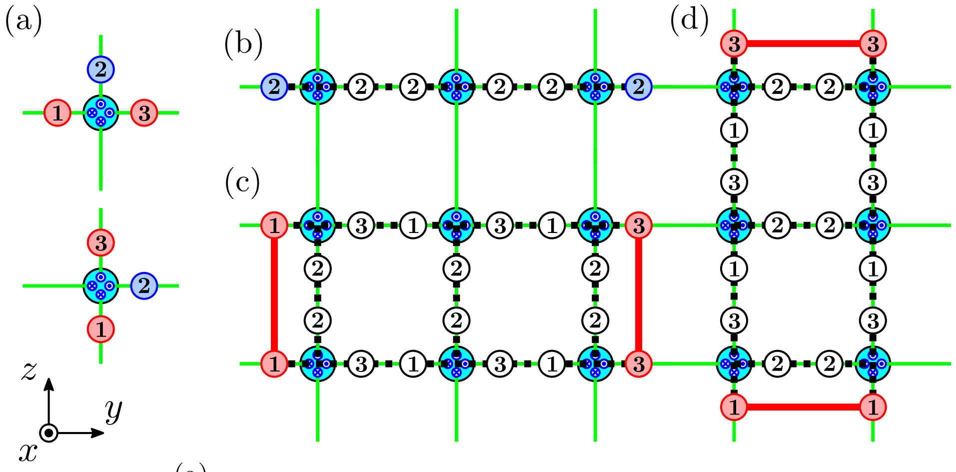
--- $l = 1$ QPs are lineons.

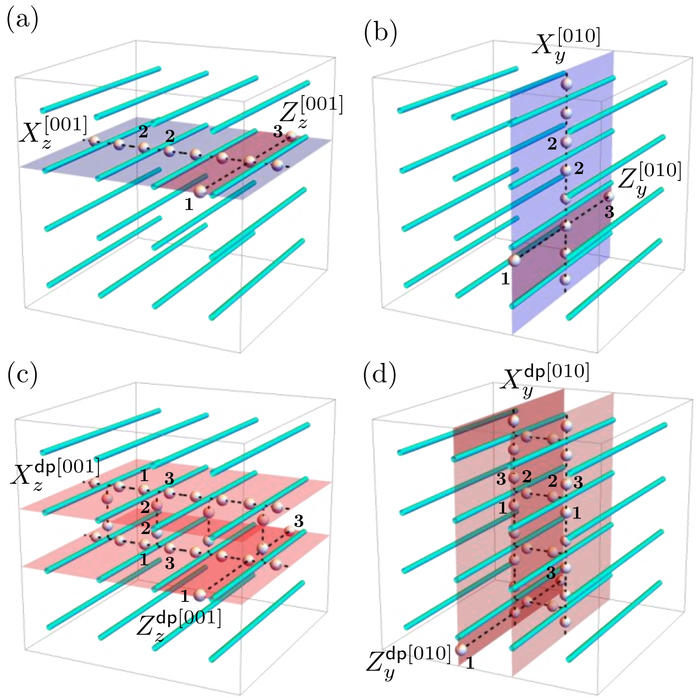
--- $l = 2$ QPs are planons in the [010] or [001] planes.

--- Dipoles of $l = 1$ lineons are also planons.

$$\mathrm{GSD} = 2^2 \cdot 4^{L_y + L_z - 2}.$$

--- Negative constant part in log(GSD) → Nontrivial fracton order





Hybrid of topological and fracton orders

Input: $$K_{\rm e} = \begin{pmatrix} 8 & & & \\ & 8 & & \\ & & -8 & \\ & & & -8 \end{pmatrix}$$

$$\boldsymbol{m}_1 = (1,1,7,7)^T,$$
$$\boldsymbol{m}_2 = (2,0,6,0)^T,$$
$$\boldsymbol{m}_3 = (4,4,0,0)^T,$$

Output:
$$\boldsymbol{\Lambda}_{\mathrm{w},1} = (2,-1,-2,1)^T,$$
$$\boldsymbol{\Lambda}_{\mathrm{w},2} = (2,-1,2,-1)^T,$$
$$\boldsymbol{\Lambda}_{\mathrm{w},3} = (2,1,-2,-1)^T,$$
$$\boldsymbol{\Lambda}_{\mathrm{w},4} = (2,1,2,1)^T.$$

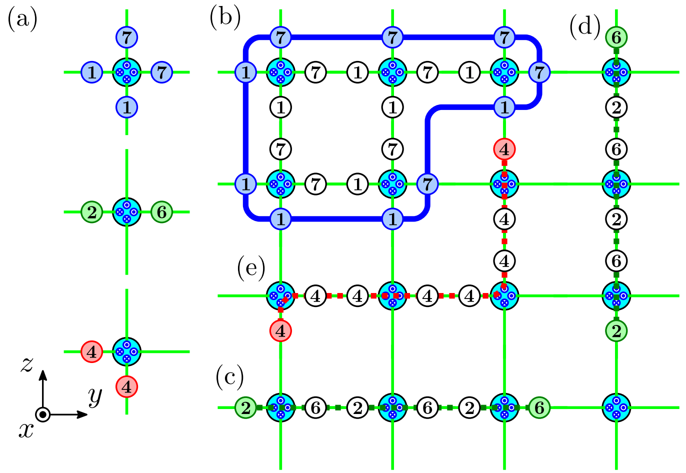
--- $l = 4$ QPs are point-like bosons in 3d.

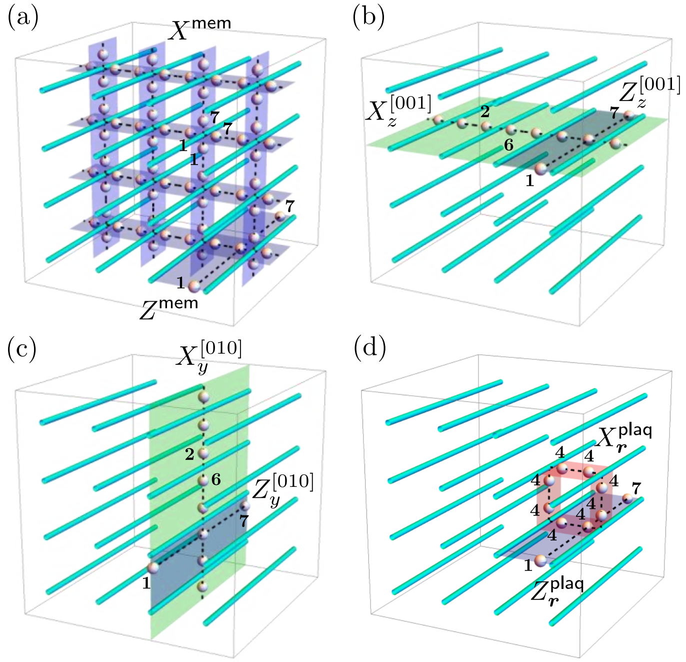
--- $l = 2$ QPs are planons in the [010] or [001] planes.

--- $l = 1$ QPs form a loop-like excitation in yz plane.

--- Extensive degeneracy arising from local plaquette loops.

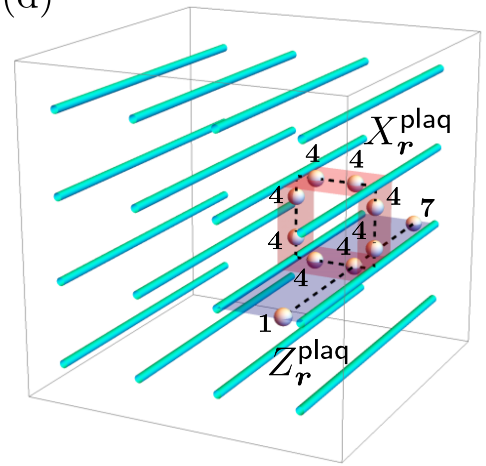
$$\mathrm{GSD} = 4 \cdot 2^{L_y L_z + L_y + L_z}.$$





Hybrid of topological and fracton orders

(d)



Extensive degeneracy can be lifted by adding local perturbations

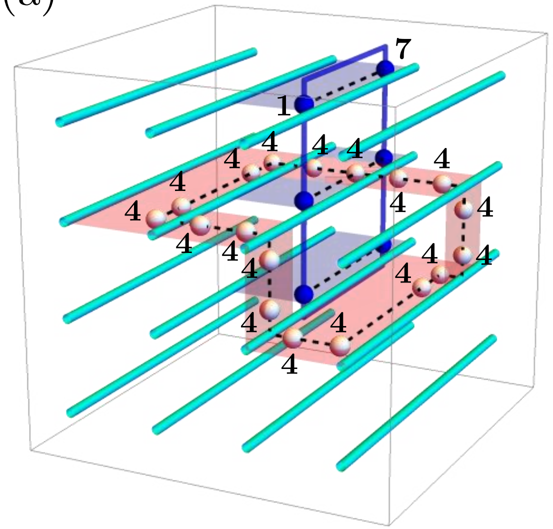
$$\mathcal{V}'_{\text{w}} = -g' \int dx \sum_{\boldsymbol{r}\in\mathbb{Z}^2} [X_{\boldsymbol{r}}^{\text{plaq}}(x) + (X_{\boldsymbol{r}}^{\text{plaq}}(x))^\dagger],$$

→ Resulting GSD becomes constant + subextensive. $\text{GSD}' = 2^3 \cdot 2^{L_y+L_z}$

→ Hybrid of Z2 gauge theory and fracton order with planons

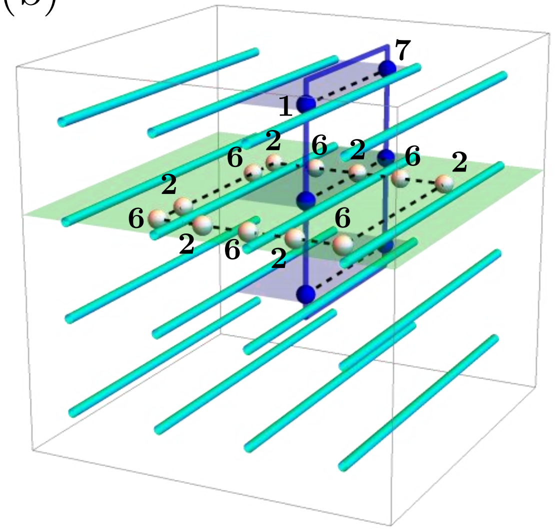
Not a simple stack of topological and fracton orders!

(a)



Point-loop braiding
→ Mutual π statistics

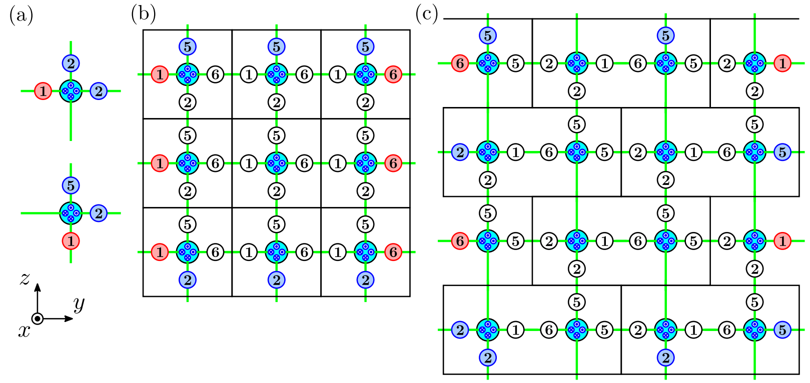
(b)



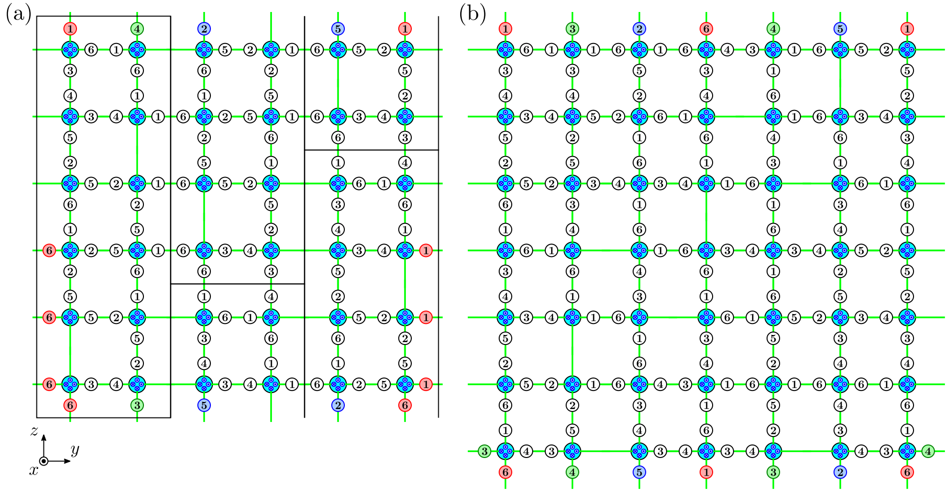
Planon-loop braiding
→ Mutual π/2 statistics

Fractal type-I fracton order with lineons

Input: $K_{\mathrm{e}} = \begin{pmatrix} 7 & & & \\ & 7 & & \\ & & -7 & \\ & & & -7 \end{pmatrix}$ $\quad \boldsymbol{m}_1 = (1,0,2,2)^T, \quad \boldsymbol{m}_2 = (0,1,2,5)^T,$



Output:
$$\boldsymbol{\Lambda}_{\mathrm{w},1} = (1,2,0,-2)^T,$$
$$\boldsymbol{\Lambda}_{\mathrm{w},2} = (0,-2,-1,-2)^T,$$
$$\boldsymbol{\Lambda}_{\mathrm{w},3} = (2,0,-2,-1)^T,$$
$$\boldsymbol{\Lambda}_{\mathrm{w},4} = (2,1,2,0)^T.$$



Lagrangian subgroup does not contain pairs of quasiparticles.

→ Conjecture: No 3d point-like excitations or planons, but only lineons.

Sparse membrane operators → Quasiparticles are created at boundaries of fractal-like operators?

Fractal type-I fracton order with lineons

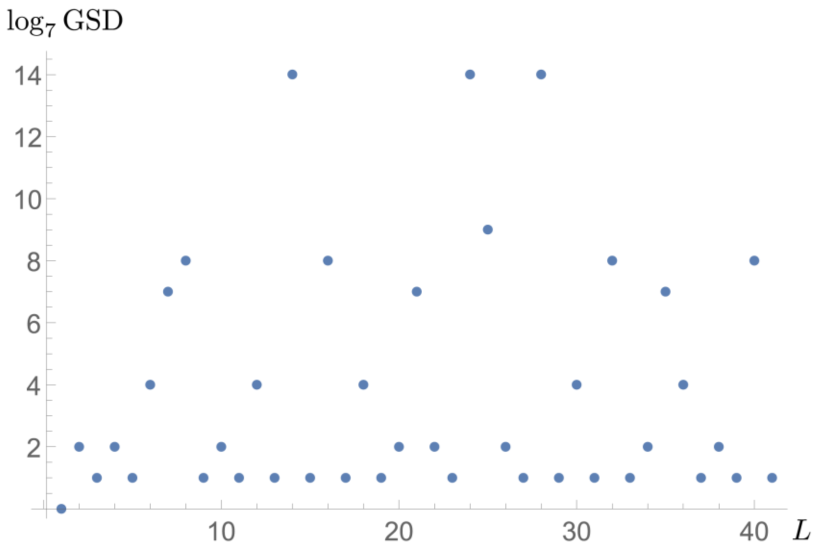
Dynamics of quasiparticles in yz plane → 2d classical seven-state Potts model

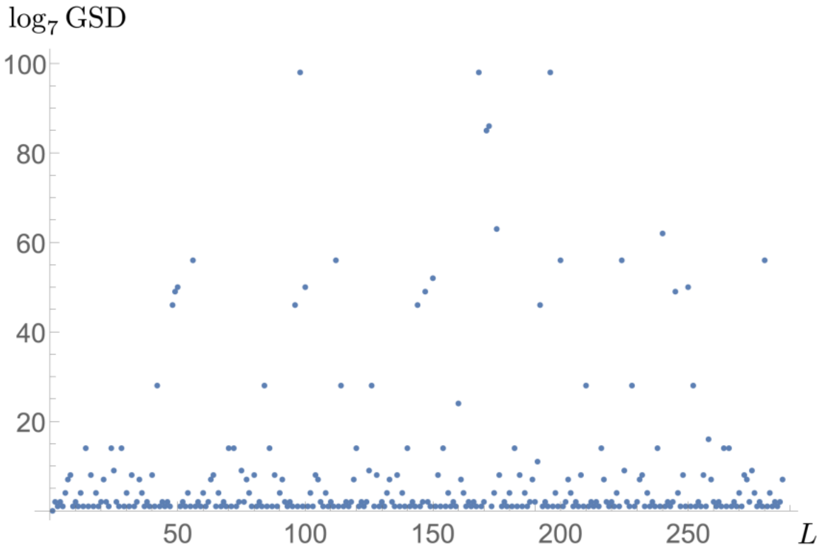
$$\mathcal{H}^{U(1)_7}_{\text{Potts}} = -J \sum_{\boldsymbol{r}\in\mathbb{Z}^2} \left[(\sigma^z_{\boldsymbol{r},1})^2 (\sigma^z_{\boldsymbol{r},2})^2 \sigma^z_{\boldsymbol{r}+\boldsymbol{e}_y,1} + (\sigma^z_{\boldsymbol{r},1})^2 (\sigma^z_{\boldsymbol{r},2})^5 \sigma^z_{\boldsymbol{r}+\boldsymbol{e}_z,2} + \text{H.c.} \right].$$

$$(\sigma^x_{\boldsymbol{r},j})^7 = (\sigma^z_{\boldsymbol{r},j})^7 = 1,$$

$$\sigma^x_{\boldsymbol{r},j}\sigma^z_{\boldsymbol{r}',j'} = \begin{cases} e^{2\pi i/7}\sigma^z_{\boldsymbol{r}',j'}\sigma^x_{\boldsymbol{r},j} & (\boldsymbol{r},j) = (\boldsymbol{r}',j') \\ \sigma^z_{\boldsymbol{r}',j'}\sigma^x_{\boldsymbol{r},j} & (\boldsymbol{r},j) \neq (\boldsymbol{r}',j') \end{cases}$$

Ground-state degeneracy on $L_x \times L \times L$ torus can be computed from the classical model.





Fractal-like structure → Fractal type-I fracton order?

Summary and outlooks

Coupled-wire construction provides exactly solvable models for cellular topological states.

They describe a variety of novel 3d topological and fracton orders:

--- 3d TQFT-type topological order

--- Foliated type-I fracton order

--- Hybrid of TQFT-type topological order and foliated type-I fracton order

--- Fractal type-I fracton order

Future directions:

--- Cellular topological states from 2d non-Abelian topological orders

--- Applications to microscopic lattice systems via bosonizations

--- Effective quantum field theories for fracton orders

--- Entanglement diagnostics of fracton orders