Lattice, Boundary, and Anomaly

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Nielsen-Ninomiya theorem

- Nielsen-Ninomiya theorem in $D = 1 + 1$ states that there are no lattice regulators for a free chiral fermion.
- The underlying assumptions are: (1) The interaction is local. (2) The chiral symmetry is exact at the lattice scale.
- All known examples of lattice constructions of a chiral fermion violates one of the assumptions. (No wonder. It's a mathematical theorem.)
- The Ginsparg-Wilson fermion satisfies (1) but not (2).

Original proof of the Nielsen-Ninomiya theorem (1)

- The original proof of the theorem by Nielsen and Ninomiya goes like this.
- The general class of lattice Hamiltonian cannot flow to a chiral fermion

$$H = \sum_{i,j} f(i-j)\psi_i\psi_j,$$

on condition that (1) It is quadratic in the fermion fields. (2) It is translation invariant. (3) The interaction is local.

Original proof of the Nielsen-Ninomiya theorem (2)

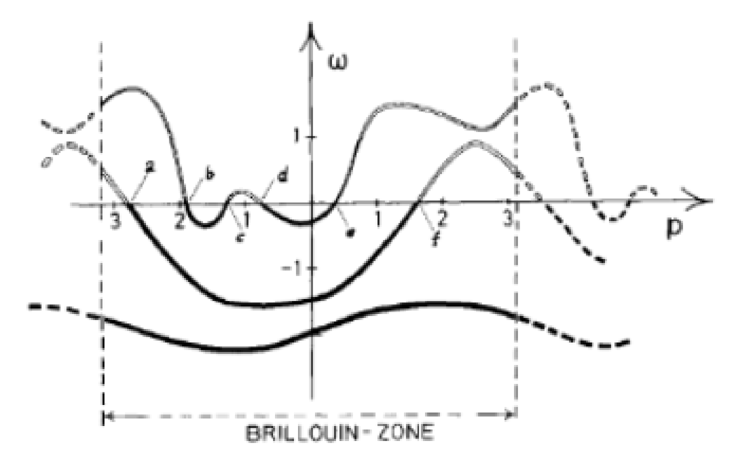
- We can Fourier transform the Hamiltonian

$$H = \sum_{i,j} f(i-j)\psi_i\psi_j = \int_0^{2\pi} dp\,\omega(p)\psi_p\psi_{-p}.$$

  $\omega(p)$ is defined on a circle (*i.e.*, Brillouin zone).

- At $a \to 0$, or in the low-energy limit, zero energy modes ($\omega = 0$) are only relevant. Here, $d\omega/dp$ determines the direction at which a low energy excitation moves.
- Since the Brillouin zone is a circle, there must accompany a left-moving mode for every right-moving mode.
- This completes the proof of the Nielsen-Ninomiya theorem.

Original proof of the Nielsen-Ninomiya theorem (3)



Today's topics

- The proof that I showed only applies to free theories.
- (1) I will therefore give a new proof of the Nielsen-Ninomiya theorem. This proof generalises the Nielsen-Ninomiya theorem to any strongly-coupled theoires.
- As it turns out, any theories with non-zero gravitational anomaly ($c_L - c_R$, same as the chiral anomaly) cannot have a lattice regulator.
- (2) The proof uses boundary conditions as an intermediate step. I will discuss general relations between boundary, lattice, and the anomaly.

New proof of the Nielsen-Ninomiya theorem

- We prove that the existence of the lattice regulator implies the vanishing gravitational anomaly (*i.e.*, $c_L - c_R = 0$).
- The sketch of the proof: (a) Any lattice theories can have a boundary condition and can be defined on a half-space. (b) This flows to a boundary CFT. (c) A boundary CFT always has $c_L - c_R = 0$.
- Take the contrapositive: A 2D CFT with non-vanishing gravitational anomaly cannot have a lattice regulator.

New proof of the Nielsen-Ninomiya theorem (a)

- (a) Any lattice theories can have a boundary condition and be defined on a half-space.
- Take any lattice theory with bounded-neighbour interactions. Call the interaction range n_{max}.
- We can always take a free boundary condition and define a theory on a half-space ($x > 0$).
- Since the interaction is bounded, the difference of the theory on a half-space is no different than a theory on a full space when $x > an_{\mathrm{max}}$.

New proof of the Nielsen-Ninomiya theorem (b)

- (b) This theory defined on a half-space flows to a boundary CFT.
- The boundedness of the interaction means that the bulk of the theory remains the same, when $x > an_{\text{max}}$.
- Take the IR limit, $a \to 0$. The bulk simply flows to a CFT.
- At $x = 0$, we have a boundary condition of a CFT.
- The boundary condition also flows under the RG flow. It is monotonic by virtue of the g-theorem (Boundary version of the c-theorem). $\log g$ is called the boundary entropy.
- The RG flow is strongly believed to terminate at a conformal boundary condition. The theory becomes a boundary CFT as a whole.
- Exception: If the lattice had infinite entropy per site, the theory will not flow to a BCFT.

New proof of the Nielsen-Ninomiya theorem (c)

- (c) A boundary CFT always has $c_L - c_R = 0$.
- $c_{L,R}$ means a left- or right-moving dof, so this is very intuitive.
- To prove this formally, use the boundary Ward identity $T(x) - \bar{T}(x) = 0$ on $x \in \mathbb{R}$.
- One can modular transform to define a boundary state,

$$(L_n - \bar{L}_{-n}) |B\rangle = 0$$

and then Virasoro algebra gets us $c_L = c_R$.

Generalised Nielsen-Ninomiya theorem

- To summarise, we have proven that

$$\boxed{\text{existence of the lattice regulator}}$$
$$\Downarrow$$
$$\boxed{\text{existence of the boundary condition}}$$
$$\Downarrow$$
$$\boxed{\text{vanishing gravitational anomaly}}$$

- We can take the contrapositive to prove the generalised Nielsen-Ninomiya theorem: A theory with a non-zero gravitational anomaly cannot have a lattice regulator.
- Of course it contains the original theorem. A left-moving free fermion has $c_L = 1/2$ and $c_R = 0$.

Comment

- We have not really used the assumption that "The chiral symmetry is exact at the lattice scale".
- Instead we used another assumption that "The lattice theory has finite entropy per site".
- This can also rule out the Ginsparg-Wilson fermion, as taking the length of the extra dimension to be infinite, the effective entropy per site in two dimensions becomes infinite.
- But I do not understand the precise relation between the two yet.
- This concludes the main part of my talk.

Boundary and anomaly

- As an intermediate step, we have proven that a boundary condition only exists for theories with vanishing gravitatonal anomalies.
- It has generalisations to other symmetries or to other dimensions.
- For example, for a continuous global continuous symmetry in 2D, we have $(J_n + \tilde{J}_{-n})\,|B\rangle = 0$, implying the vanishing anomaly.
- More generally, it is proven by Thorngren and Wang for (1) Gravitational anomaly in any dimensions. (2) Anomaly for any continuous symmetries in any dimensions. (3) Anomaly for any discrete symmetries in any dimensions.

Lattice and anomaly

- Even then, this does not mean that we can generalise the Nielsen-Ninomiya theorem to the case with global symmetries.
- There is a folklore that says that "Any anomalous symmetry cannot be realised on-site". This is not true in general.
- Example: the Villain action for the free compact boson in 2D,

$$S = \frac{\beta}{2} \sum_{\text{link}} (\Delta_\mu \phi - 2\pi n_\mu)^2 + i \sum_{\text{plq}} \tilde{\phi}(\Delta_x n_y - \Delta_y n_x)$$

  where $\phi \sim \phi + 2\pi$ and $\tilde{\phi} \sim \tilde{\phi} + 2\pi$.
- Even though it has the mixed anomaly between the momentum $U(1)_m$ and the winding $U(1)_w$ symmetry, they are both realised on site, as $\phi \mapsto \phi + c_m$ and $\tilde{\phi} \mapsto \tilde{\phi} + c_w$ on a closed manifold.

Boundary condition of the Villain lattice

- It still should break either $U(1)_m$ or $U(1)_w$ at the boundary, even though the Villain lattice realises both of them on-site.
- For example, place a free boundary condition at the boundary.
- We can insert an operator $e^{iW\tilde{\phi}}$ in the middle and computing the partition function. They are non-zero for any $W \in \mathbb{Z}$.
- The free boundary condition preserves $U(1)_m$ completely while breaking $U(1)_w$ completely.
- It is an interesting future direction to study other symmetry preserved at the boundary of the Villain lattice, such as $\mathbb{Z}_M \times \mathbb{Z}_W$, where M and W coprime.

Boundary condition and the entanglement entropy

- Entanglement entropy is a basic object in quantum information theory. It can be combined with AdS/CFT to compute some observables in quantum gravity, for example.
- In order to define entanglement, we separate the physical space into two (L and R), with Hilbert space factorised as $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$.
- This notion of factorisation is meaningless without a boundary condition between L and R.
- This means that gravitational anomaly obstructs us from talking about entanglement entropy of *e.g.*, a chiral fermion.

Summary

- We have proven the Nielsen-Ninomiya theorem: Any 2D theory with non-vanishing gravitational anomaly cannot have a lattice regulator.
- We used the existence of the boundary condition to prove this.
- We argued that the following generalisation is *not* true: When a global symmetry G is anomalous, G cannot be realised on-site.
- This was in spite of the fact that G-symmetric boundary conditions do not exist when G is anomalous.
- It would be interesting to explore relation between boundary, lattice and anomaly more. (1) Can we always find a boundary condition for a non-anomalous theory? (2) Does the existence of the boundary condition implies the lattice construction? (3)...