Curved domain-wall fermion and its anomaly inflow

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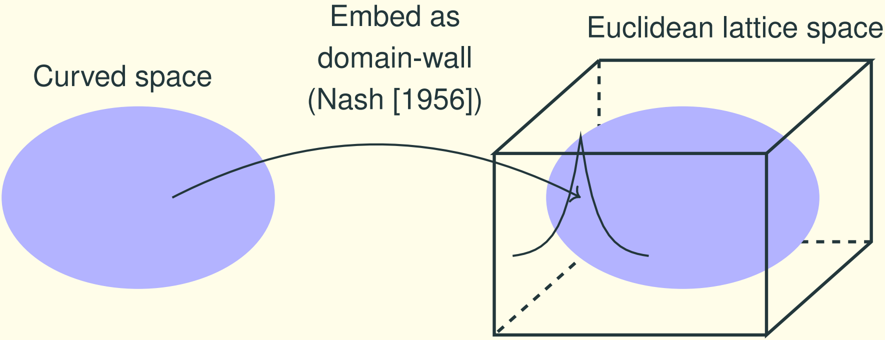
Motivation

Every curved manifold can be isometrically embedded into some higher-dimensional Euclidean spaces.



Localize the edge modes of the curved domain-wall fermion.
= they feel "gravity" by the equivalence principle.

Embedding a curved space

For any n-dim. Riemann space (Y, g), there is an embedding $f : Y \to \mathbb{R}^m$ $(m \gg n)$ such that Y is identified as

$$x^\mu = x^\mu(y^1, \cdots, y^n)\ (\mu = 1, \cdots, m)$$

$$\left(\begin{array}{ll} x^\mu & \text{: Cartesian coordinates of } \mathbb{R}^m \\ y^i & \text{: coordinates of } Y \end{array} \right.$$

and the metric is written as

$$g_{ij} = \sum_{\mu\nu} \delta_{\mu\nu} \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\nu}{\partial y^j}.$$

$\longrightarrow$ vielbein and spin connection are also induced!

(Y, g) can be identified as a submanifold of $\mathbb{R}^m$!

Cf. Nash [1956].

Our Work

We consider a Hermitian Dirac operator

$$H = \bar{\gamma}\left(\sum_{i=1}^{n+1} \gamma^i \frac{\partial}{\partial x^i} + m\,\mathsf{sign}(f)\right)$$

$$\{\gamma^I, \gamma^J\} = 2\delta^{IJ}, \{\bar{\gamma}, \gamma^J\} = 0,\ \bar{\gamma}^2 = 1\ , (I, J = 1, \cdots, n+1)$$

where the smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ determines the domain-wall $Y = \{\, f = 0 \,\}$. The edge modes are

- localized at Y,
- the chiral eigenstate of $\gamma_{\text{normal}} = \boldsymbol{n} \cdot \boldsymbol{\gamma}$,
- and feel gravity throguth the spin connection on Y.

$\longrightarrow$ We confirm the above properties on a square lattice .

$Y = S^1$ or S^2 in this work

Cf. Continuum analysis in condensed matter physics: [Imura et al. [2012], Parente et al. [2011]]

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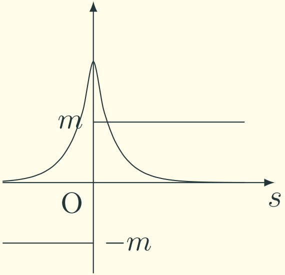
Flat Domain-wall (review)

Domain-wall=A bounday where a sign of mass is flipped.

We put a mass term $m(s) = m\text{sign}(s)$ in 5-dim space.

EoM : $(\sum_{i=1}^{4} \gamma^i \partial_i + \gamma^s \partial_s + m(s))\psi(x,s) = 0$

Sol : $\psi(x,s) = \eta_+(x) e^{-m|s|}, \ \gamma^s \eta_+ = +\eta_+$



A state with $\not{D}_4 = \sum_{i=1}^{4} \gamma^i \partial_i = 0$ and $\gamma^s = +1$ is localized at $s = 0$!

Domain-wall

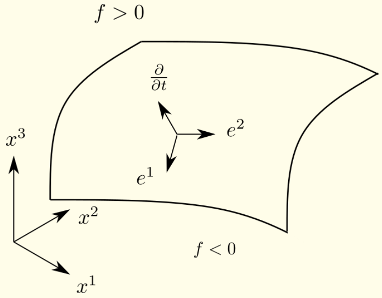
Curved case

Curved domain-wall case:

$$D = \sum_{i=1}^{2m+1} \gamma^i \frac{\partial}{\partial x^i} + m\text{sign}(f)$$

$$\simeq \gamma^{2m+1} \frac{\partial}{\partial t} + F + m\text{sign}(f)$$

$$+ \underbrace{\gamma^a \left(e_a + \frac{1}{4} \sum_{bc} \omega_{bc,a} \gamma^b \gamma^c \right)}_{\not{D}^Y}$$



In the large m limit, $D \to \not{D}^Y_+ = \not{D}^Y \frac{1}{2}(1 + \gamma^{2m+1})$

$\longrightarrow$ A zero mode of $\not{D}^Y$ only appear as the edgemode of D

Hermitian Dirac operator

We consider a Hermitian Dirac operator

$$H = \bar{\gamma}\left(\sum_{i=1}^{n+1} \gamma^i \frac{\partial}{\partial x^i} + m\text{sign}(f)\right) = \bar{\gamma}(\not{D} + m\text{sign}(f))$$

$$\gamma^a = -\sigma_2 \otimes \tilde{\gamma}^a,\ \gamma^{n+1} = \sigma_1 \otimes 1,\ \bar{\gamma} = \sigma_3 \otimes 1$$

$$\{\tilde{\gamma}^a, \tilde{\gamma}^b\} = 2\delta^{a,b},\ (a, b = 1, \cdots, n)$$

where the smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$. The edge modes are

- localized at the domain-wall $Y = \{f = 0\}$,
- the chiral eigenstate of $\gamma_{\text{normal}} = \boldsymbol{n} \cdot \boldsymbol{\gamma}$,
- and feel gravity throguth the spin connection on Y.

Induced spin connection

We take an appropriate coordinate $(y^1, \cdots, y^n, t)$ and vielbein

$$\Big(\underbrace{e^1, \cdots, e^n}_{\text{vielbein on } Y}, \frac{\partial}{\partial t}\Big).$$

We put $\psi = \left(g^{IJ}\frac{\partial f}{\partial x^I}\frac{\partial f}{\partial x^J}\right)^{\frac{1}{4}}\psi'$, H act on ψ' as

$$H' = \begin{pmatrix} \epsilon m & i\not{D}^Y + \frac{\partial}{\partial t} + F \\ i\not{D}^Y - \frac{\partial}{\partial t} - F & -\epsilon m \end{pmatrix},$$

$$i\not{D}^Y = i\sum_{a=1}^{n}\tilde{\gamma}^a\left(e_a + \frac{1}{4}\sum_{bc}\omega_{bc,a}\tilde{\gamma}^b\tilde{\gamma}^c\right)$$

$$F = \frac{1}{4}\frac{\partial}{\partial t}\left(\log\left(g^{IJ}\frac{\partial f}{\partial x^I}\frac{\partial f}{\partial x^J}\right)\right) - \underbrace{\frac{1}{2}\operatorname{tr} h}_{\text{mean curvature}}$$

Spin connection on Y is induced!

Edge mode

In the large m limit, we find an edgemode as

$$\psi = \left(g^{IJ} \frac{\partial f}{\partial x^I} \frac{\partial f}{\partial x^J} \right)^{\frac{1}{4}} e^{-m|t|} \exp\left(-\int_0^t dt' F(y,t') \right) \begin{pmatrix} \chi(y) \\ \chi(y) \end{pmatrix}$$

$$\left(F = \frac{1}{4} \frac{\partial}{\partial t} \left(\log \left(g^{IJ} \frac{\partial f}{\partial x^I} \frac{\partial f}{\partial x^J} \right) \right) - \frac{1}{2} \operatorname{tr} h \right),$$

where χ is a massless Dirac fermion:

$$i \not{D}^Y |_{t=0} \chi = i \sum_{a=1}^{n} \tilde{\gamma}^a \left(e_a + \frac{1}{4} \sum_{bc} \omega_{bc,a} \tilde{\gamma}^b \tilde{\gamma}^c \right)_{t=0} \chi = \lambda \chi$$

ψ is a eigenstate: $H\psi = \lambda \psi$ and $\gamma^{n+1} \psi = (\sigma_1 \otimes 1) \psi = +\psi$

Spin connection on Y is induced and detected by solving the eigenvalue problem of H!

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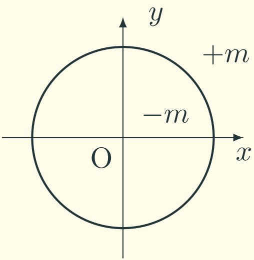
S^1 domain-wall

Domain wall:

$$\epsilon(r) = \text{sign}(r - r_0) = \begin{cases} -1 & (r < r_0) \\ 1 & (r \geq r_0) \end{cases},$$



Hermitian Dirac operator:

$$H = \sigma_3 \left(\sum_{i=1,2} \left(\sigma_i \frac{\partial}{\partial x^i} \right) + m\epsilon \right)$$
$$= \begin{pmatrix} m\epsilon & e^{-i\theta}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \theta}\right) \\ -e^{i\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right) & -m\epsilon \end{pmatrix}.$$

Spectrum of Edge modes

Effective Dirac operator:

$$i\mathbb{D}^{S^1}_{eff} = \frac{1}{r_0}\left(-i\frac{\partial}{\partial\theta} + \underset{\sim}{\frac{1}{2}}\right)$$

Spin^c connection

$\longrightarrow$The edge modes is effectively anit-periodic spinor.

(trivial element of the spin bordism group)

Eigenvalue:

$$E = \pm\frac{n+\frac{1}{2}}{r_0}\ (n = 0, 1, \cdots).$$

$\longrightarrow$ Gravity appears as the gap of the spectrum

Lattice domain-wall fermion

Let $(\mathbb{Z}/N\mathbb{Z})^2$ be a two-dim. lattice.

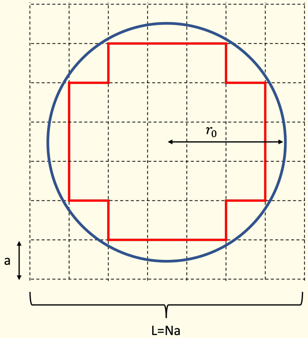
The domain-wall is given by

$$\epsilon(x) = \begin{cases} -1 & (r < r_0) \\ 1 & (r \geq r_0) \end{cases},$$

and the (Wilson) Dirac op is

$$H = \sigma_3 \left(\sum_{i=1,2} \left[\sigma_i \frac{\nabla_i - \nabla_i^\dagger}{2} + \frac{1}{2} \nabla_i \nabla_i^\dagger \right] + \epsilon m a \right),$$

$$(\nabla_i \psi)_x = \psi_{x+\hat{i}} - \psi_x,\ (\nabla_i^\dagger \psi)_x = \psi_{x-\hat{i}} - \psi_x$$



$+$ PBC for all direction.

Cf. Kaplan [1992] studied a flat domain-wall in $\mathbb{R}^{2m+1}$

Spectrum

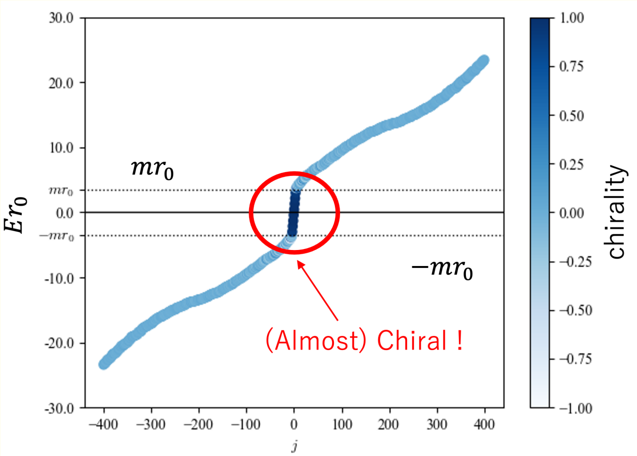
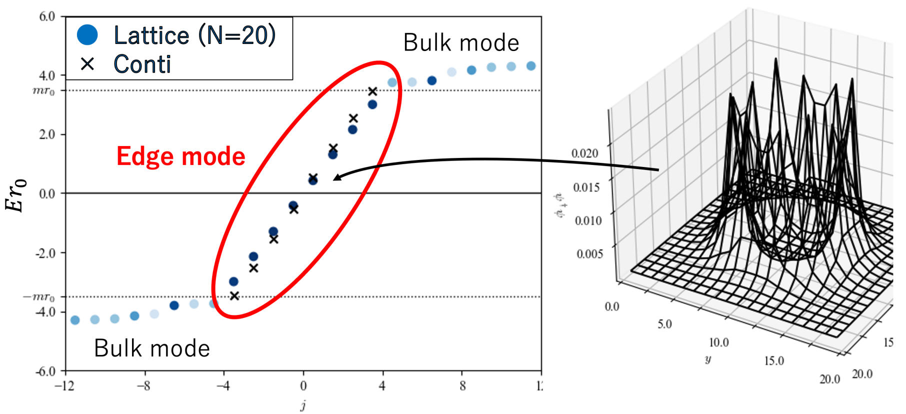


Fig 1: The Dirac eigenvalue spectrum: $ma = 0.7, r_0 = L/4, N = 20$

The color = chirality: $\gamma_{\text{normal}} = \frac{x}{r}\sigma_1 + \frac{y}{r}\sigma_2$

Edge modes

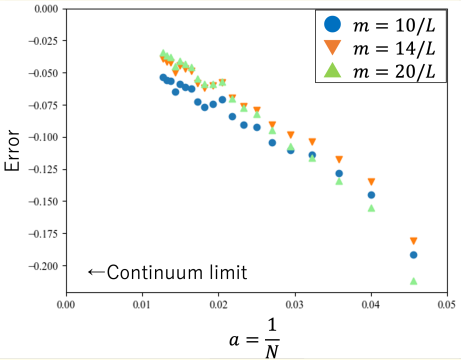


The edge modes

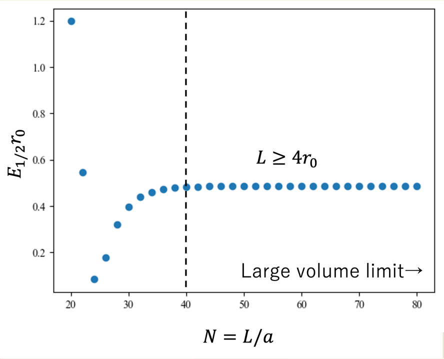
- are chiral: $\gamma_{\text{normal}} = \frac{x}{r}\sigma_1 + \frac{y}{r}\sigma_2$
- have a gap from zero (as a gravitational effect)
- agree well with the continuum prediction

Continuum limit and Finite-volume effect

Continuum limit $a = 1/N \to 0$



Large volume limit $L = Na \to \infty$



Fixed parameter:

$L = Na, r_0 = Na/4, m = 14/L$

Agree well with the conti. prediction!

Fixed parameter:

$r_0 = 10a$

Saturates when $L \geq 4r_0$!

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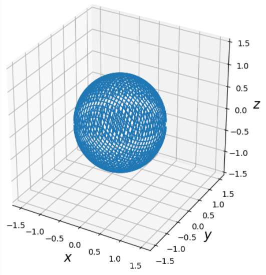
S^2 domain-wall

Domain-wall:



$$\epsilon(r) = \text{sign}(r - r_0) = \begin{cases} -1 & (r < r_0) \\ 1 & (r \geq r_0) \end{cases},$$

Hermitian Dirac operator

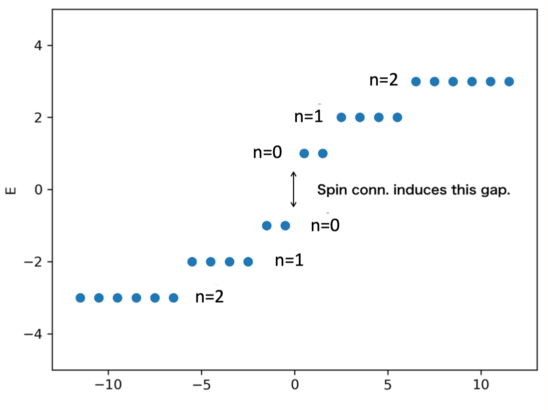
$$H = \bar{\gamma}\left(\gamma^j \frac{\partial}{\partial x^j} + m\epsilon\right) = \begin{pmatrix} m\epsilon & \sigma^j \partial_j \\ -\sigma^j \partial_j & -m\epsilon \end{pmatrix}$$

$$(\bar{\gamma} = \sigma_3 \otimes 1,\ \gamma^j = \sigma_1 \otimes \sigma^j)$$

acts on two-flavors of two-component spinors.

Spectrum of Edgemodes

$$i\mathbb{D}^{S^2}_{eff} = -\frac{\sigma_3}{r_0}\left(\sigma_1\frac{\partial}{\partial\theta} + \sigma_2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} \underbrace{+\frac{i}{2\sin\theta} - i\frac{\cos\theta}{2\sin\theta}\sigma_3}_{\text{Spin}^c\text{ connection}}\right)\right)$$



Eigenvalue:

$$E = \pm\frac{n+1}{r_0}$$

$$(n = 0, 1, \cdots)$$

Degeneracy:

$$2n+2$$

Lattice Domain-wall Fermion

Let $(\mathbb{Z}/N\mathbb{Z})^3$ be a three-dim. lattice.

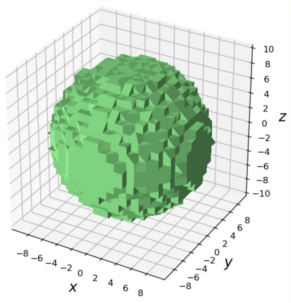
The domain-wall is given by

$$\epsilon(x) = \begin{cases} -1 & (r < r_0) \\ 1 & (r \geq r_0) \end{cases},$$

and the (Wilson) Dirac op is

$$H = \bar{\gamma}\left(\sum_{i=1,2,3}\left[\gamma^i \frac{\nabla_i - \nabla_i^\dagger}{2} + \frac{1}{2}\nabla_i \nabla_i^\dagger\right] + \epsilon m a\right).$$

$$(\nabla_i \psi)_x = \psi_{x+\hat{i}} - \psi_x,\ (\nabla_i^\dagger \psi)_x = \psi_{x-\hat{i}} - \psi_x$$



$+$PBC for all direction

Spectrum

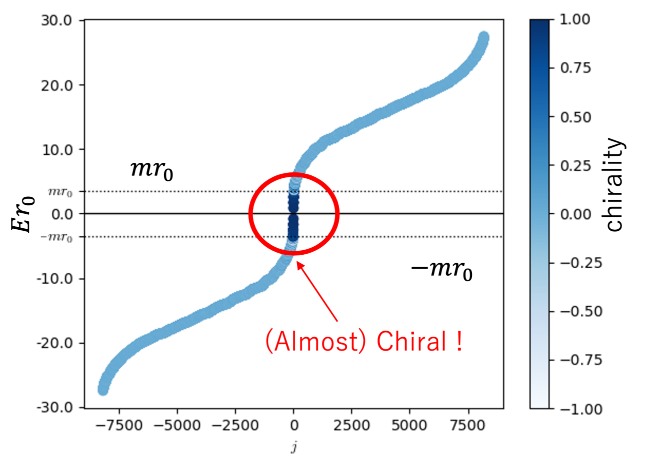
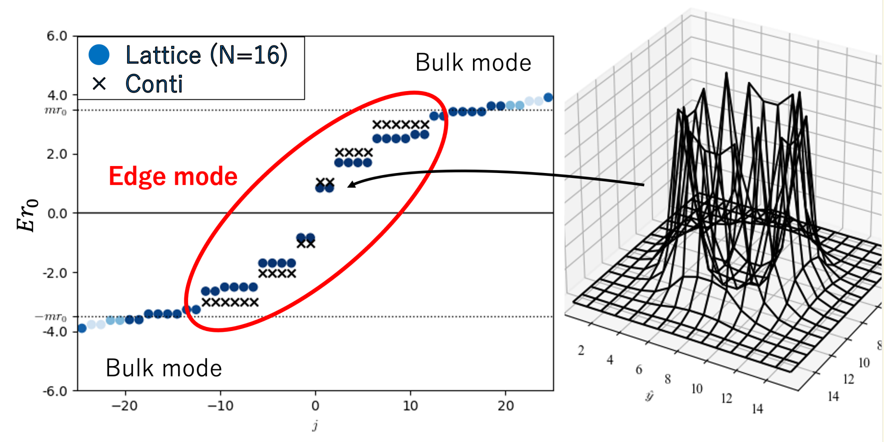


Fig 2: The Dirac eigenvalue spectrum: $ma = 0.7, r_0 = L/4, N = 16$

The color = chirality: $\gamma_{\text{normal}} = \frac{x}{r}\gamma^1 + \frac{y}{r}\gamma^2 + \frac{z}{r}\gamma^3$

Edge modes

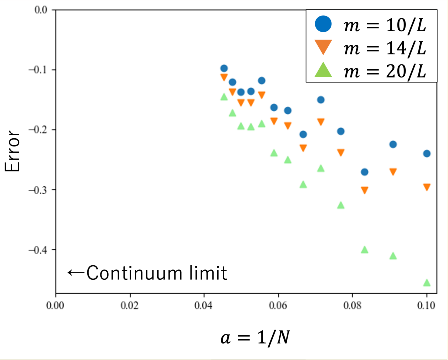


The edge modes

- are chiral: $\gamma_{\text{normal}} = \frac{x}{r}\gamma^1 + \frac{y}{r}\gamma^2 + \frac{z}{r}\gamma^3$
- have a gap from zero (as a gravitational effect)
- agree well with the continuum prediction

Continuum limit and Finite volume effect

Continuum limit $a = 1/N \to 0$

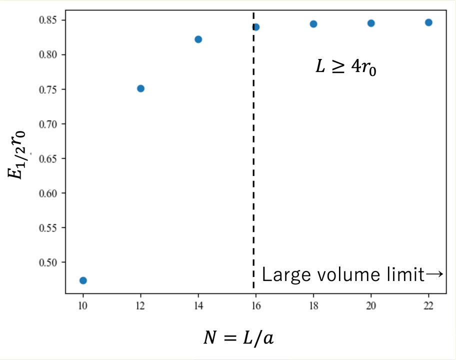


Fixed parameter:

$$L = Na, r_0 = Na/4, m = 14/L$$

Agree well with the conti. prediction!

Large volume limit $L = Na \to \infty$



Fixed parameter:

$$r_0 = 4a$$

Saturates when $L \geq 4r_0$!

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T-Anomaly

"Anomaly" is a phenomenon in which a partition function $Z[A]$ does not have a symmetry of the classical action $S[A]$.

We assume $S[A] = \int_Y \bar{\psi} \not{D}^Y \psi$ has time reversal symmetry. However, the partition function

$$Z_{reg}[A] = \prod_{\lambda} \frac{i\lambda}{i\lambda + M_{PV}} = |Z[A]| \exp\left(-i\frac{\pi}{2}\eta(i\not{D}^Y)\right)$$

breaks T-symmetry since PV regulator has no T-symm.

$$\eta(i\not{D}^Y) = \lim_{\epsilon \to +0} \lim_{s \to 0} \sum_{\lambda \in \mathrm{Spec}\left(i\not{D}^Y\right)} \frac{\lambda + \epsilon}{|\lambda + \epsilon|^{1+s}}$$

$$= \sum_{\lambda \neq 0} \mathrm{sign}(\lambda) + \#\{\lambda = 0\}$$

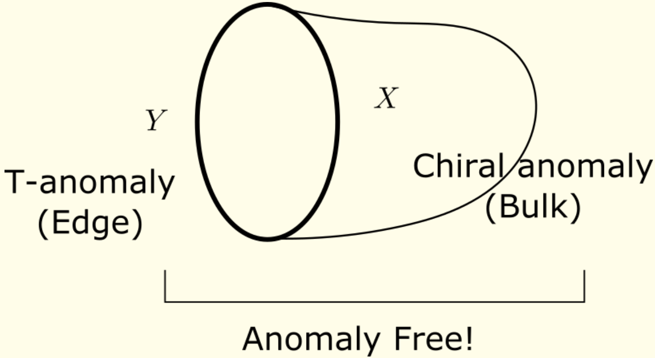
Anomaly inflow

T-anomaly is cancelled by $\exp\left(i\pi \int_X ch(F)\right)$, so

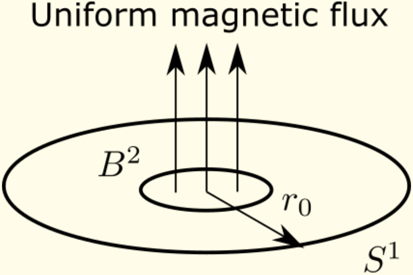
$$Z[A,X] = |Z[A]| \exp\left[i\pi \left(\underbrace{\int_X ch(F) - \frac{1}{2}\eta(i\not{D}^Y)}_{\text{APS index} \in \mathbb{Z}} \right) \right]$$

has T-symmetry!

$\longrightarrow$ Anomaly inflow (Cf. Witten [2016])



B^2 with magnetic flux (continuum)



$U(1)$ connection:

$$A = \begin{cases} -a\left(\frac{r}{r_0/2}\right)^2 d\theta & (r < r_0/2) \\ -a d\theta & (r_0/2 < r \leq r_0) \end{cases}$$

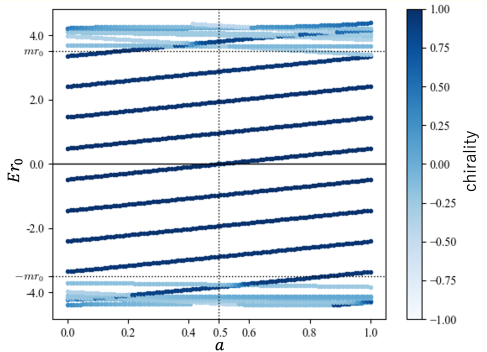
1-st Chern class:

$$\frac{1}{2\pi}\int_{S^1} dA = -a < 0$$

We consider $\not{D}^{B^2} = \sum \sigma^j(\frac{\partial}{\partial x^j} - iA_j)$, then APS index is

$$\begin{aligned} \mathrm{Ind}_{\mathrm{APS}}(B^2) =& \frac{1}{2\pi}\int dA - \frac{1}{2}\eta(i\not{D}^{S^1}_{eff}) \\ =& \frac{1}{2\pi}\int dA - \frac{1}{2}\eta\left(\frac{1}{r_0}\left(-i\frac{\partial}{\partial\theta} + a + \frac{1}{2}\right)\right) = -[a + \frac{1}{2}]. \end{aligned}$$

S^1 domain-wall with magnetic flux (lattice)



covariant derivative:

$$(\nabla_i \psi)_x = e^{-iA_i a}\psi_{x+\hat{i}} - \psi_x$$

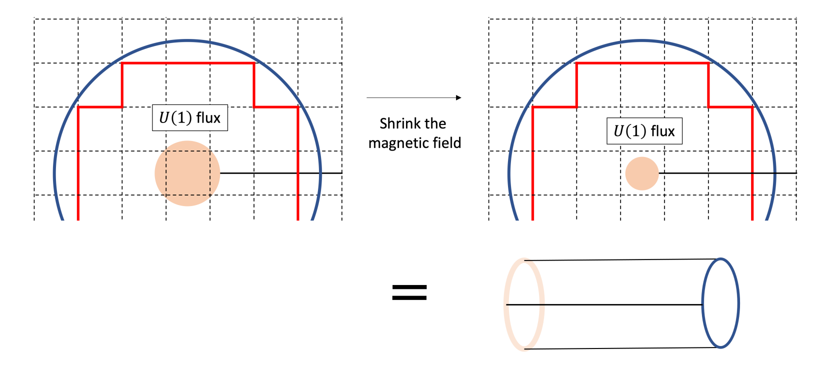
$$H = \sigma_3 \sum_{i=1,2}\left[\sigma_i \frac{\nabla_i - \nabla_i^\dagger}{2} + \frac{1}{2}\nabla_i \nabla_i^\dagger\right] + \sigma_3 \epsilon m a$$

APS index is given by (Cf. Fukaya et al. [2020], Yamaguchi-san's talk)

$$\mathrm{Ind}_{\mathrm{APS}}(B^2) = -\frac{1}{2}\eta(H) = -[a + \frac{1}{2}].$$

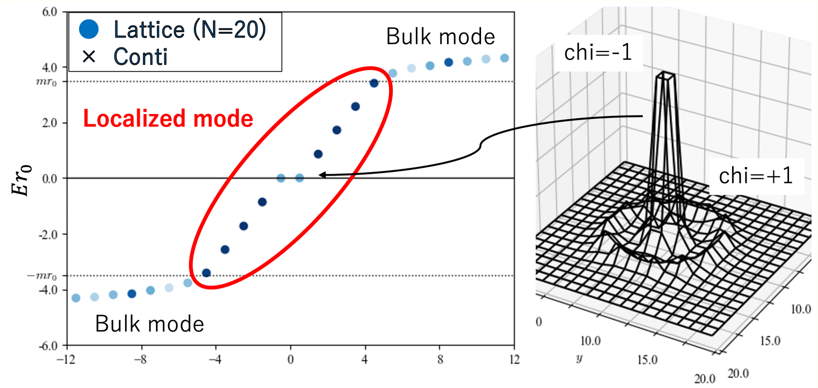
⟶ Agree well with continuum prediction !

Shrink the $U(1)$ flux when $a = 0.5$



Two zero-modes appear at the center and the edge?

Spectrum and zero modes



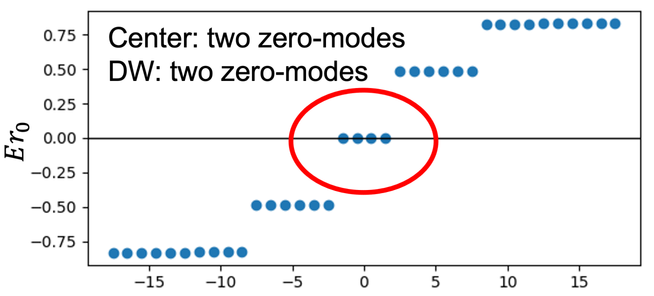
- Two zero modes appear at the domain-wall and the flux!!
- The flux mode tunnels with an edge mode.
- The flux has chirality -1 (Edge modes has chirality $+1$).

Euler number of S^2

We can consider $H \to (d + d^\dagger)_{S^2}$ $(m \to \infty)$, but

$$\text{Ker}(H) = 4 = 2 \times \sum_i \beta^i, \ (\beta^i : \text{Betti number (cf. Misumi-san's talk)})$$

Extra zero-modes appear at the center!



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Summary and Outlook

[Summary]

In cases S^1 and S^2, we embodied Nash's thm in domain-wall.

- Masless chiral edge sates appear on the domain-wall.
- Edge states feel gravity through the induced spin connection.
- We can see "Anomaly inflow" !

[Outlook]

- Gravitational anomaly inflow
- Index theorem with a nontrivial curvature
- Formulate real projective space

Reference i

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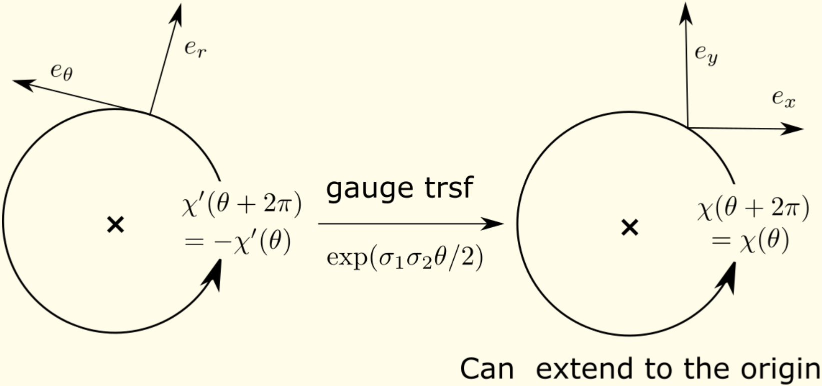
Contents

Periodicity of Edge modes

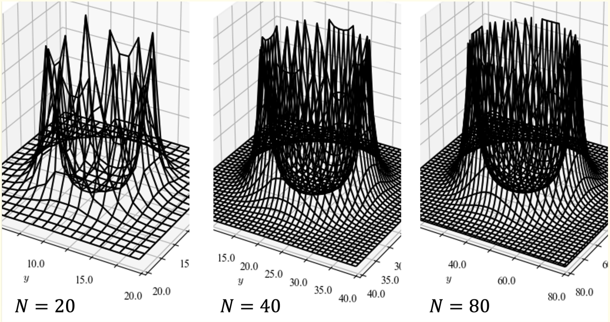
S^1 admits two spin structures:

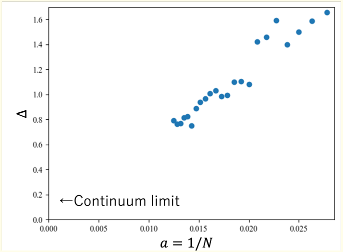
$\longrightarrow$ periodic spinor and anti-periodic spinor.



Only anti-periodic spinors appear at the boundary.

Recovery of Rotational symmetry in the continuum limit (S^1)





$$\Delta = (\max(\text{peak}) - \min(\text{peak}))/a^2$$

The rotational symmetry automatically recovers in the continuum limit!

Effective Dirac op for S^2 domain-wall

We consider a normalized edge state as

$$\psi_{\text{edge}} = \rho(r)\begin{pmatrix}\chi(\theta,\phi)\\ \frac{\boldsymbol{x}\cdot\boldsymbol{\sigma}}{r}\chi(\theta,\phi)\end{pmatrix}$$

$$\int_0^\infty dr r^2 2\rho^2 = 1,\ \int_{S^2}\chi^\dagger\chi = 1,$$

and we assume $2r^2\rho^2 \to \delta(r-r_0)$ $(m\to\infty)$. Thus

$$\int dx^3\psi^\dagger_{\text{edge}}H\psi_{\text{edge}} = \int_0^\infty dr 2r^2\rho^2\int_{S^2}\chi^\dagger\frac{1}{r}(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)\chi$$

$$\to \int_{S^2}\chi^\dagger\underbrace{\frac{1}{r_0}(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)}\chi\ (m\to\infty),$$

Effective Dirac op H_{S^2} !!

where $\boldsymbol{L}$ is an orbital angular momentum.

Effective Dirac op and Dirac op. of S^2

The gauge transformation using

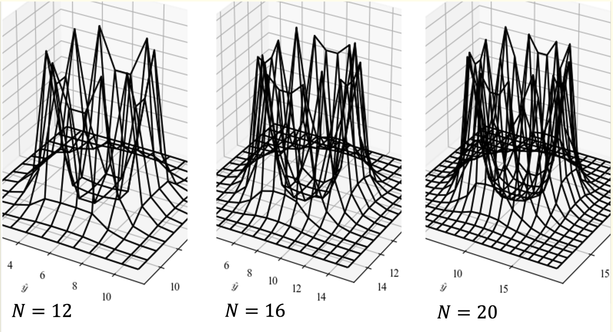
$$s = \begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\left(\frac{\theta}{2}\right) & -e^{-i\frac{\phi}{2}}\sin\left(\frac{\theta}{2}\right) \\ e^{i\frac{\phi}{2}}\sin\left(\frac{\theta}{2}\right) & e^{i\frac{\phi}{2}}\cos\left(\frac{\theta}{2}\right) \end{pmatrix} e^{i\frac{\phi}{2}}$$

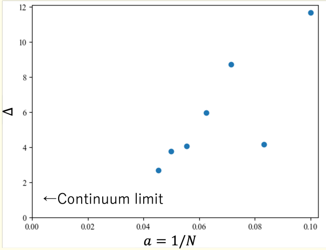
changes $\chi \to s^{-1}\chi$ and

$$\begin{aligned} H_{S^2} \to & s^{-1} H_{S^2} s \\ &= -\frac{\sigma_3}{r_0}\left(\sigma_1 \frac{\partial}{\partial\theta} + \sigma_2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} \underbrace{+\frac{i}{2\sin\theta} - \frac{\cos\theta}{2\sin\theta}\sigma_1\sigma_2}_{\text{Spin conn. of } S^2}\right)\right) \\ &= -\frac{\sigma_3}{r_0} \not{D}_{S^2}. \end{aligned}$$

Edge states are affected by the spin connection of the spherical domain-wall [Takane and Imura [2013]].

Recovery of Rotational symmetry in the continuum limit (S^2)





(slice at $z = N/2$)

$$\Delta = (\max(\text{peak}) - \min(\text{peak}))/a^3$$

The rotational symmetry automatically recovers in the continuum limit!

Construct H

We consider matrix valued function and two action of Pauli matrix:

$$\begin{aligned} \psi =& \psi_0 + \psi_i \sigma^i, \\ \sigma^{i,L}\psi =& \sigma^i \psi,\ \sigma^{i,R}\psi = \psi\sigma^i \end{aligned}$$

Hermitian Dirac operator H is defined by

$$\begin{aligned} D =& \left(\sum_{i=1,2,3} \left[\sigma^{i,L} \frac{\nabla_i - \nabla_i^\dagger}{2} + \frac{1}{2}\nabla_i \nabla_i^\dagger \right] + \epsilon m a \right) \\ D_1 =& D\frac{x^i \sigma^{i,R}}{r} + \frac{x^i \sigma^{i,R}}{r} D \\ H =& \begin{pmatrix} 0 & D_1 \\ D_1^\dagger & 0 \end{pmatrix} \to (d + d^\dagger)_{S^2} \end{aligned}$$