Spontaneous mass generation and chiral symmetry breaking

Yukimi Goto

Faculty of Mathematics, Kyushu University

Joint work with T. Koma

Lattice and Continuum Field Theories 2022

Yukawa Institute for Theoretical Physics, Kyoto University

QCD Theory

- QCD Lagrangian: $\psi = (u, d, s, \dots) = N_f$-quark, $t^a = \mathrm{SU}(3)$-color matrix

$$\mathcal{L}_{\mathrm{QCD}} := \underbrace{\bar{\psi}(i\gamma^\mu D_\mu - m)\psi}_{\text{quark}} - \underbrace{\mathrm{tr}\, F^a_{\mu\nu} F^{\mu\nu}_a / 4}_{\text{Yang-Mills}}, \quad D_\mu := \partial_\mu - igt^a A^a_\mu$$

$$F^a_{\mu\nu} := \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu \quad (a, b, c = 1, 2, \dots, 8)$$

- Chiral transformation: For $U_{R/L} \in \mathrm{SU}(N_f)$

$$\psi_L = \frac{1-\gamma_5}{2}\psi \to U_L \psi_L, \quad \psi_R = \frac{1+\gamma_5}{2}\psi \to U_R \psi_R$$

$\mathcal{L}_{\mathrm{QCD}}$ is invariant under the chiral transformation when $m = 0$.

Effective (Low-Energy) Theory

A simplest effective theory: Nambu-Jona-Lasinio (NJL) model.

$$\mathcal{L}_{\mathrm{NJL}} := \bar{\psi}(i\gamma^\mu D_\mu - m)\psi + g \underbrace{\left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2\right]}_{\text{4-fermion}} \quad \text{(without gluon)}$$

$\mathcal{L}_{\mathrm{NJL}}$ has the chiral symmetry when $m = 0$.
In physics literature, it shows that the dynamical mass $M \sim \langle\bar{\psi}\psi\rangle \neq 0$ when $g > g_c \to$ Spontaneous breakdown of chiral symmetry $\simeq$ quark mass generation.

Main issue in this talk

Proof of mass generation in a mathematically rigorous way.

- Lattice model: Some known results...

Staggered Fermion: Lattice NJL

We consider the staggered fermion + four-fermion interaction (NJL).
Two formalism:

- Lagrangian: $\psi, \bar{\psi}$: Grassmann
  When $m = 0$, the action is invariant under

$$\psi(x) \to e^{i\alpha\varepsilon(x)}\psi(x), \quad \bar{\psi}(x) \to e^{i\alpha\varepsilon(x)}\bar{\psi}(x), \quad \varepsilon(x) = (-1)^{\sum_{\mu=1}^{\nu} x_\mu}, \quad \alpha \in \mathbb{R}$$

Theorem (Salmhoter-Seiler '91)

In $\Lambda \to \mathbb{Z}^\nu$, the mass generation $\langle \bar{\psi}\psi \rangle \neq 0$ occurs when $\nu \geq 3$.

- Hamiltonian: ψ : Fermion operator

Kogut-Susskind Hamiltonian

$\Lambda = [-L+1, L]^\nu$, $\{\psi^\dagger(x), \psi(y)\} = \delta_{x,y}$, $\{\psi(x), \psi(y)\} = 0$, $\kappa^{-1} = a =$ lattice spacing.

$$H(m) := i\kappa \sum_{x\in\Lambda} \sum_{\mu} (-1)^{\theta_\mu(x)} [\psi^\dagger(x)\psi(x+e_\mu) - \psi^\dagger(x+e_\mu)\psi(x)]$$

$$+ m \sum_x (-1)^{\sum_\mu x^{(\mu)}} \rho(x) + g \sum_{x,\mu} \rho(x)\rho(x+e_\mu), \quad \rho(x) = \psi^\dagger(x)\psi(x) - \frac{1}{2}$$

$$\theta_\mu(x) := \begin{cases} x^{(1)} + \cdots + x^{(\mu-1)} & \text{for } x^{(\mu)} \neq L \\ x^{(1)} + \cdots + x^{(\mu-1)} + 1 & \text{for } x^{(\mu)} = L, \end{cases} \quad x^{(0)} := 0$$

- $H(0)$ does not have chiral symmetry.
- $H(0)$ is invariant under a certain discrete chiral transformation.

Particle-Hole Symmetry and Mass Generation

$$U_{\mathrm{PH}} := \prod_{x\in\Lambda}\prod_{y\neq x}(-1)^{\psi^\dagger(y)\psi(y)}(\psi^\dagger(x)+\psi(x)) \quad \Rightarrow \quad U_{\mathrm{PH}}^\dagger\psi(x)U_{\mathrm{PH}} = \psi^\dagger(x)$$

$$\therefore \quad U_{\mathrm{PH}}^\dagger H(0) U_{\mathrm{PH}} = H(0) \Rightarrow \langle\rho(x)\rangle_{\beta,m=0} = 0, \quad \langle A\rangle_{\beta,m} := Z_{\beta,m}^{-1}\,\mathrm{tr}(Ae^{-\beta H(m)})$$

Theorem (in preparation)

Assume $\nu \geq 3$. For $|\kappa|/g \leq g_c$, $\beta \geq \beta_c$ we can show that

$$\lim_{m\to 0}\lim_{\Lambda\to\mathbb{Z}^\nu}\langle\mathcal{O}_\Lambda\rangle_{\beta,m} := \left\langle \sum_x (-1)^{\sum_\mu x^{(\mu)}}\rho(x)\right\rangle_{\beta,m} \neq 0.$$

- Breaking the discrete symmetry
- If we can take the continuum limit, then the chiral symmetry is broken.

Strategy of Proof 1: Reflection Positivity

- Fermion Reflection Positivity (RP): Jaffe-Pedrocchi '15, Koma '21.
  Divide $\Lambda = \Lambda_- \cup \Lambda_+$ and take r s.t. $r(\Lambda_\pm) = \Lambda_\mp$.
  E.g. $\Lambda_- = \{x : -L+1 \leq x^{(1)} \leq 0\}$, $r(x^{(1)}) = -x^{(1)}$.
  Define $\mathcal{A}_\pm$ = polynomials of $\psi(x), \psi^\dagger(y)$, $x, y \in \Lambda_\pm$, and anti-linear $\theta : \mathcal{A}_\pm \to \mathcal{A}_\mp$ s.t. $(\theta\psi)(x) = \psi(\theta(x))$ etc.
  In usual, RP is

$$\mathrm{tr}(A\theta(A)e^{-H}) \geq 0 \quad A \in \mathcal{A}_\pm$$

  if $H = H_+ + H_- + H_0$, with $H_\pm \in \mathcal{A}_\pm$, $H_0 = \sum_i A_i\theta(A_i)$, $A_i \in \mathcal{A}_\pm$.
  This fails for fermions.
  Fermion RP: $\mathrm{tr}(A\theta(A)) \geq 0$ holds for $A \in \mathcal{A}_\pm$.
  For functions h_μ, the modified interaction is defined as

$$H_{\mathrm{int}} \to H_{\mathrm{int}}(h) = \sum_{x,\mu} (\rho(x) + \rho(x+e_\mu) + (-1)^{\sum_i x^{(i)}} h_\mu(x))^2.$$

  $\Rightarrow$ Gaussian domination: $\mathrm{tr}\exp(-\beta H(m,h)) \leq \mathrm{tr}\exp(-\beta H(m))$.

Strategy of Proof 2: Infrared Bound

- Infrared Bound (IB): Dyson-Lieb-Simon for Heisenberg anti-ferro. '78.
  Define the Duhamel two-point function:
  $$(A,B) = Z_{\beta,m}^{-1} \int_0^1 ds e^{-s\beta H(m)} A e^{-(1-s)\beta H(m)} B.$$
  Then for $\rho_p = |\Lambda|^{-1/2} \sum_x e^{ipx} \rho(x)$, we have the IB ($\simeq$ BEC):
  $$\lim_{\Lambda \to \mathbb{Z}^\nu} |\Lambda|^{-1} \sum_{p \neq 0} (\rho_p, \rho_{-p}) \leq \frac{C}{\beta g} \int_{|p_i| \leq \pi} \frac{dp}{E_p} \quad (E_p = v - \sum_i \cos p^{(i)} \sim |p|^2)$$
- Long-Range Order: IB implies
  $$\lim_{\Lambda \to \infty} m_{\mathrm{LRO}}^{(\Lambda)} = \lim_{\Lambda \to \infty} |\Lambda|^{-1} \left\langle \mathcal{O}_\Lambda^2 \right\rangle_{\beta, m=0}^{1/2} > 0.$$
- Mass Generation: By Koma-Tasaki '93
  $$\lim_{m \to 0} \lim_{\Lambda \to \mathbb{Z}^\nu} \langle \mathcal{O}_\Lambda \rangle_{\beta, m} \geq C \lim_{\Lambda \to \infty} m_{\mathrm{LRO}}^{(\Lambda)} > 0.$$

Summary and Other Problems

Kogut-Susskind (KS) Hamiltonian: Lattice version of NJL model.

- KS Hamiltonian does not have chiral symmetry.
- Mass term $\langle \rho(x) \rangle = 0$ when $m = 0$ in KS Hamiltonian.
- For a large coupling constant, we can show the mass generation in the infinite-volume limit.
- Fermion reflection positivity is crucial.
- If we can take the continuum limit, the chiral symmetry is broken.

I don't know

- KS Hamiltonian $\leftrightarrow$ Grassmann theory?
- Proof of mass gap (spectral gap).
- Existence of the continuum limit.