

Gravitational Waves Induced by a Particle Orbiting around a Rotating Black Hole

— *Effect of Orbital Precession* —

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We have calculated the energy and the angular momentum fluxes of gravitational waves induced by a test particle of mass μ , the orbital radius r_0 and the Carter constant C moving around a Kerr black hole of mass $M \gg \mu$ and the angular momentum aM . In this case the orbital plane of the particle precesses around the symmetric axis due to the spin-orbit interaction. It is found that the energy flux is approximately written as a linear function of $\cos \theta_i$ for $\cos \theta_i \gtrsim 0.8$, where θ_i is the inclination of the orbital plane to the equatorial plane. It is also found that the orbital precession drastically modifies the wave pattern of gravitational waves. We propose a method to calculate the back reaction to the Carter constant and show a table of the fluxes of the Carter constant as well as the energy, angular momentum for various combinations of a , r_0 and C . Using those fluxes, we perform a rough estimation of the time evolution of the inclination angle. It seems that the inclination angle of the orbital plane gradually becomes small for $a < 0$ case and becomes large for $a > 0$ case due to the back reaction of the gravitational radiation.

§ 1. Introduction

We know there exist at least three binary neutron star systems in our Galaxy, PSR1913+16,¹⁾ PSR2127+11C²⁾ and PSR1534+12,³⁾ which will merge within the Hubble time radiating gravitational waves. Statistical studies using the above binary pulsars^{4),5)} showed that coalescence of binary neutron stars may occur about 3 events per year within a distance of 200Mpc. They also showed that the coalescence of black hole-neutron star binaries may occur approximately at the same rate. In the final few minutes before merging, such a binary emits gravitational waves of the frequency 10–1000 Hz and the effective amplitude $\sim 10^{-21} - 10^{-22}$.⁶⁾ Furthermore the wave form is so clean in the sense that the effects except for the gravitation are not important⁷⁾ that it is predictable. Hence coalescing compact stars binary, such as neutron star-neutron star and neutron star-black hole binary, is regarded as one of the most promising sources of planned laser interferometric gravitational wave observatory, such as LIGO⁸⁾ and VIRGO.⁹⁾

Recently to prepare the theoretical template in the frequency band 10–1000Hz,¹⁰⁾ efforts are paid to calculate accurate wave forms of gravitational waves during the inspiral of such a binary. As pointed out by several authors, there is a possibility to determine not only the various parameters of the binary, such as mass, radius, spin, etc., but also the cosmological parameters such as the Hubble parameter from the relativistic wave form of gravitational waves from binary.^{7),10)} Hence it is important to investigate the relativistic features of gravitational waves in detail. To see various relativistic effects, in previous two papers^{11),12)} we calculated the energy flux of gravitational waves induced by a test particle orbiting around nonrotating and

rotating black holes. We found that the relativistic effects such as the contribution of the higher multipole parts ($l \geq 3$) of the energy flux, curvature scattering effect of gravitational waves and the spin-orbit interaction effect are important.

In the case that the particle moves around a nonrotating body, the orbital plane does not change. Even in the case that the particle moves around a spinning body, if it moves on the equatorial plane, it leaves on it. However, if the particle moves on a plane except for the equatorial plane around the spinning body, the orbital plane precesses around the symmetric axis due to the spin-orbit interaction effect.¹³⁾ Since a companion does not always move on the equatorial plane of the primary star in a binary system, we should consider the case in which it has a precession orbit. To study the effect of the precession to the orbital evolution of a binary, it is appropriate to investigate gravitational waves induced by a particle on a Kerr spacetime because the particle has the fourth constant of motion, the so-called Carter constant, which specifies the degree of the precession of the particle.^{14),15)} Hence in this paper to investigate the precession effect, we perform numerical calculations of gravitational waves from a Kerr black hole of mass M and angular momentum aM induced by a test particle of mass $\mu \ll M$ with the Carter constant C . As a first step we restrict the case that the particle moves at a constant radius ($r = r_0$).

In §2 we review the Sasaki-Nakamura formalism and show the formula of calculation of the energy flux and all components of the angular momentum fluxes. We propose a method of calculating the change rate of the Carter constant using those fluxes. In §3 we mention numerical methods and the accuracy. In §4 the numerical results are shown, and from those we consider the orbital evolution of a companion which precesses around a spinning star. Section 5 is devoted to the discussion.

§ 2. Formulation

2.1. Basic equations

To calculate a perturbation on a Kerr spacetime, the Teukolsky formalism¹⁶⁾ is often adopted, in which the perturbation is expressed as the Newman-Penrose quantities.¹⁷⁾ In the Newman-Penrose formalism, the perturbation of the outgoing gravitational waves is described by Weyl scalar ψ_4 . On a Kerr spacetime, the equation for ψ_4 is completely separable and ψ_4 can be written as

$$\psi_4 = (r - ia \sin \theta)^{-4} \int d\omega e^{-i\omega t} \sum_{l,m} e^{im\varphi} \frac{{}_2S_{lm}^{a\omega}}{\sqrt{2\pi}}(\theta) R_{lm\omega}(r). \quad (2.1)$$

${}_2S_{lm}^{a\omega}$ is a spin-weighted spheroidal function, which obeys

$$\begin{aligned} \frac{1}{\sin \theta} \frac{d}{d\theta} \left\{ \sin \theta \frac{d({}_2S_{lm}^{a\omega})}{d\theta} \right\} + \left\{ -a^2 \omega^2 \sin^2 \theta - \frac{(m - 2 \cos \theta)^2}{\sin^2 \theta} + 4a\omega \cos \theta \right. \\ \left. - 2 + 2ma\omega + \lambda \right\} {}_2S_{lm}^{a\omega} = 0, \end{aligned} \quad (2.2)$$

and is normalized as

$$\int_0^\pi |_{-2} S_{lm}^{a\omega}|^2 \sin\theta d\theta = 1. \quad (2.3)$$

On the other hand, $R_{lm\omega}(r)$ obeys the Teukolsky equation

$$\Delta^2 \frac{d}{dr} \left(\frac{1}{\Delta} \frac{dR_{lm\omega}}{dr} \right) - V(r) R_{lm\omega} = T_{lm\omega}(r). \quad (2.4)$$

Here, λ is the separation constant, $T_{lm\omega}(r)$ is a source term, $\Delta = r^2 - 2Mr + a^2$ and

$$V(r) = -\frac{K^2 + 4i(r-M)K}{\Delta} + 8i\omega r + \lambda, \quad (2.5)$$

where $K = (r^2 + a^2)\omega - ma$. To calculate gravitational waves from a test particle with a circular orbit, Detweiler¹⁸⁾ solved the Teukolsky equation by the Green function method. The solution of it at infinity is written as

$$R_{lm\omega} = R_{\text{out}}^{(0)}(\infty) W_r^{-1} \int_{r_+}^{\infty} T_{lm\omega} R_{\text{in}}^{(0)} \Delta^{-2} dr, \quad (2.6)$$

where r_+ denotes the radius of event horizon, $R_{\text{in}}^{(0)}$ and $R_{\text{out}}^{(0)}$ are two homogeneous solutions, and

$$W_r = -\Delta^{-1} R_{\text{out}}^{(0)} dR_{\text{in}}^{(0)}/dr + \Delta^{-1} R_{\text{in}}^{(0)} dR_{\text{out}}^{(0)}/dr = \text{constant}. \quad (2.7)$$

$R_{\text{out}}^{(0)}$ and $R_{\text{in}}^{(0)}$ satisfy the following boundary conditions at infinity,

$$\begin{aligned} R_{\text{out}}^{(0)} &\rightarrow r^3 e^{i\omega r^*}, \\ R_{\text{in}}^{(0)} &\rightarrow r^3 C_{\text{out}} e^{i\omega r^*} + r^{-1} C_{\text{in}} e^{-i\omega r^*}, \end{aligned} \quad (2.8)$$

where C_{out} and C_{in} are two complex constants and r^* is the tortoise coordinate defined by

$$\frac{dr^*}{dr} = \frac{r^2 + a^2}{\Delta}. \quad (2.9)$$

From Eqs. (2.7) and (2.8), W_r becomes $2i\omega C_{\text{in}}$. Hence to obtain gravitational waves at infinity, we must carry out the accurate numerical calculation to solve $R_{\text{in}}^{(0)}$ and C_{in} . However it seems difficult to obtain C_{in} accurately because of the bad behavior of $R_{\text{in}}^{(0)}$ at infinity. Thus, instead of the Teukolsky equation we use the Sasaki-Nakamura (SN) equation,^{19)~21)} which is transformed from the Teukolsky equation. In their formalism the equation becomes

$$\left[\frac{d^2}{dr^{*2}} - F(r) \frac{d}{dr^*} - U(r) \right] X(r) = S(r), \quad (2.10)$$

where

$$F(r) = \frac{\gamma, r}{\gamma} \frac{\Delta}{r^2 + a^2}, \quad (2.11)$$

$$\gamma = c_0 + c_1/r + c_2/r^2 + c_3/r^3 + c_4/r^4,$$

$$c_0 = -12i\omega M + \lambda(\lambda + 2) - 12a\omega(a\omega - m),$$

$$\begin{aligned}
c_1 &= 8ia[3a\omega - \lambda(a\omega - m)], \\
c_2 &= -24iaM(a\omega - m) + 12a^2[1 - 2(a\omega - m)^2], \\
c_3 &= 24ia^3(a\omega - m) - 24Ma^2, \\
c_4 &= 12a^4
\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}
U(r) &= \frac{\Delta U_1}{(r^2 + a^2)^2} + G^2 + \frac{\Delta G_{,r}}{r^2 + a^2} - FG, \\
G &= -\frac{2(r - M)}{r^2 + a^2} + \frac{r\Delta}{(r^2 + a^2)^2}, \\
U_1 &= V + \frac{\Delta^2}{\beta} \left[\left(2\alpha + \frac{\beta_{,r}}{\Delta} \right)_{,r} - \frac{\gamma_{,r}}{\gamma} \left(\alpha + \frac{\beta_{,r}}{\Delta} \right) \right], \\
\alpha &= -i\frac{K\beta}{\Delta^2} + 3iK_{,r} + \lambda + \frac{6\Delta}{r^2}, \\
\beta &= 2\Delta \left(-iK + r - M - \frac{2\Delta}{r} \right).
\end{aligned} \tag{2.13}$$

The potentials $F(r)$ and $U(r)$ behave, respectively, as $O(r^{-n})$ and $(iK/(r^2 + a^2))^2 + O(r^{-n})$, $n \geq 2$, at $r^* \rightarrow \pm\infty$. This means that the boundary conditions of the homogeneous equation at $r^* \rightarrow \pm\infty$ become

$$\begin{aligned}
X_{\text{out}}^{(0)} &\rightarrow e^{i\omega r^*}, \\
X_{\text{in}}^{(0)} &\rightarrow A_{\text{out}} e^{i\omega r^*} + A_{\text{in}} e^{-i\omega r^*}, \quad r^* \rightarrow \infty, \\
X_{\text{out}}^{(0)} &\rightarrow B_{\text{out}} e^{ikr^*} + B_{\text{in}} e^{-ikr^*}, \\
X_{\text{in}}^{(0)} &\rightarrow e^{ikr^*}, \quad r^* \rightarrow -\infty,
\end{aligned} \tag{2.14}$$

where $k = \omega - ma/2Mr_+$. $S(r)$ is related to the source term of the Teukolsky equation by

$$S = \frac{\Delta \gamma W}{r^2 (r^2 + a^2)^{3/2}} \exp\left(-i \int \frac{K}{\Delta} dr\right), \tag{2.15}$$

where

$$W_{,rr} = \frac{Tr^2}{\Delta^2} \exp\left(i \int \frac{K}{\Delta} dr\right). \tag{2.16}$$

The solution of gravitational waves at infinity is obtained by the Green function method,

$$X_{lm\omega} = X_{\text{out}}^{(0)}(\infty) \frac{c_0}{2i\omega A_{\text{in}}} \int_{-\infty}^{\infty} \frac{S X_{\text{in}}^{(0)}}{\gamma} dr^*. \tag{2.17}$$

Therefore to obtain gravitational waves at infinity, we must also perform the accurate numerical calculation to solve $X_{\text{in}}^{(0)}$ and A_{in} . Contrary to the Teukolsky equation, in

the SN's one, the absolute values of the ingoing and outgoing parts of $X_{\text{in}}^{(0)}$ are comparable for large $r \gg 2\pi/\omega$, so that it is possible to obtain A_{in} accurately.

2.2. Source term

A test particle obeys the following equations of motion,

$$\begin{aligned}\Sigma \frac{d\theta}{d\tau} &= \pm \left[C - \cos^2 \theta \left\{ a^2(1 - E^2) + \frac{L_z^2}{\sin^2 \theta} \right\} \right]^{1/2} \equiv \Theta(\theta), \\ \Sigma \frac{d\varphi}{d\tau} &= - \left(aE - \frac{L_z^2}{\sin^2 \theta} \right) + \frac{a}{\Delta} (E(r^2 + a^2) - aL_z), \\ \Sigma \frac{dt}{d\tau} &= - \left(aE - \frac{L_z^2}{\sin^2 \theta} \right) a \sin^2 \theta + \frac{a^2 + r^2}{\Delta} (E(r^2 + a^2) - aL_z), \\ \Sigma \frac{dr}{d\tau} &= \pm \sqrt{R},\end{aligned}\tag{2.18}$$

where

$$\begin{aligned}\Sigma &= r^2 + a^2 \cos^2 \theta, \\ R &= [E(r^2 + a^2) - aL_z]^2 - \Delta[(Ea - L_z)^2 + r^2 + C].\end{aligned}$$

In this paper we consider the case in which a particle moves along a constant radius $r = r_0$, but precesses around the symmetric axis. Then the energy momentum tensor of a test particle is written as

$$T^{\mu\nu} = \frac{\mu}{\Sigma \sin \theta dt/d\tau} \frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau} \delta(r - r_0) \delta(\theta - \theta(t)) \delta(\varphi - \varphi(t)).\tag{2.19}$$

If r_0 and C are given, the energy E and the z -component of the angular momentum L_z of a test particle are obtained by the two equations, $R=0$ and $dR/dr=0$.

The source term of the Teukolsky equation is written as

$$T = 4 \int d\Omega dt \rho^{-5} \bar{\rho}^{-1} (B'_2 + B'^*_2) e^{-im\varphi + i\omega t} \frac{-2S_{lm}^{a\omega}}{\sqrt{2\pi}},\tag{2.20}$$

where

$$\begin{aligned}B'_2 &= -\frac{1}{2} \rho^8 \bar{\rho} L_{-1} [\rho^{-4} L_0 (\rho^{-2} \bar{\rho}^{-1} T_{mn})] \\ &\quad - \frac{1}{2\sqrt{2}} \rho^8 \bar{\rho} \Delta^2 L_{-1} [\rho^{-4} \bar{\rho}^2 J_+ (\rho^{-2} \bar{\rho}^{-2} \Delta^{-1} T_{\bar{m}\bar{n}})], \\ B'^*_2 &= -\frac{1}{4} \rho^8 \bar{\rho} \Delta^2 J_+ [\rho^{-4} J_+ (\rho^{-2} \bar{\rho} T_{\bar{m}\bar{n}})] \\ &\quad - \frac{1}{2\sqrt{2}} \rho^8 \bar{\rho} \Delta^2 J_+ [\rho^{-4} \bar{\rho}^2 \Delta^{-1} L_{-1} (\rho^{-2} \bar{\rho}^{-2} T_{\bar{m}\bar{n}})],\end{aligned}\tag{2.21}$$

$$\rho = (r - i a \cos \theta)^{-1},\tag{2.22}$$

$$L_s = \partial_\theta + \frac{m}{\sin\theta} - a\omega\sin\theta + \text{scot}\theta,$$

$$J_+ = \partial_r + \frac{iK}{\Delta}, \quad (2.23)$$

and $\bar{Q}$ denotes the complex conjugate of Q . From the equations of motion of a test particle, T_{nn} , $T_{\bar{m}n}$ and $T_{\bar{m}\bar{m}}$ are written as

$$T_{nn} = -\frac{C_{nn}}{\sin\theta} \delta(r-r_0) \delta(\theta-\theta(t)) \delta(\varphi-\varphi(t)),$$

$$T_{\bar{m}n} = -\frac{C_{\bar{m}n}}{\sin\theta} \delta(r-r_0) \delta(\theta-\theta(t)) \delta(\varphi-\varphi(t)),$$

$$T_{\bar{m}\bar{m}} = -\frac{C_{\bar{m}\bar{m}}}{\sin\theta} \delta(r-r_0) \delta(\theta-\theta(t)) \delta(\varphi-\varphi(t)), \quad (2.24)$$

where

$$C_{nn} = \frac{\mu}{4\Sigma^3 \dot{t}} [E(r^2 + a^2) - aL_z]^2,$$

$$C_{\bar{m}n} = -\frac{\mu\rho}{2\sqrt{2}\Sigma^2 \dot{t}} [E(r^2 + a^2) - aL_z] \left[i\sin\theta \left(aE - \frac{L_z}{\sin^2\theta} \right) + \Theta(\theta) \right],$$

$$C_{\bar{m}\bar{m}} = \frac{\mu\rho^2}{2\Sigma \dot{t}} \left[i\sin\theta \left(aE - \frac{L_z}{\sin^2\theta} \right) + \Theta(\theta) \right]^2 \quad (2.25)$$

and $\dot{t} = dt/d\tau$. Substituting Eq. (2.21) into Eq. (2.20) and performing integral by part, we obtain the following expression,

$$T = \frac{4}{\sqrt{2}\pi} \int dt e^{i\omega t - im\varphi} \left[-\frac{1}{2} L_1^+ \{ \rho^{-4} L_2^+ (\rho^3 S) \} C_{nn} \rho^{-2} \bar{\rho}^{-1} \delta(r-r_0) \right. \\ \left. + \frac{\Delta^2 \bar{\rho}^2}{\sqrt{2}\rho} (S_{,\theta} + ia(\bar{\rho} - \rho) \sin\theta S + f_2 S) J_+ \{ C_{\bar{m}n} \rho^{-2} \bar{\rho}^{-2} \Delta^{-1} \delta(r-r_0) \} \right. \\ \left. + \frac{1}{2\sqrt{2}} L_2^+ \{ \rho^3 S (\bar{\rho}^2 \rho^{-4})_{,r} \} C_{\bar{m}n} \Delta \rho^{-2} \bar{\rho}^{-2} \delta(r-r_0) \right. \\ \left. - \frac{1}{4} \rho^3 \Delta^2 S J_+ \{ \rho^{-4} J_+ (\bar{\rho} \rho^{-2} C_{\bar{m}\bar{m}} \delta(r-r_0)) \} \right], \quad (2.26)$$

where

$$L_s^+ = \partial_\theta - \frac{m}{\sin\theta} + a\omega\sin\theta + \text{scot}\theta,$$

$$f_s = -\frac{m}{\sin\theta} + a\omega\sin\theta + \text{scot}\theta, \quad (2.27)$$

and S denotes ${}_{-2}S_{lm}^{a\omega}(\theta(t))$. Noting the relations

$$\theta(t + \Delta t) = \theta(t), \quad \varphi(t + \Delta t) = \varphi(t) + \Delta\varphi, \quad (2.28)$$

where Δt is a period of the orbital rotation and $\Delta\varphi$ is the phase shift during Δt , we can rewrite the source term of the Teukolsky equation as follows,

$$\begin{aligned} T &= T_{\text{one}} \sum_k e^{ik(\omega\Delta t - m\Delta\varphi)}, \\ &= \frac{2\pi}{\Delta t} T_{\text{one}} \sum_n \delta(\omega - \omega_n), \end{aligned} \quad (2.29)$$

where T_{one} denotes the source term due to the contribution from one period and

$$\omega_n = \frac{2\pi n + m\Delta\varphi}{\Delta t}. \quad (n=0, \pm 1, \pm 2, \dots) \quad (2.30)$$

Using those equations, T can be written as the following form,

$$\begin{aligned} T &= \frac{2\pi}{\Delta t} \sum_k \delta(\omega - \omega_k) [(A_{nn} + A_{\bar{m}n2} + A_{\bar{m}m3}) \delta(r - r_0) \\ &\quad + (A_{\bar{m}n1} + A_{\bar{m}m2}) \delta(r - r_0)_{,r} + A_{\bar{m}m3} \delta(r - r_0)_{,rr}], \end{aligned} \quad (2.31)$$

where A_{nn} et al. are shown in Appendix A. Substituting this equation into Eq. (2.16), we find

$$\begin{aligned} W &= \frac{2\pi}{\Delta t} \sum_k \delta(\omega - \omega_k) [(E_0 - E_{1,r} + E_{2,rr})|_{r=r_0} \mathbf{H}(r_0 - r)(r_0 - r) \\ &\quad - (E_1 - 2E_2)|_{r=r_0} \mathbf{H}(r_0 - r) + E_2 \delta(r_0 - r)], \end{aligned} \quad (2.32)$$

where

$$\begin{aligned} E_0 &= (A_{nn} + A_{\bar{m}n2} + A_{\bar{m}m3}) \frac{r^2}{\Delta^2} \exp\left(i \int \frac{K}{\Delta} dr\right), \\ E_1 &= (A_{\bar{m}n1} + A_{\bar{m}m2}) \frac{r^2}{\Delta^2} \exp\left(i \int \frac{K}{\Delta} dr\right), \\ E_2 &= A_{\bar{m}m1} \frac{r^2}{\Delta^2} \exp\left(i \int \frac{K}{\Delta} dr\right). \end{aligned} \quad (2.33)$$

$\mathbf{H}(r_0 - r)$ is a step function, which is defined by

$$\begin{aligned} \mathbf{H}(r_0 - r) &= 1 \quad \text{for } r < r_0 \\ &= 0 \quad \text{for } r > r_0. \end{aligned} \quad (2.34)$$

Then, substituting W into Eq. (2.15), we obtain the source term of the SN equation.

2.3. Wave form, energy and angular momentum fluxes of gravitational waves

ψ_4 at infinity is written as

$$\psi_4 = \frac{8\pi}{r\Delta t} \sum_{lmn} \omega_n^2 \frac{X_{lm\omega_n}}{c_0} \frac{-2S_{lm}^{a\omega_n}}{\sqrt{2\pi}} e^{i\omega_n(r^* - t) + im\varphi}. \quad (2.35)$$

Then the wave form of gravitational waves is given by

$$h_+ - ih_\times = -\frac{16\pi}{r\Delta t} \sum_{lmn} \frac{X_{lm\omega n}}{c_0} \frac{-2S_{lm}^{a\omega n}}{\sqrt{2\pi}} e^{i\omega n(r^*-t) + im\varphi}. \quad (2.36)$$

Note that in the above equation we can make use of the relations of the symmetry as ${}_{-2}S_{lm}^{a\omega n}(\theta) = {}_{-2}S_{l-m}^{a\omega-n}(\pi - \theta)$ and $X_{l-m-n}/c_0 = (-1)^n \bar{X}_{lmn}/\bar{c}_0$.

The energy and the z -component of the angular momentum flux are, respectively, calculated from

$$\frac{dE}{dt} = \frac{1}{16\pi} \int (\dot{h}_+ \dot{h}_+ + \dot{h}_\times \dot{h}_\times) r^2 d\Omega \quad (2.37)$$

and

$$\frac{dJ_z}{dt} = -\frac{1}{16\pi} \int (\dot{h}_+ h_{+, \varphi} + \dot{h}_\times h_{\times, \varphi}) r^2 d\Omega. \quad (2.38)$$

Using Eq. (2.36), these are rewritten as follows,

$$\frac{dE}{dt} = \frac{16\pi}{(\Delta t)^2} \sum_{lmn} \int d\omega d\omega' \omega \omega' \frac{X_{lm\omega}}{c_0} \frac{\bar{X}_{lm\omega'}}{\bar{c}_0'} \delta(\omega - \omega_n) \delta(\omega' - \omega'_n) e^{i(r^*-t)(\omega - \omega')}, \quad (2.39)$$

$$\frac{dJ_z}{dt} = \frac{16\pi}{(\Delta t)^2} \sum_{lmn} \int d\omega d\omega' \omega m \frac{X_{lm\omega}}{c_0} \frac{\bar{X}_{lm\omega'}}{\bar{c}_0'} \delta(\omega - \omega_n) \delta(\omega' - \omega'_n) e^{i(r^*-t)(\omega - \omega')}, \quad (2.40)$$

where $c'_0 = -12i\omega' M + \lambda(\lambda+2) - 12a\omega'(a\omega' - m)$. These formulae are the time dependent energy and angular momentum fluxes. In the numerical calculations, however, we use the averaged energy fluxes defined as

$$\begin{aligned} \left\langle \frac{dE}{dt} \right\rangle &= \lim_{T \rightarrow \infty} \frac{E_{\text{total}}}{T} \\ &= \frac{16\pi}{(\Delta t)^2} \sum_{lmn} \omega_n^2 \left| \frac{X_{lmn}}{c_0} \right|^2, \\ \left\langle \frac{dJ_z}{dt} \right\rangle &= \lim_{T \rightarrow \infty} \frac{J_{z, \text{total}}}{T} \\ &= \frac{16\pi}{(\Delta t)^2} \sum_{lmn} \omega_n m \left| \frac{X_{lmn}}{c_0} \right|^2, \end{aligned} \quad (2.41)$$

where we used the relation,

$$\lim_{T \rightarrow \infty} \frac{2\pi}{T} [\delta(\omega - \omega_n)]^2 = \delta(\omega - \omega_n). \quad (2.42)$$

For a test particle staying on an equatorial plane, we need only these fluxes to consider the evolution of the trajectory of a test particle due to the back reaction of gravitational waves. However when an orbital plane of a test particle is not the equatorial plane and precesses around the symmetric axis, we must consider the change rate of the Carter constant as well as those of the energy and the z -component of the angular momentum. It is non-trivial problem to calculate the change rate of the Carter constant because it cannot be simply related to dE/dt and dJ_z/dt . Thus let us first consider the meaning of the Carter constant. The square of the total angular

momentum L^2 is written as

$$\begin{aligned} L^2 &= p_\theta^2 + \frac{L_z^2}{\sin^2 \theta} \\ &= C + L_z^2 - a^2(1-E^2)\cos^2 \theta. \end{aligned} \quad (2.43)$$

For $a=0$, C is equal to $L_x^2 + L_y^2$, and both L_x and L_y of the system are lost by gravitational waves. Hence for $a \neq 0$, C should also decrease mainly according to the loss of x and y components of the angular momentum due to the emission of gravitational waves. The time dependent loss rate of C is written as

$$\begin{aligned} \frac{dC}{dt} &= 2L_x(t) \frac{dL_x}{dt} + 2L_y(t) \frac{dL_y}{dt} + a^2 \frac{d\{(1-E^2)\cos^2 \theta(t)\}}{dt}, \\ &= L_+(t) \frac{dL_-}{dt} + L_-(t) \frac{dL_+}{dt} + a^2 \frac{d\{(1-E^2)\cos^2 \theta(t)\}}{dt}, \end{aligned} \quad (2.44)$$

where $L_\pm = L_x \pm iL_y$. We note that dL_i/dt denotes the total change rate of the particle angular momentum. In Eq. (2.44), $L_\pm$ are not constants of motion in the case $a \neq 0$ unlike E and L_z , so that the loss rate $dL_\pm/dt$ cannot be simply written by the fluxes of gravitational waves, $dJ_\pm/dt$. If we write $dL_\pm/dt$ as

$$\frac{dL_\pm}{dt} = \frac{dJ_\pm}{dt} + \frac{dl_\pm}{dt},$$

where $dl_\pm/dt$ denote the change rates of $L_\pm$ by the precession, then Eq. (2.44) becomes

$$\begin{aligned} \frac{dC}{dt} &= L_+(t) \frac{dl_-}{dt} + L_-(t) \frac{dl_+}{dt} + a^2(1-E^2) \frac{d\cos^2 \theta(t)}{dt} \\ &\quad + L_+(t) \frac{dJ_-}{dt} + L_-(t) \frac{dJ_+}{dt} - a^2 \cos^2 \theta(t) \frac{dE^2}{dt}. \end{aligned} \quad (2.45)$$

In general case, it seems impossible to estimate the averaged loss rate of the Carter constant by the fluxes of gravitational waves using the above expression. However, we now consider the nearly quasi-periodic orbit of a test particle, i.e., the time scale of the back reaction of gravitational radiation ($\propto r^4/M^2/\mu$) is much larger than Δt ($\propto r^{1.5}$) and the period of the precession of the orbital plane ($\propto r^3/aM$). In this case the back reaction of the gravitational radiation to E and $L_\pm$ can be ignored. This means that sum of the first three terms of Eq. (2.45) becomes zero because the Carter constant is the constant of motion if the back reaction of the gravitational radiation is not taken into account. Furthermore if we note the relation that Δt is much shorter than the period of the precession, we can define the averaged loss rate of C due to the gravitational radiation as

$$\left\langle \frac{dC}{dt} \right\rangle = \langle L_+ \rangle \left\langle \frac{dJ_-}{dt} \right\rangle + \langle L_- \rangle \left\langle \frac{dJ_+}{dt} \right\rangle - 2a^2 E \left\langle \frac{dE}{dt} \right\rangle \langle \cos^2 \theta \rangle, \quad (2.46)$$

where the quantities expressed by $\langle Q \rangle$ denotes those averaged over the period Δt . It may be rough estimation of the loss of C , but if we assume the quasi-periodic evolution of a test particle for $a \rightarrow 0$, it is a correct expression of the loss rate of the

Carter constant.

However there is another problem; if we calculate $\langle dJ_{\pm}/dt \rangle$ by the same method as that for $\langle dE/dt \rangle$ and $\langle dJ_z/dt \rangle$ (Eq. (2.41)), using

$$\frac{dJ_{\pm}}{dt} = -\frac{1}{16\pi} \int (\dot{h}_{\pm} \partial_{\pm} h_{\pm} + \dot{h}_{\times} \partial_{\pm} h_{\times}) r^2 d\Omega, \quad (2.47)$$

where $\partial_{\pm} = e^{\pm i\varphi} (\pm i\partial_{\theta} - \cot\theta \partial_{\varphi} \pm 2i/\sin\theta)$,²³⁾ they become zero. This is because the orbital plane precesses around the z -axis and, if we average x and y components of the angular momentum fluxes for the long period $\gg \Delta t$, they cancel out. Thus we define the following fluxes,

$$\frac{dJ_{\pm}}{dt} = -\frac{1}{16\pi} \int \exp\left(\mp i \frac{\delta\varphi}{\Delta t} (t - r^*)\right) (\dot{h}_{\pm} \partial_{\pm} h_{\pm} + \dot{h}_{\times} \partial_{\pm} h_{\times}) r^2 d\Omega, \quad (2.48)$$

where $\delta\varphi \equiv \Delta\varphi - 2\pi$. These fluxes are interpreted as those observed by an observer comoving with the precession of the orbital plane around the z -axis. Note that we can use these expressions only in the case in which the time scale of the radiation reaction of gravitational waves and a period of the precession are much longer than a period of the orbital rotation (we now consider such a situation).

Using the above expressions, the averaged fluxes are written as

$$\begin{aligned} \left\langle \frac{dJ_{\pm}}{dt} \right\rangle = & \frac{8\pi}{(\Delta t)^2} \sum_{l'mn} \omega_n \left[-\frac{\bar{X}_{lmn}}{\bar{c}_0} \frac{X_{l'm\mp 1n\pm 1}}{c_0'} (\pm B_{l'mn}^{\mp} - (m\mp 1)C_{l'mn}^{\mp}) \right. \\ & \left. + \frac{X_{lmn}}{c_0} \frac{\bar{X}_{l'm\pm 1n\mp 1}}{\bar{c}_0'} (\pm B_{l'mn}^{\pm} + (m\pm 1)C_{l'mn}^{\pm}) \right], \end{aligned}$$

where

$$\begin{aligned} B_{l'mn}^{\pm} &= \int d\theta S_{lmn} (\sin\theta S_{l'm\pm 1n\mp 1, \theta} \mp 2S_{l'm\pm 1n\mp 1}), \\ C_{l'mn}^{\pm} &= \int d\theta \cos\theta S_{lmn} S_{l'm\pm 1n\mp 1}, \end{aligned} \quad (2.49)$$

and S_{lmn} denotes $_{-2}S_{lm}^{\omega n}$.

§ 3. Numerical methods and accuracy

Methods adopted in the numerical calculations are as follows. 1) To obtain a particle trajectory, the fourth order Runge-Kutta method is used to solve the equations of motion (Eq. (2.18)). 2) To perform the numerical integrals in Eq. (2.17), (2.26), (2.49) and (A.1), we use the trapezoidal rule. 3) Method to solve the eigenvalue problem in Eq. (2.2) is shown in Appendix B.

Although the contributions of the higher multipoles or the higher frequency modes to the fluxes are important when a test particle moves near the event horizon or the inclination of the orbital plane increases, in the numerical calculation, $X_{lm\omega n}$ are solved up to $l=6$ and at most $n=\pm 10$ to save the computational time. If an inclination angle of a test particle is large ($\theta_i \lesssim 90^\circ$), contributions of large n modes to the energy and angular momentum fluxes are important, so that the neglect of the large

n modes causes an inaccuracy. Hence in this paper, we consider the case $\theta_i \lesssim 30^\circ$. In order to see the effects of the precession of the orbital plane, it is sufficient to calculate such cases.

As shown in a previous paper,¹²⁾ the errors incurred by neglecting the higher multipole modes $l \geq 7$ are less than 10^{-4} for $r \geq 10M$. To estimate the size of the errors by neglecting large n modes in the present calculations ($\theta_i \lesssim 30^\circ$), we compared the energy fluxes for various values of $C (\neq 0)$ and $a=0$ with those for $a=0$ calculated in a previous paper¹²⁾ at various radii. We find that the relative error is less than 10^{-4} . We also compare the energy flux for various values of a and $C=0$ with those in a previous paper,¹²⁾ as a code check. We confirmed the relative error of the energy flux are $\lesssim 10^{-4}$ for $r \geq 30M$. Hence we believe that the numerical accuracy is $\lesssim 10^{-4}$ in total.

§ 4. Numerical results

In Table I, we show the parameters of the orbits of a test particle adopted in the calculation and the averaged fluxes. Each column shows the orbital radius r_0/M , the spin parameter of a Kerr spacetime a/M , the Carter constant C , the inclination angle of an orbital plane to the equatorial plane θ_i (see Fig. 1), $\langle dE/dt \rangle$, $\langle dJ_z/dt \rangle$, $\langle dJ_x/dt \rangle$ and $\langle dC/dt \rangle$, respectively. We omit $\langle dJ_y/dt \rangle$ because it does not contribute to $\langle dC/dt \rangle$ within the numerical accuracy in the present calculation. Since we determine that at $t=0$, a particle is located at $(\theta, \varphi) = (\theta_i, 0)$, $\langle L_x \rangle$ is negative for $0 \leq t \leq \Delta t$ and $\langle dJ_x/dt \rangle$ is also negative. Note that $|\delta\varphi|/2\pi \sim 0.01$ for $r=30M$ and $a=\pm 0.9$, and it becomes small for large r or small a . Therefore for most of the orbit, a period of the precession is much longer than a period of the orbital rotation (Δt).

From Table I, it is found that the energy flux decreases for $a < 0$ and increases for $a > 0$ as the Carter constant increases. The post-Newtonian formula of energy flux induced by a spinning binary calculated by Kidder et al.²⁴⁾ is convenient to understand these results. Using their formula, the energy flux including the spin-orbit interaction is approximately written as

$$\frac{dE}{dt} = -\frac{32M^2\mu^2}{5}v^2 \left\{ 1 - \frac{S \cos \theta_i}{12Mrv} \left(37v^2 + \frac{12M}{r} \right) \right\}, \quad (4.1)$$

where S denotes a spin angular momentum of a compact star. As the inclination of the orbital plane increases, the spin-orbit interaction effect is weakened and the energy flux decreases (increases) for $a < 0$ ($a > 0$). To see the dependency of the energy flux on the inclination angle in the present calculation, as an example, we show the relation between the energy flux and the inclination angle for $r=30M$, $70M$ and $a=\pm 0.9M$, $\pm 0.3M$ in Fig. 2. Although the coefficients of $\cos \theta_i$ do not agree with those in Eq. (4.1) probably due to the lack of the higher post-Newtonian terms, we can see that the energy flux is denoted as a linear function of $\cos \theta_i$.

The spin-orbit interaction also affects the place of the last stable orbit of a test particle. In Fig. 3 the radius of the last stable orbit for $a=\pm 0.9M$ is shown and it is found that for larger C , it approaches $r=6M$, i.e., the last stable orbit in the case $a=0$. This also indicates that for large C , the spin-orbit interaction effect is

Table I. The energy and each component of the angular momentum fluxes are shown. Each column denotes the orbital radius r_0/M , a/M , C , $\langle dE/dt \rangle$, $\langle dJ_z/dt \rangle$, $\langle dJ_x/dt \rangle$ and $\langle dC/dt \rangle$.

r/M	a/M	C	θ_i	dE/dt	dJ_z/dt	dJ_x/dt	dC/dt
30	-0.9	1	9.79	2.5810E-07	4.1548E-05	-7.2833E-06	1.4543E-05
30	-0.9	3	17.15	2.5774E-07	4.0205E-05	-1.2598E-05	4.3573E-05
30	-0.9	6	24.67	2.5719E-07	3.8107E-05	-1.7789E-05	8.7015E-05
30	-0.9	10	32.65	2.5643E-07	3.5130E-05	-2.2922E-05	1.4474E-04
30	-0.3	1	9.92	2.5163E-07	4.0650E-05	-7.1439E-06	1.4285E-05
30	-0.3	3	17.36	2.5153E-07	3.9361E-05	-1.2369E-05	4.2841E-05
30	-0.3	6	24.96	2.5137E-07	3.7348E-05	-1.7482E-05	8.5629E-05
30	-0.3	10	33.03	2.5115E-07	3.4488E-05	-2.2551E-05	1.4260E-04
30	0.3	1	10.03	2.4577E-07	3.9843E-05	-7.0176E-06	1.4033E-05
30	0.3	3	17.55	2.4585E-07	3.8600E-05	-1.2155E-05	4.2101E-05
30	0.3	6	25.24	2.4598E-07	3.6652E-05	-1.7198E-05	8.4243E-05
30	0.3	10	33.39	2.4617E-07	3.3878E-05	-2.2215E-05	1.4048E-04
30	0.9	1	10.13	2.4047E-07	3.9122E-05	-6.9037E-06	1.3790E-05
30	0.9	3	17.73	2.4066E-07	3.7909E-05	-1.1969E-05	4.1412E-05
30	0.9	6	25.49	2.4098E-07	3.6011E-05	-1.6943E-05	8.2902E-05
30	0.9	10	33.71	2.4144E-07	3.3305E-05	-2.1901E-05	1.3835E-04
50	-0.9	1	7.82	1.9954E-08	6.9707E-06	-9.6391E-07	1.9270E-06
50	-0.9	3	13.63	1.9946E-08	6.8341E-06	-1.6684E-06	5.7769E-06
50	-0.9	6	19.47	1.9935E-08	6.6238E-06	-2.3592E-06	1.1553E-05
50	-0.9	10	25.49	1.9919E-08	6.3333E-06	-3.0435E-06	1.9241E-05
50	-0.9	15	31.83	1.9899E-08	5.9514E-06	-3.7243E-06	2.8836E-05
50	-0.9	21	38.63	1.9873E-08	5.4586E-06	-4.4027E-06	4.0333E-05
50	-0.3	1	7.86	1.9730E-08	6.9040E-06	-9.5559E-07	1.9111E-06
50	-0.3	3	13.70	1.9728E-08	6.7698E-06	-1.6550E-06	5.7327E-06
50	-0.3	6	19.57	1.9724E-08	6.5635E-06	-2.3397E-06	1.1462E-05
50	-0.3	10	25.63	1.9720E-08	6.2777E-06	-3.0208E-06	1.9104E-05
50	-0.3	15	32.00	1.9714E-08	5.9022E-06	-3.6979E-06	2.8642E-05
50	-0.3	21	38.83	1.9706E-08	5.4173E-06	-4.3738E-06	4.0084E-05
50	0.3	1	7.90	1.9521E-08	6.8420E-06	-9.4789E-07	1.8957E-06
50	0.3	3	13.77	1.9523E-08	6.7100E-06	-1.6418E-06	5.6873E-06
50	0.3	6	19.67	1.9525E-08	6.5070E-06	-2.3215E-06	1.1372E-05
50	0.3	10	25.76	1.9529E-08	6.2257E-06	-2.9979E-06	1.8960E-05
50	0.3	15	32.15	1.9535E-08	5.8548E-06	-3.6729E-06	2.8449E-05
50	0.3	21	39.02	1.9541E-08	5.3768E-06	-4.3459E-06	3.9829E-05
50	0.9	1	7.94	1.9325E-08	6.7847E-06	-9.4081E-07	1.8810E-06
50	0.9	3	13.84	1.9330E-08	6.6545E-06	-1.6297E-06	5.6434E-06
50	0.9	6	19.77	1.9337E-08	6.4537E-06	-2.3056E-06	1.1291E-05
50	0.9	10	25.88	1.9347E-08	6.1757E-06	-2.9779E-06	1.8828E-05
50	0.9	15	32.30	1.9361E-08	5.8104E-06	-3.6473E-06	2.8241E-05
50	0.9	21	39.20	1.9379E-08	5.3369E-06	-4.3190E-06	3.9570E-05
70	-0.9	1	6.68	3.7184E-09	2.1595E-06	-2.5408E-07	5.0806E-07
70	-0.9	3	11.63	3.7178E-09	2.1291E-06	-4.3999E-07	1.5239E-06
70	-0.9	6	16.56	3.7170E-09	2.0828E-06	-6.2211E-07	3.0472E-06
70	-0.9	10	21.60	3.7157E-09	2.0192E-06	-8.0328E-07	5.0794E-06
70	-0.9	15	26.80	3.7141E-09	1.9371E-06	-9.8314E-07	7.6139E-06
70	-0.9	21	32.25	3.7122E-09	1.8337E-06	-1.1628E-06	1.0655E-05
70	-0.3	1	6.70	3.6937E-09	2.1473E-06	-2.5278E-07	5.0554E-07

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70	-0.3	3	11.67	3.6935E-09	2.1172E-06	-4.3797E-07	1.5171E-06
70	-0.3	6	16.62	3.6932E-09	2.0713E-06	-6.1932E-07	3.0339E-06
70	-0.3	10	21.67	3.6928E-09	2.0085E-06	-7.9938E-07	5.0555E-06
70	-0.3	15	26.89	3.6924E-09	1.9271E-06	-9.7903E-07	7.5831E-06
70	-0.3	21	32.35	3.6918E-09	1.8247E-06	-1.1582E-06	1.0615E-05
70	0.3	1	6.73	3.6702E-09	2.1358E-06	-2.5156E-07	5.0311E-07
70	0.3	3	11.70	3.6703E-09	2.1060E-06	-4.3587E-07	1.5099E-06
70	0.3	6	16.67	3.6705E-09	2.0605E-06	-6.1640E-07	3.0197E-06
70	0.3	10	21.73	3.6709E-09	1.9983E-06	-7.9570E-07	5.0323E-06
70	0.3	15	26.97	3.6713E-09	1.9177E-06	-9.7463E-07	7.5495E-06
70	0.3	21	32.45	3.6718E-09	1.8162E-06	-1.1532E-06	1.0569E-05
70	0.9	1	6.74	3.6479E-09	2.1250E-06	-2.5042E-07	5.0076E-07
70	0.9	3	11.74	3.6483E-09	2.0954E-06	-4.3391E-07	1.5029E-06
70	0.9	6	16.72	3.6489E-09	2.0503E-06	-6.1366E-07	3.0058E-06
70	0.9	10	21.80	3.6498E-09	1.9886E-06	-7.9221E-07	5.0095E-06
70	0.9	15	27.04	3.6509E-09	1.9086E-06	-9.7045E-07	7.5158E-06
70	0.9	21	32.54	3.6522E-09	1.8076E-06	-1.1489E-06	1.0528E-05
90	-0.9	1	5.93	1.0612E-09	9.0023E-07	-9.3796E-08	1.8757E-07
90	-0.9	3	10.31	1.0611E-09	8.9036E-07	-1.6240E-07	5.6250E-07
90	-0.9	6	14.66	1.0610E-09	8.7533E-07	-2.2969E-07	1.1251E-06
90	-0.9	10	19.07	1.0608E-09	8.5486E-07	-2.9652E-07	1.8752E-06
90	-0.9	15	23.59	1.0606E-09	8.2860E-07	-3.6308E-07	2.8121E-06
90	-0.9	21	28.26	1.0603E-09	7.9596E-07	-4.2953E-07	3.9362E-06
90	-0.3	1	5.94	1.0564E-09	8.9680E-07	-9.3471E-08	1.8694E-07
90	-0.3	3	10.33	1.0564E-09	8.8699E-07	-1.6184E-07	5.6062E-07
90	-0.3	6	14.69	1.0563E-09	8.7206E-07	-2.2891E-07	1.1214E-06
90	-0.3	10	19.11	1.0563E-09	8.5175E-07	-2.9554E-07	1.8691E-06
90	-0.3	15	23.64	1.0562E-09	8.2569E-07	-3.6190E-07	2.8032E-06
90	-0.3	21	28.33	1.0561E-09	7.9327E-07	-4.2818E-07	3.9243E-06
90	0.3	1	5.95	1.0518E-09	8.9353E-07	-9.3163E-08	1.8632E-07
90	0.3	3	10.35	1.0518E-09	8.8379E-07	-1.6131E-07	5.5878E-07
90	0.3	6	14.72	1.0519E-09	8.6896E-07	-2.2817E-07	1.1178E-06
90	0.3	10	19.15	1.0519E-09	8.4878E-07	-2.9460E-07	1.8632E-06
90	0.3	15	23.69	1.0520E-09	8.2288E-07	-3.6077E-07	2.7945E-06
90	0.3	21	28.38	1.0521E-09	7.9068E-07	-4.2688E-07	3.9124E-06
90	0.9	1	5.97	1.0474E-09	8.9043E-07	-9.2872E-08	1.8573E-07
90	0.9	3	10.37	1.0475E-09	8.8074E-07	-1.6081E-07	5.5702E-07
90	0.9	6	14.75	1.0476E-09	8.6599E-07	-2.2747E-07	1.1143E-06
90	0.9	10	19.19	1.0477E-09	8.4592E-07	-2.9370E-07	1.8574E-06
90	0.9	15	23.74	1.0479E-09	8.2018E-07	-3.5969E-07	2.7859E-06
90	0.9	21	28.44	1.0481E-09	7.8814E-07	-4.2563E-07	3.9005E-06
110	-0.9	1	5.38	3.8999E-10	4.4759E-07	-4.2269E-08	8.4533E-08
110	-0.9	3	9.35	3.8997E-10	4.4356E-07	-7.3222E-08	2.5363E-07
110	-0.9	6	13.28	3.8994E-10	4.3744E-07	-1.0354E-07	5.0721E-07
110	-0.9	10	17.26	3.8990E-10	4.2917E-07	-1.3362E-07	8.4506E-07
110	-0.9	15	21.31	3.8986E-10	4.1858E-07	-1.6370E-07	1.2679E-06
110	-0.9	21	25.47	3.8980E-10	4.0553E-07	-1.9364E-07	1.7746E-06
110	0.9	1	5.41	3.8625E-10	4.4398E-07	-4.1962E-08	8.3920E-08
110	0.9	3	9.40	3.8627E-10	4.4001E-07	-7.2693E-08	2.5180E-07
110	0.9	6	13.35	3.8629E-10	4.3399E-07	-1.0280E-07	5.0359E-07
110	0.9	10	17.34	3.8632E-10	4.2582E-07	-1.3275E-07	8.3951E-07

(continued)

110	0.9	15	21.41	3.8636E-10	4.1539E-07	-1.6256E-07	1.2591E-06
110	0.9	21	25.58	3.8640E-10	4.0252E-07	-1.9232E-07	1.7625E-06
130	-0.9	1	4.96	1.6948E-10	2.5012E-07	-2.1766E-08	4.3529E-08
130	-0.9	3	8.62	1.6948E-10	2.4821E-07	-3.7700E-08	1.3059E-07
130	-0.9	6	12.24	1.6947E-10	2.4532E-07	-5.3314E-08	2.6117E-07
130	-0.9	10	15.88	1.6946E-10	2.4141E-07	-6.8820E-08	4.3524E-07
130	-0.9	15	19.58	1.6945E-10	2.3645E-07	-8.4267E-08	6.5270E-07
130	-0.9	21	23.36	1.6943E-10	2.3034E-07	-9.9709E-08	9.1380E-07
130	0.9	1	4.98	1.6822E-10	2.4855E-07	-2.1643E-08	4.3284E-08
130	0.9	3	8.65	1.6823E-10	2.4667E-07	-3.7487E-08	1.2985E-07
130	0.9	6	12.28	1.6823E-10	2.4381E-07	-5.3016E-08	2.5971E-07
130	0.9	10	15.94	1.6824E-10	2.3995E-07	-6.8440E-08	4.3284E-07
130	0.9	15	19.65	1.6825E-10	2.3503E-07	-8.3808E-08	6.4915E-07
130	0.9	21	23.45	1.6827E-10	2.2899E-07	-9.9176E-08	9.0894E-07

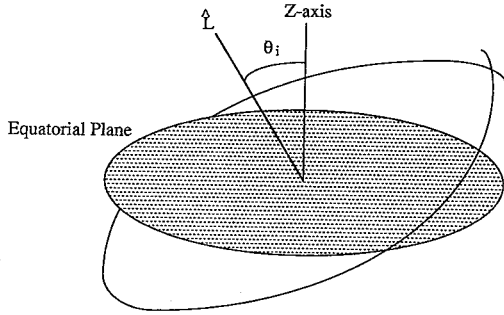


Fig. 1. A schematic particle trajectory is shown. θ_i denotes the inclination of the orbital plane to the equatorial plane.

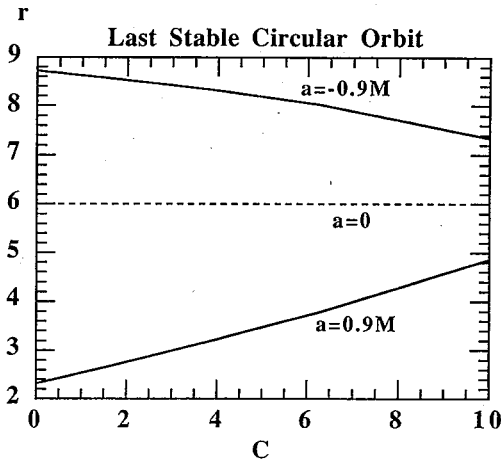


Fig. 3. Dependence of the last stable radius for $a = \pm 0.9M$ on C is shown.

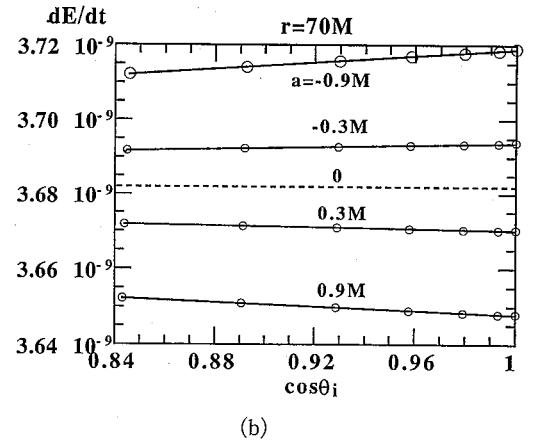
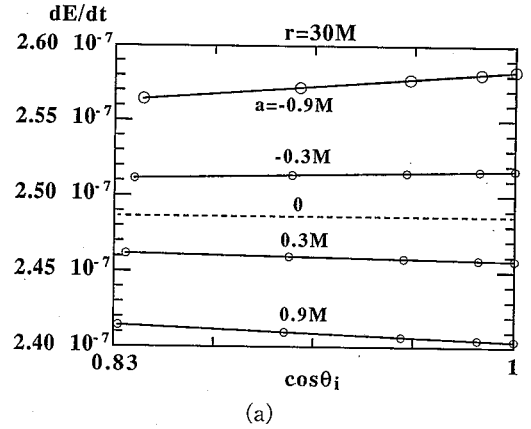


Fig 2. (a) The energy flux dE/dt for $r=30M$, $a = \pm 0.9M, \pm 0.3M$ and $C=1, 3, 6, 10$. A transversal axis denotes $\cos \theta_i$. We can see that dE/dt depends on a linear function of $\cos \theta_i$. (b) The same as in (a), but for $r=70M$, $a = \pm 0.9M, \pm 0.3M$ and $C=1, 3, 6, 10, 15, 21$.

weakened.

It is interesting to see an evolution of the inclination angle due to the back reaction of the gravitational radiation. Since the inclination angle is determined by

$$C = \sin^2 \theta_i \left\{ a^2 (1 - E^2) + \frac{L_z^2}{\cos^2 \theta_i} \right\}, \quad (4.2)$$

the time variation of it is obtained from

$$\dot{\theta}_i = \frac{2a^2 E y^2 \dot{E} - \dot{C} + y(\dot{C} - 2a^2 E \dot{E} + 2L_z \dot{L}_z)}{2\{2a^2(1 - E^2)y - C - a^2(1 - E^2) - L_z^2\} \sin \theta_i \cos \theta_i}, \quad (4.3)$$

where $y \equiv \sin^2 \theta_i$ and $\dot{Q}$ denotes dQ/dt . Using values shown in Table I, we calculated $\dot{\theta}_i$. In Fig. 4, we show $\delta\theta_i \equiv \dot{\theta}_i(E - \mu)/\dot{E}$, which denotes the change of the inclination angle in a logarithmic interval of the energy, for various parameters. It is found that for $a < 0$, $\dot{\theta}_i$ is always negative, and for $a > 0$, it is always positive. These results seem reasonable because of the following reason: In the case $a > 0$ ($a < 0$), the spin-orbit interaction works as the repulsive (attractive) force and the interaction energy is higher (lower) than that in the case in which there is no spin-orbit interaction. Hence to decrease the interaction energy, the inclination angle should increase for $a > 0$ case and should decrease for $a < 0$ case.

We note that $\delta\theta_i$ depends roughly on r_0^{-n} , $n \gtrsim 2$. Hence it seems that the change of the inclination angle occurs at the $P^n N$ level below the quadrupole level. This suggests that if we consider the post-Newtonian back reaction only up to the $P^{3/2} N$ level, θ_i does not change.²⁵⁾ This result seems to show that our treatment for the loss rate of the Carter constant is fairly good.

To see the effects of the orbital precession to the wave form, we show the wave forms of gravitational waves for $r = 30M$, $a = -0.9M$ and $C = 1, 10$ in Figs. 5(a)~(c). Wave forms in Figs. 5(a) and (b) are for an observer in the direction of $\theta = 3\pi/4$. For $C = 1$ (Fig. 5(a)), the wave forms are slightly deformed from the sine curve due to the higher multipole modes, $l \geq 3$, but since

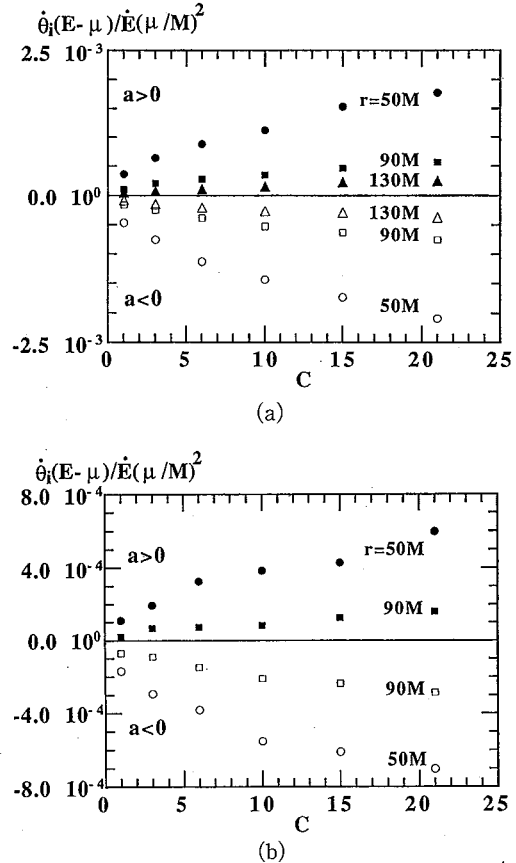


Fig. 4. (a) Change rates of the inclination angle due to the back reaction of the gravitational radiation in a logarithmic interval of the energy, $\delta\theta_i = \dot{\theta}_i(E - \mu)/\dot{E}$, for $a = \pm 0.9M$, $r = 50, 90, 130M$ and $C = 1, 3, 6, 10, 15, 21$ are shown. (b) The same as in (a), but for $a = \pm 0.3M$, $r = 50, 90M$ and $C = 1, 3, 6, 10, 15, 21$.

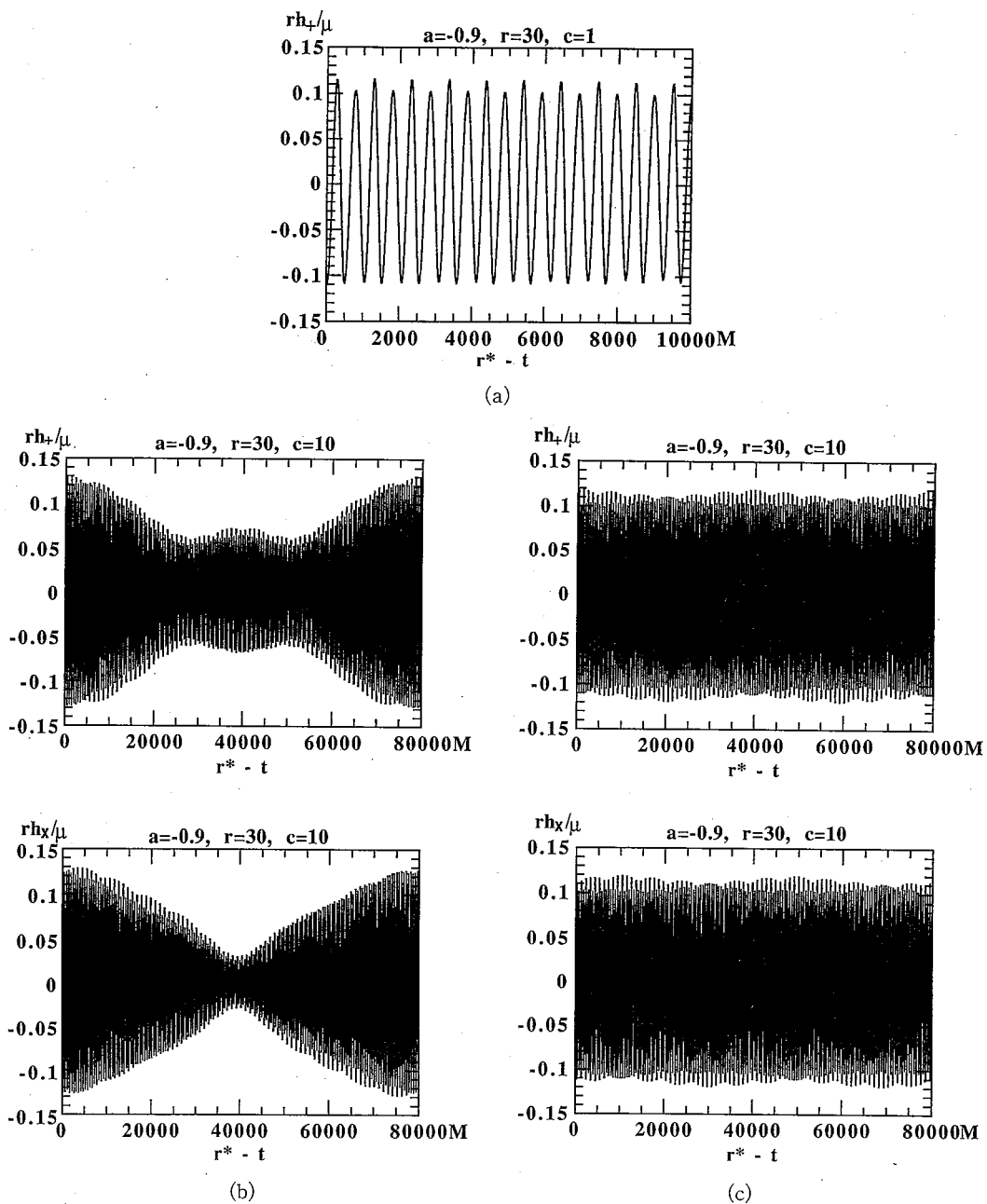


Fig. 5. (a) The wave forms of gravitational waves for $(r, a, C) = (30M, -0.9M, 1)$ observed by an observer in the direction of $\theta = 3\pi/4$ are shown.

(b) The same as in (a), but for $C = 10$.

(c) The same as in (b), but for an observer along the symmetric axis.

the effect of the orbital precession is small, we can see the regular wave pattern. In contrast, for $C = 10$ (Fig. 5(b)), a beating wave form is seen. The reason is as follows. If the inclination angle is small, dominant modes of gravitational waves are only $(l, m, n) = (2, \pm 2, 0)$ modes. However if the inclination angle is large, other modes such

as $(2, \pm 1, \pm 1)$ modes also become large. The frequency of the former and latter modes are not the same, and the difference of them are roughly $\sim P_p^{-1}$, where P_p is a precession period of the orbital plane and it is about 100 times as long as the orbital period for $r=30M$. Therefore the beating occurs.

Since $P_p \lesssim T_r$ for $r \gtrsim 30M$, where T_r is the radiation reaction time scale $\sim 5r^4/(256M^2\mu)$ and $\mu \leq M/4$, wave forms in Fig. 5(b) are realistic during the inspiral of a binary. Therefore the inclination of the orbital plane may be determined from the beating wave forms of gravitational waves.

However we should note that it cannot always be seen. In Fig. 5(c) we also show the wave forms for $C=10$ seen by an observer along the symmetric axis. In contrast to Fig. 5(b), beating wave forms are not seen. This is because the spheroidal harmonics for $(2, \pm 1, \pm 1)$ modes, etc. vanishes at the symmetric axis (Fig. 6). Hence whether we can observe the beating wave forms or not strongly depends on the direction of an observation.

§ 5. Discussion

If θ_i is large, the spin-orbit interaction effect is weakened, and the difference between the energy fluxes of gravitational waves for $a \neq 0$ and those for $a=0$ becomes small. Both our perturbation study and the post-Newtonian study by Kidder et al.²⁴⁾ show that the energy flux is written as

$$\frac{dE}{dt} = \frac{dE}{dt}\bigg|_{a=0} (1 - f(a) \cos \theta_i). \quad (5.1)$$

In a previous paper we find that for $\theta_i=0$, the difference of the accumulated phase of gravitational waves is $\Delta N \sim 15$ for $\Delta a \sim 0.1$ due to the effects of the second term of Eq. (5.1). However we should consider it is the maximum difference of the accumulated phase because such a difference becomes small, if the orbital plane inclines. Can we specify the parameters a and θ_i from a signal of gravitational waves? When we detect the beating wave forms, the Fourier spectrum is different from that without beating, so the parameters may be specified even if the accumulated phases of the two cases are the same. However if we detect gravitational waves along the symmetric axis, the beating wave forms cannot be seen. Therefore it seems that the measurement accuracy of the spin parameter is worsened if such a situation occurs.

In this paper we consider the case in which a test particle moves along a constant radius. Contrary to the case in which a particle moves on the equatorial plane in a Schwarzschild spacetime,²⁶⁾ it is not clear that the particle keeps the circular orbit, r

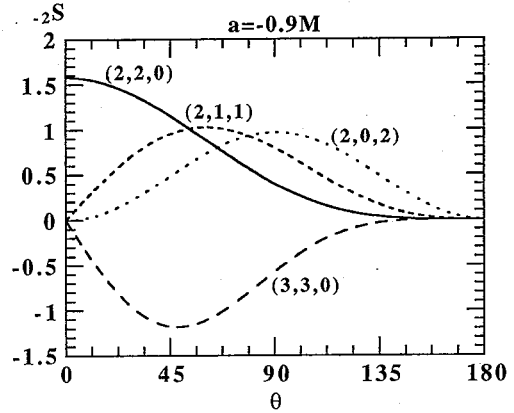


Fig. 6. The spin-weighted spheroidal harmonics $-2S_{lm}^{a\omega n}$ for modes which mainly contribute to the wave form is shown. The parameters shown denote (l, m, n) . Note $-2S_{lm}^{a\omega n}(\theta) = -2S_{l-m}^{a\omega-n}(\pi - \theta)$.

$=r_0(t)$, in spite of the radiation reaction of gravitational waves in the case when the orbital plane of a test particle precesses around the symmetric axis. To investigate the orbital evolution of a test particle on a Kerr spacetime in a general case, we should consider gravitational waves induced by a particle with an eccentric orbit. We now plan to calculate the energy and angular momentum fluxes induced by a particle moving on the equatorial plane with an eccentric orbit on a Kerr spacetime as a first step. We will report it in a future paper.

The author would like to thank Professor T. Nakamura, Professor M. Sasaki and T. Tanaka for valuable discussions and comments. The numerical calculations are mainly performed on a YHP-S720 work station.

Appendix A

In this appendix we show the A_{nn} , etc. in Eq. (2.33).

$$\begin{aligned}
 A_{nn} &= -\frac{2}{\sqrt{2\pi}} \int_0^{\Delta t} dt e^{i\omega t - im\varphi(t)} C_{nn} \rho^{-2} \bar{\rho}^{-1} \\
 &\quad \times [S_{,\theta\theta} \rho^{-1} - 2iasin\theta S_{,\theta} - 3iacos\theta \dot{S} + f_2 \rho^{-1} S_{,\theta} + ia f_2 sin\theta S \\
 &\quad + f_{2,\theta} \rho^{-1} S + f_1 (\rho^{-1} S_{,\theta} - 3iasin\theta S + S f_2 \rho^{-1})], \\
 A_{\bar{m}n1} &= \frac{2\Delta}{\sqrt{\pi}} \int_0^{\Delta t} dt e^{i\omega t - im\varphi(t)} \rho^{-3} C_{\bar{m}n} [S_{,\theta} - iasin\theta(\rho - \bar{\rho})S + f_2 S], \\
 A_{\bar{m}n2} &= \frac{2}{\sqrt{\pi}} \int_0^{\Delta t} dt e^{i\omega t - im\varphi(t)} \left[\frac{\Delta^2 \bar{\rho}^2}{\rho} J_+ \left(\frac{C_{\bar{m}n}}{\rho^2 \bar{\rho}^2 \Delta} \right) \{S_{,\theta} - iasin\theta(\rho - \bar{\rho})S + f_2 S\} \right. \\
 &\quad \left. + \Delta \bar{\rho} \rho^{-2} C_{\bar{m}n} \{(S_{,\theta} + f_2 S)(r + 3iacos\theta) - 6a^2 sin\theta cos\theta \bar{\rho}\} S \right], \\
 A_{\bar{m}m1} &= -\frac{\Delta^2}{\sqrt{2\pi}} \int_0^{\Delta t} dt e^{i\omega t - im\varphi(t)} \rho^{-3} \bar{\rho} C_{\bar{m}m} S, \\
 A_{\bar{m}m2} &= -\frac{2\Delta^2}{\sqrt{2\pi}} \int_0^{\Delta t} dt e^{i\omega t - im\varphi(t)} \rho^{-3} \bar{\rho} C_{\bar{m}m} S \left(-\bar{\rho} + 4\rho + \frac{C_{\bar{m}m,r}}{C_{\bar{m}m}} + \frac{iK}{\Delta} \right), \\
 A_{\bar{m}m3} &= -\frac{\Delta^2}{\sqrt{2\pi}} \int_0^{\Delta t} dt e^{i\omega t - im\varphi(t)} \rho^{-3} \bar{\rho} C_{\bar{m}m} S \left[\left(-\bar{\rho} + 2\rho + \frac{C_{\bar{m}m,r}}{C_{\bar{m}m}} + \frac{iK}{\Delta} \right),_r \right. \\
 &\quad \left. + \left(-\bar{\rho} + 6\rho + \frac{C_{\bar{m}m,r}}{C_{\bar{m}m}} + \frac{iK}{\Delta} \right) \left(-\bar{\rho} + 2\rho + \frac{C_{\bar{m}m,r}}{C_{\bar{m}m}} + \frac{iK}{\Delta} \right) \right], \quad (A \cdot 1)
 \end{aligned}$$

where S denotes $_{-2}S_{lm}^{a\omega}$.

Appendix B

In this appendix, we show a numerical method to determine an eigenvalue and an eigenfunction of Eq. (2.2). At first to guarantee the regularity at the symmetric axis, we use a variable defined as

$$f_{lm}^{a\omega} \equiv \frac{-2S_{lm}^{a\omega}}{(1+\cos\theta)^{k_1}(1-\cos\theta)^{k_2}}, \quad (\text{B}\cdot 1)$$

where $k_1=|m+2|/2$ and $k_2=|m-2|/2$. Then Eq. (2.2) becomes

$$f_{,\theta\theta} + \{2(k_1+k_2)+1\}f_{,\theta}\cot\theta - 2(k_1-k_2)\frac{f_{,\theta}}{\sin\theta} + \{-a^2\omega^2\sin^2\theta + 4a\omega\cos\theta + 2 + \lambda + 2am\omega - (k_1+k_2)(k_1+k_2+1)\}f = 0. \quad (\text{B}\cdot 2)$$

Here we omit subscripts l, m, a, ω of f . Using the finite difference method with the boundary conditions $f_{,\theta}=0$ at the symmetric axis, Eq. (B.2) becomes the matrix equation as $A_{ij}(\lambda)f_j=0$, where

$$A = \begin{pmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ c_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & c_{n-2} & a_{n-1} & b_{n-1} \\ 0 & \cdots & 0 & c_{n-1} & a_n \end{pmatrix}. \quad (\text{B}\cdot 3)$$

Since Eq. (B.3) has nontrivial solutions only when the determinant of $A(\lambda)$ is equal to zero, the eigenvalues are calculated from $\det A(\lambda)=0$. In the case the matrix is tri-diagonal, the determinant is calculated from

$$p_i = a_i p_{i-1} - b_{i-1} c_{i-1} p_{i-2}, \quad (\text{B}\cdot 4)$$

where

$$p_i = \det \begin{pmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ c_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & c_{i-2} & a_{i-1} & b_{i-1} \\ 0 & \cdots & 0 & c_{i-1} & a_i \end{pmatrix}, \quad (\text{B}\cdot 5)$$

and $p_0=1, p_1=a_1$.

If λ_1 and λ_2 satisfy

$$p_n(\lambda_1) < 0 \quad \text{and} \quad p_n(\lambda_2) > 0,$$

λ has at least one solution between λ_1 and λ_2 . Hence solutions can be obtained if we repeat this procedure until $|\lambda_2 - \lambda_1|$ becomes sufficiently small.

Once we obtain an eigenvalue, the eigenfunction is solved by the inverse iteration method as follows. 1) We put an arbitrary initial vector x_i . We here put $x_i = 1/\sqrt{n}$, where n is the grid number of θ direction. 2) We solve the matrix equation $A_{ij}y_j = x_i$ for y_i . 3) We make the new $x_i = y_i/|y_i|$. The procedure 1), 2), 3) is repeated until convergence, $|x_i^{(k)} - x_i^{(k-1)}| \ll |x_i^{(k)}|$. If $|x_i|$ converges, it is an eigenfunction.

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