Mystery of relativistic cylindrically symmetric system

―about Apostolatos-Thorne shell model―

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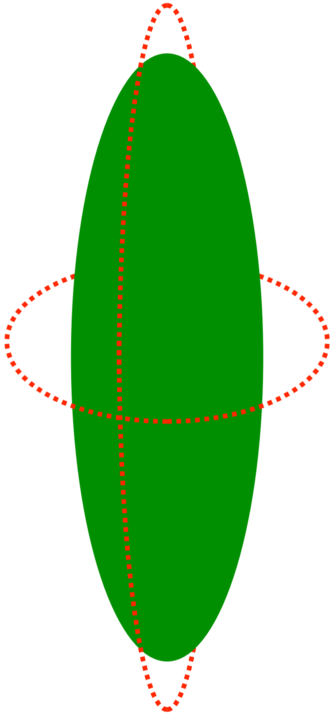
Ref) PRD77, 044021(arXiv:0711.0243), arXiv:1112.4252

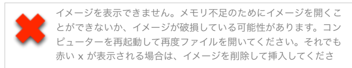
Molecule Workshop 2012 @ YITP

§Hoop conjecture and spindle collapse

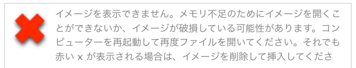


Mass M

Black hole with horizon forms
when and only when mass M gets compacted into
a region whose circumference C in every direction



by Thorne (1972)

Gravitational Collapse of 

(e.g. highly elongated configuration)

$\longrightarrow$ ***No Horizon = Naked Singularities***

Analysis of initial data

Nakamura, Shapiro & Teukolsky(1988)

One of the examples supporting the hoop conjecture

- time symmetric initial data

$$^{(3)}R = 16\pi G\rho$$

- homogeneous spheroid

In the case of
highly elongated distribution,
No marginally trapped surface

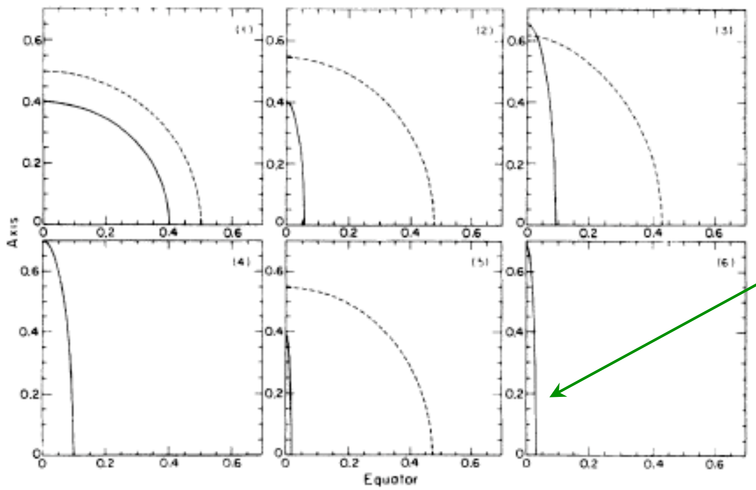


FIG. 1. Representative cases of fully relativistic, momentarily static prolate spheroids. The solid line shows the matter surface. The dashed line shows the apparent horizon if it is present. The coordinates are in units of M. Parameters for the specific cases are given in Table I. Note that whenever any dimension exceeds $\approx 0.5M$, no apparent horizon forms (hoop conjecture).

After this work, many similar analyses appeared and further will appear.

Analysis of dynamical evolution

Shapiro & Teukolsky (1991)

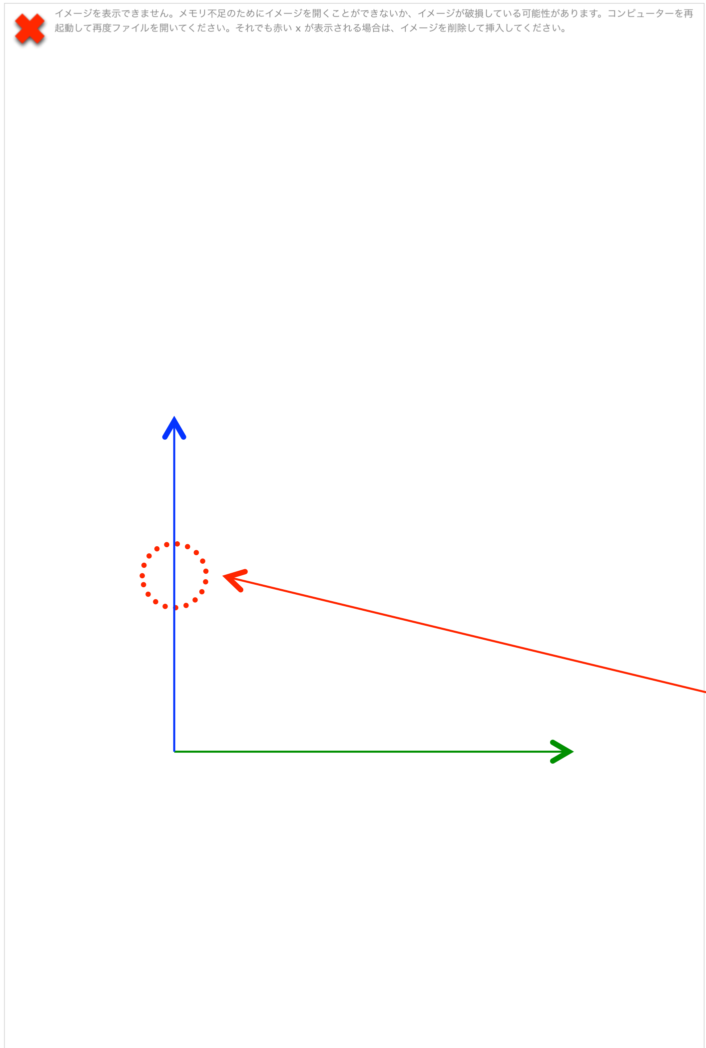
System composed of collisionles particles

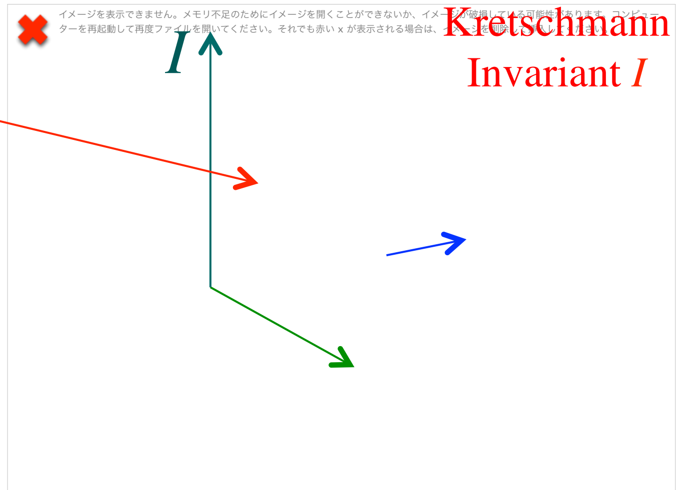
[Nakamura, Maeda, Miyama, Sasaki (1981),
Nakamura, Sato (1982)]

Spindle collapse occurred in their numerical simulation.

No trapped surface was found.

Is this truly naked?





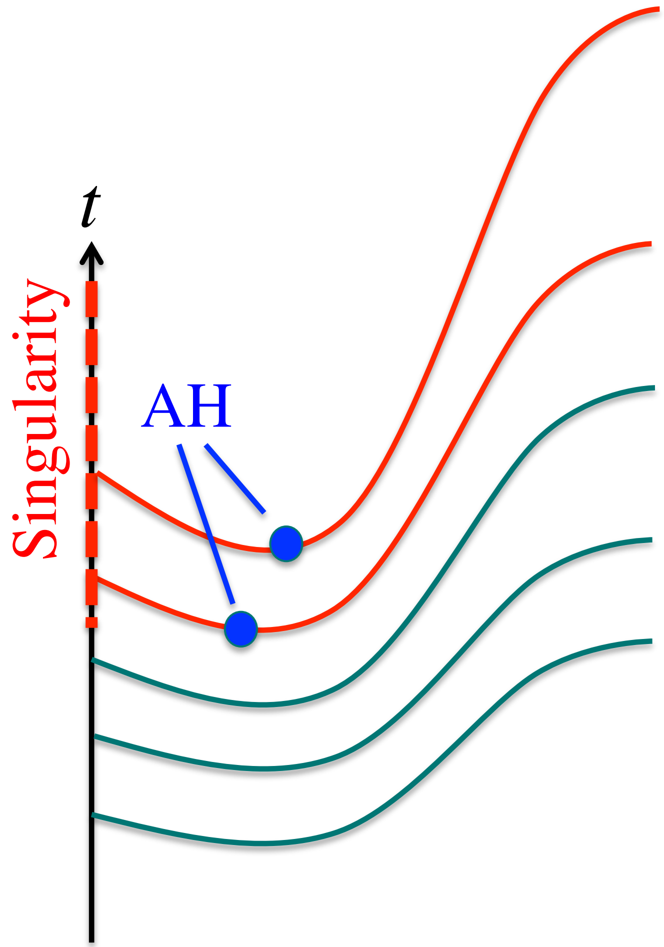
D=5 version was studied by Yamada and Shinkai, PRD83 (2011) 064006

There are two objections.

1) Time slicing condition is wrong.

Family of spacelike hypersurfaces does not hit the Apparent Horizon (AH), before it hits singularities.

Wald and Iyer, PRD (1991),
Pelath, Tod and Wald, CQG (1998)



2) Rotational motion halts the collapse

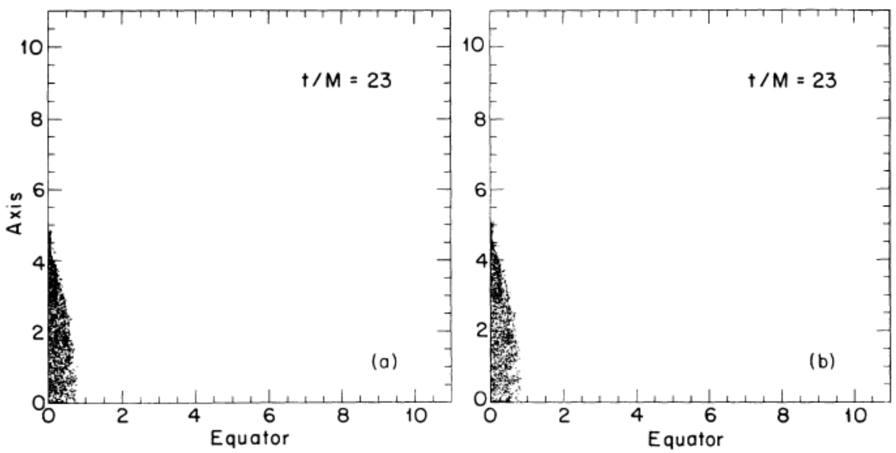
In the case of an infinitely long cylinderical matter distribution, the rotational motion of matter seems to halt its collapse.
So, also in Shapiro and Teukolsky case…

Apostolatos and Thorne, PRD (1992)

After AT paper, Shapiro and Teukolsky performed numerical simulations with rotational motion of constituent particles.

Their result was the same: the naked singularity will appear.

But, it is still controversial.



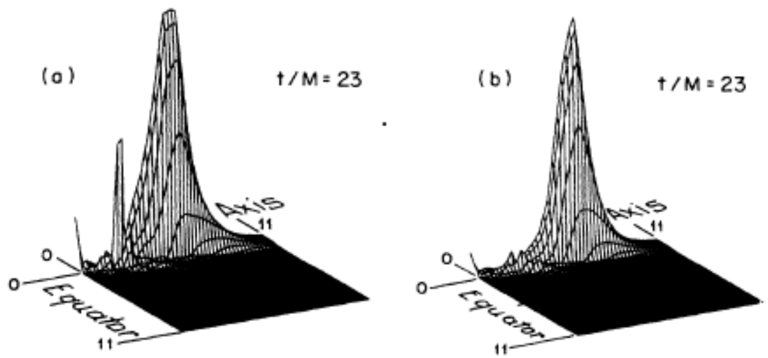


FIG. 8. Profile of I in a meridional plane for the cases shown in Figs. 5(a) and 5(b). For the case of 32 angular zones shown here, the peak value of I is $31/M^4$ for case (a) and $54/M^4$ for case (b). It occurs on the axis just outside the matter.

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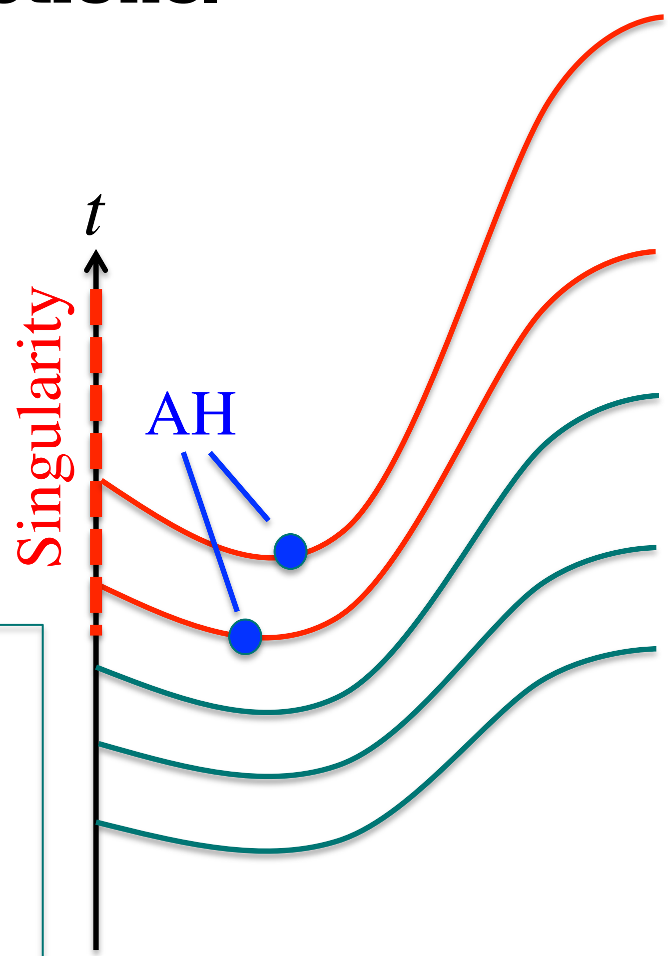

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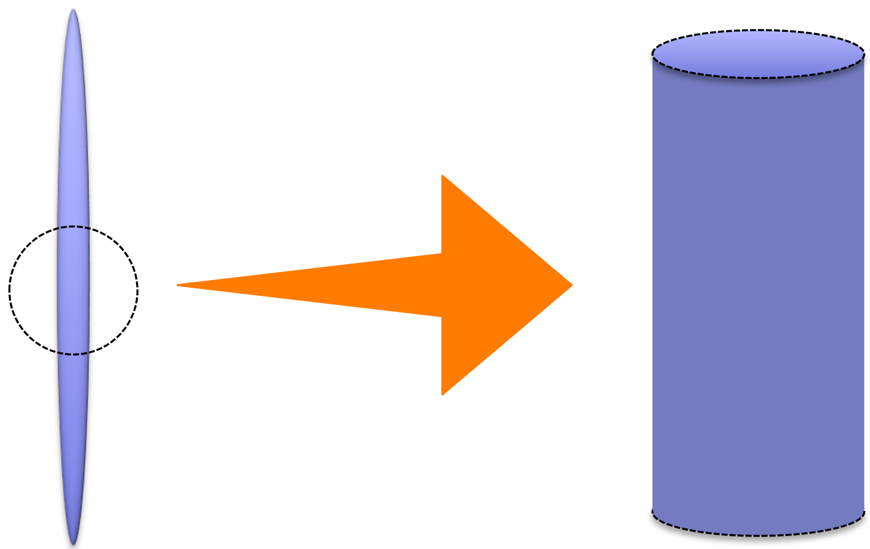
Apostolatos and Thorne, PRD (1992)

Cylindrically symmetric gravitational collapse

・Highly elongated gravitational collapse might be approximated by cylindrically symmetric gravitational collapse.



・Infinitely long cylindrical matter is not enclosed by a horizon. → Singularity is naked.

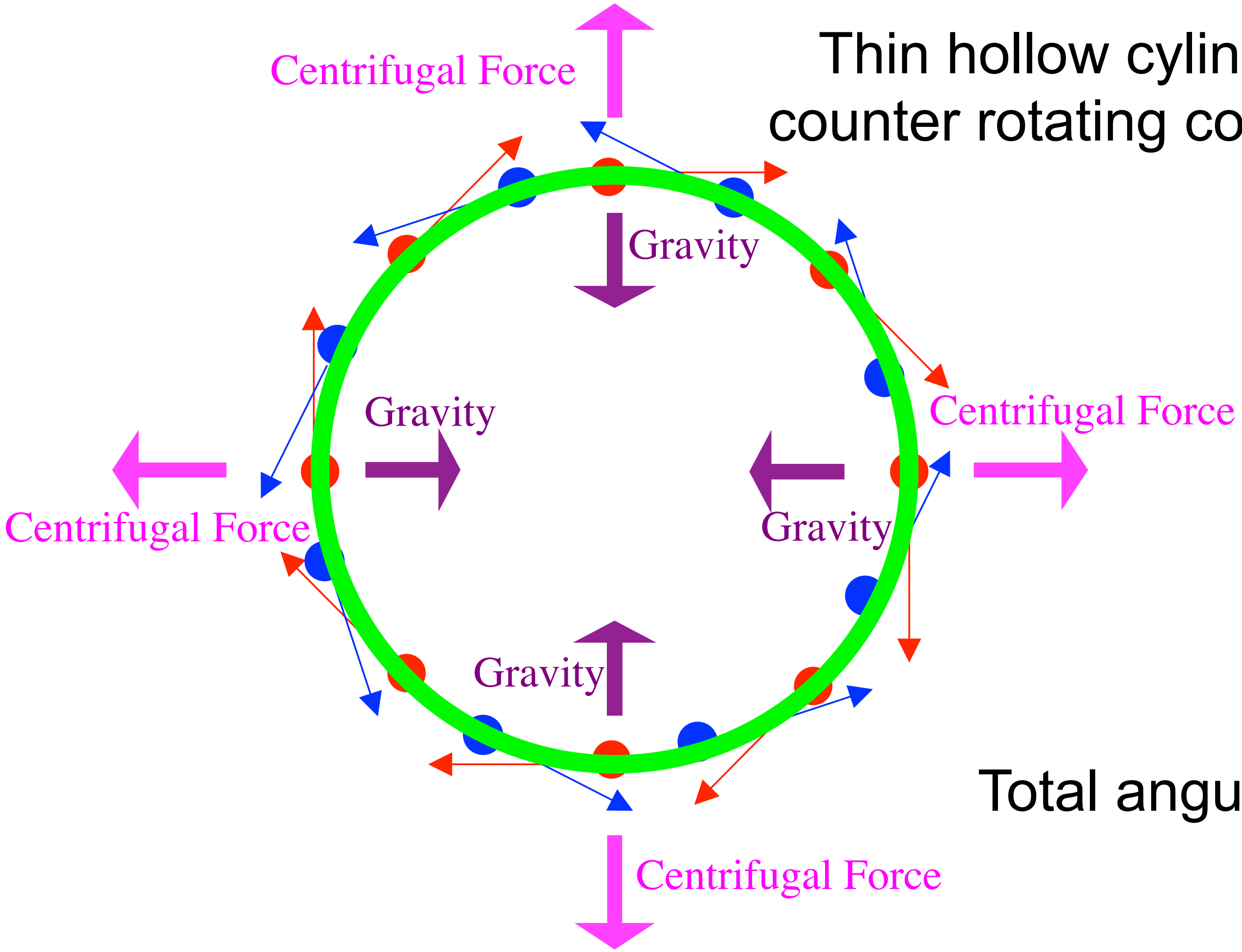
K. Thorne (1972), S. Hayward (2000)

・This is a very simple system → Detailed analysis is possible.

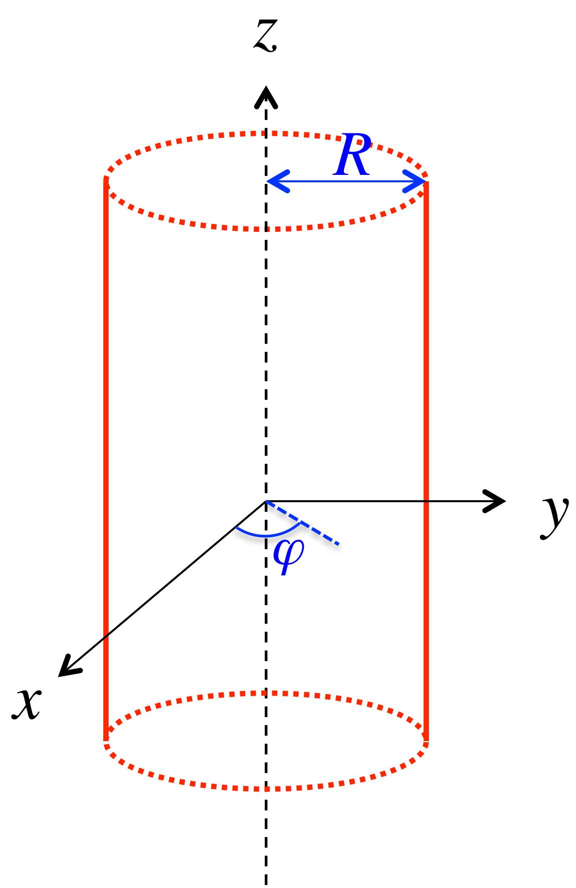
Appostolatos- Thorne (AT) Shell Model

[PRD46,p.2435,(1992)]

Thin hollow cylinder composed of counter rotating collisionless particles



Total angular momentum=0



Stress-energy tensor of AT-shell

$$T^{\mu\nu} = S^{\mu\nu}\delta\left(r - R(\tau)\right)$$

Surface stress-energy tensor of AT-shell

$$S^{\mu\nu} = \sigma u^{\mu}u^{\nu} + Te^{\mu}_{(\varphi)}e^{\nu}_{(\varphi)}$$

$$\sigma = \frac{\lambda\sqrt{1+u^2}}{2\pi R} = \frac{1+u^2}{u^2}T$$

where

$$u = \frac{l}{e^{-\psi}R}$$

; Rotational velocity of each constituent particle (φ-component of 4-velocity)

Conserved quantities

λ ; rest mass per unit Killing length

l ; specific angular momentum of each constituent particle

Einstein Equations in Vacuum Region

Line element

$$ds^2 = e^{2(\gamma - \psi)}(-dt^2 + dr^2) + e^{-2\psi} r^2 d\varphi^2 + e^{2\psi} dz^2$$

Two metric variables; $\gamma = \gamma(t, r)$, $\psi = \psi(t, r)$

$$\partial_t \gamma = 2r(\partial_t \psi)\partial_r \psi$$

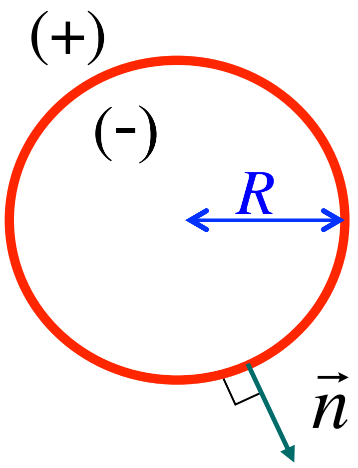
$$\partial_r \gamma = r\left[(\partial_t \psi)^2 + (\partial_r \psi)^2\right]$$

Contraint equations

$$\left(\partial_t^2 - \partial_r^2 - \frac{1}{r}\partial_r\right)\psi = 0$$

Evolution equation

Metric Junction Condition



$$K^{+}_{\mu\nu} - K^{-}_{\mu\nu} = 8\pi G\left(S_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}S^{\alpha}_{\alpha}\right) \qquad \text{at } r=R$$

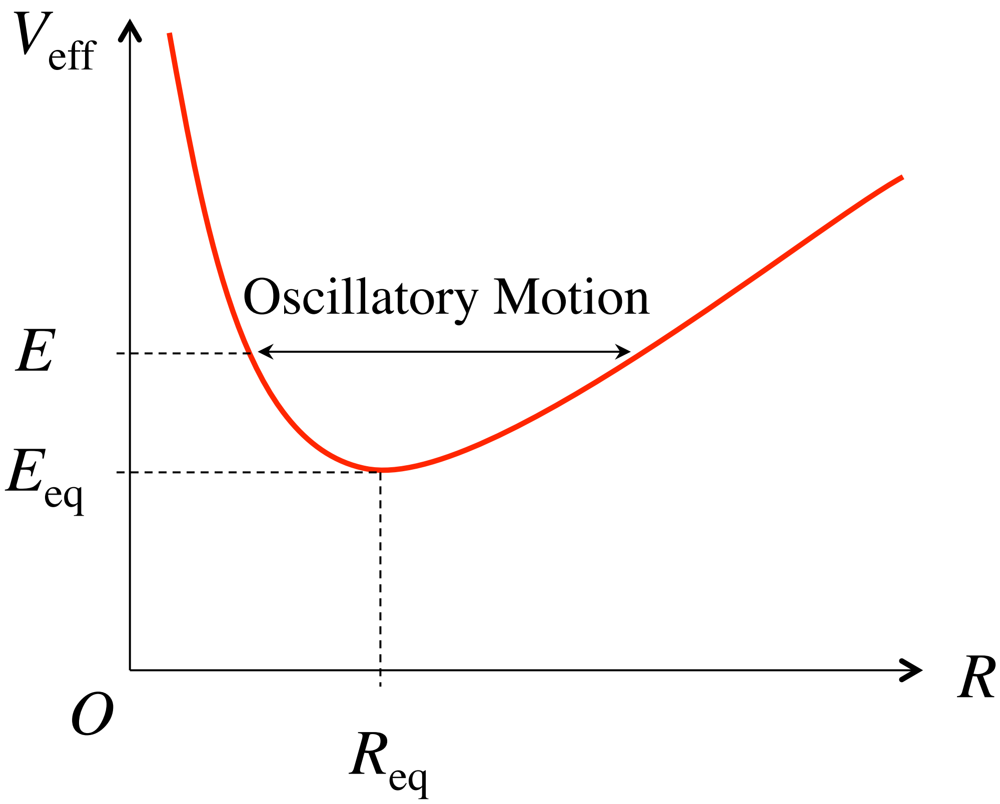
$$\begin{cases} (\partial_n\psi)_+ - (\partial_n\psi)_- = -\dfrac{2\lambda}{R\sqrt{1+u^2}} \\ \sqrt{e^{-2(\gamma_+ - \psi_s)} + \left(\dfrac{dR}{d\tau}\right)^2} - \sqrt{e^{-2(\gamma_- - \psi_s)} + \left(\dfrac{dR}{d\tau}\right)^2} = -4\lambda\sqrt{1+u^2} \\ \dfrac{d^2R}{d\tau^2} = (\text{positive quantity}) \times \left(\Lambda_{\text{eq}}(u) - \Lambda\right) \quad \text{: EOM for the shell} \end{cases}$$

where

$\Lambda \equiv e^{-\psi_s}\lambda$: rest mass per unit proper z-coordinate

$\Lambda_{\text{eq}}(u) \equiv \dfrac{u^2\sqrt{1+u^2}}{(1+2u^2)^2}$: rest mass per unit proper z-coordinate for static configuration

Newtonian Situation



Centrifugal Potential

$$V_{\text{eff}} = \frac{l^2}{2R^2} + \lambda \ln R$$

Gravitational Potential

R : radius of cylindrical shell
λ : rest mass per unit length
l : specific angular momentum

If initially $E > E_{\text{eq}}$ holds, the cylindrical shell will be oscillating.

If initially $E = E_{\text{eq}}$ holds, the cylindrical shell will remain at $R = R_{\text{eq}}$.

Relativistic Situation **(Indication by Apostolatos & Thorne)**

Even if the cylindrical shell is initially collapsing with $E>E_{eq}$,
it will bounce by the centrifugal potential.

↓

As in the Newtonian case, the cylindrical shell will oscillate.

↓

By contrast to the Newtonian case, gravitational waves will be generated.

↓

Kinetic energy E of the cylindrical shell will be released by the gravitational waves.

↓

Finally, the cylindrical shell will rest at an equilibrium position.

Is this always true? **No!**

Proposition

Initial data which does not settle down in any static configuration exists.

<u>Sketch of proof</u>

1) Momentarily Static and Radiation Free (MSRF) initial data can be obtained analytically.

MSRF condition: $\partial_t \psi = 0,\ \partial_t^2 \psi = 0$ and $\dfrac{dR}{d\tau} = 0$

$$\gamma = 0, \qquad \psi = \psi_{\mathrm{i}} \qquad \text{for } (-) \text{ region}$$

$$\gamma = \gamma_{\mathrm{i}} + \kappa^2 \ln\frac{r}{R}, \qquad \psi = \psi_{\mathrm{i}} - \kappa \ln\frac{r}{R} \qquad \text{for } (+) \text{ region}$$

where $\gamma_{\mathrm{i}} = -\ln\left(1 - 4\Lambda_{\mathrm{i}}\sqrt{1 + u_{\mathrm{i}}^2}\right)$, $\kappa = \dfrac{2\Lambda_{\mathrm{i}}}{\left(1 - 4\Lambda_{\mathrm{i}}\sqrt{1 + u_{\mathrm{i}}^2}\right)\sqrt{1 + u_{\mathrm{i}}^2}}$ and $\Lambda_{\mathrm{i}} := e^{-\psi_{\mathrm{i}}}\lambda$

2) Consider a STATC CONFIGURATION with the same conserved quantities λ and l as those of the MSRF initial data: such a static configuration is unique.

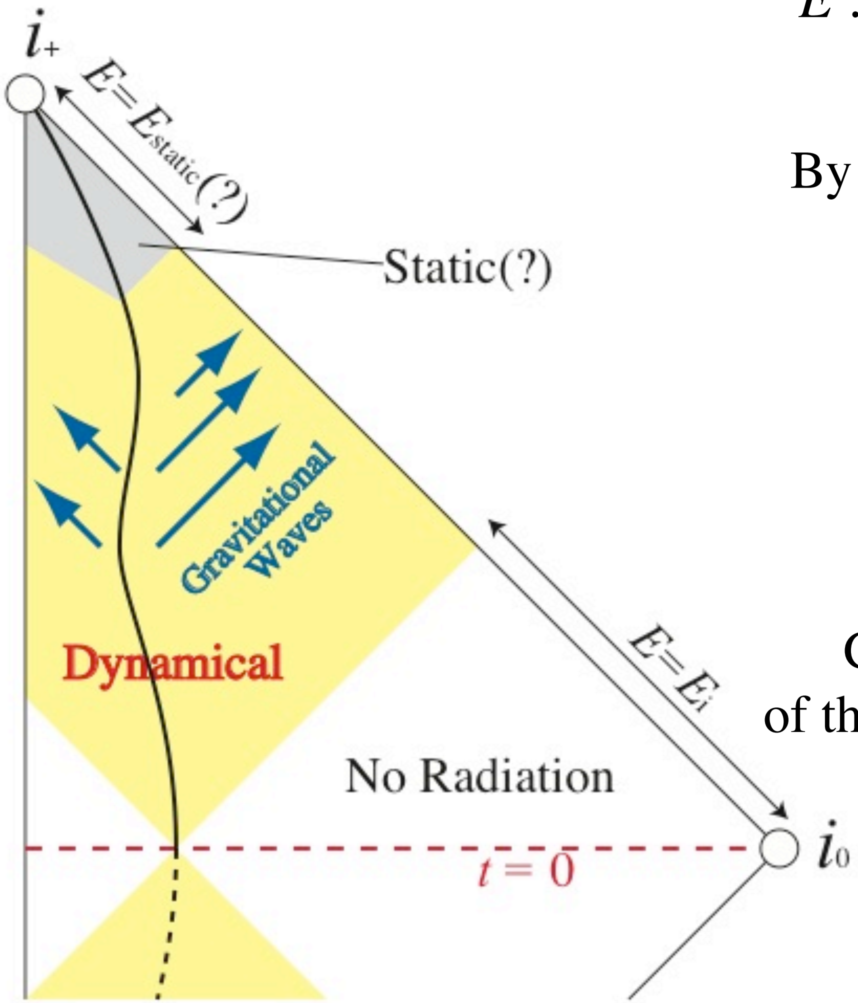
$$\gamma = 0, \qquad \psi = \psi_{\rm f} \qquad \text{for } (-) \text{ region}$$

$$\gamma = \gamma_{\rm f} + \kappa^2 \ln\frac{r}{R}, \qquad \psi = \psi_{\rm f} - \kappa \ln\frac{r}{R} \qquad \text{for } (+) \text{ region}$$

where $\gamma_{\rm f} = 2\ln\left(1 + 2u_{\rm f}^{\ 2}\right)$, $\psi_{\rm f} = \ln\dfrac{\lambda}{\Lambda_{\rm eq}(u_{\rm f})}$ and $\Lambda_{\rm eq}(u) \equiv \dfrac{u^2\sqrt{1+u^2}}{\left(1+2u^2\right)^2}$

Note $u_{\rm f} := \dfrac{l}{e^{-\psi_{\rm f}} R}$ is a free parameter.

C-energy: quasi-local energy per unit Killing length (Thorne 1965)



$$E := \frac{\gamma}{4G}$$ in vacuum region

By Einstein's eqs.,

$$\frac{\partial E}{\partial w} = -\frac{r}{2G}\left(\frac{\partial \psi}{\partial w}\right)^2 \leq 0,$$

where $w := t - r$ is a returded time.

C-energy is a non-increasing function of the returded time at the future null infinity.

$E_i \geq E_{static}$ should hold.

3) Find C-energy of MSRF initial data $E_{\rm i}$ and C-energy of corresponding to that of the static configuration $E_{\rm static}$.
Then, show that there is a MSRF initial data with $E_{\rm i} < E_{\rm static}$.

$$E_{\rm i} = \frac{1}{4G}\left[-\ln\left(1-4\Lambda_{\rm i}\sqrt{1+u_{\rm i}^2}\right)+\kappa^2\ln\frac{r}{R_{\rm i}}\right]\xrightarrow{r\to\infty}\infty$$

$$E_{\rm static} = \frac{1}{4G}\left[-\ln\left(1-4\Lambda_{\rm eq}(u_{\rm f})\sqrt{1+u_{\rm f}^2}\right)+\kappa^2\ln\frac{r}{R_{\rm f}}\right]\xrightarrow{r\to\infty}\infty$$

Total C-energy usually diverges.

where $u_{\rm f} = \sqrt{\frac{\lambda}{\left(1-4\lambda\sqrt{1+u_{\rm i}^2}\right)\sqrt{1+u_{\rm i}^2}}}$

However

$$E_{\rm i} - E_{\rm static} = \frac{1}{4G}\ln\frac{1-4\Lambda_{\rm eq}(u_{\rm f})\sqrt{1+u_{\rm f}^2}}{1-4\Lambda_{\rm i}\sqrt{1+u_{\rm i}^2}}\xrightarrow{r\to\infty}\text{finite}$$

3) Find C-energy of MSRF initial data E_{i} and C-energy of corresponding static configuration E_{static} . Then, show that there is a MSRF initial data with $E_{\mathrm{i}} < E_{\mathrm{static}}$.

Since the C-energy is non-increasing function of the returded time, such a MSRF initial data does not settle down in a static configuration.

Q.E.D.

For the details of the proof,
please see PRD77, 044021 (2008) or arXiv:0711.0243.

This result suggest that a static AT-shell may unstable.

We can see that the condition $E_{\mathrm{static}} - E_{\mathrm{i}} > 0$ holds for a MSRF initial data, if and only if $R_{\mathrm{i}} > R_{\mathrm{static}}$ is satisfied.

What is a final state?

Will it escape to infinity or form a naked singularity?

We need to study the dynamical evolution.

Liner Perturbations in Static AT-shell

Kurita and KN, arXiv:1112.4252

Line element

$$ds^2 = e^{2(\gamma - \psi)}(-dt^2 + dr^2) + e^{-2\psi}\beta^2 d\varphi^2 + e^{2\psi} dz^2$$

Three metric variables; $\gamma = \gamma(t,r)$, $\psi = \psi(t,r)$, $\beta = \beta(t,r)$

Einstein's equations

$$\dot\beta\beta'' - \beta'\dot\beta' + \left(\beta'^2 - \dot\beta^2\right)\dot\gamma + \beta\dot\beta\left(\dot\psi^2 + \psi'^2\right) - 2\beta\beta'\dot\psi\psi' = 0$$

$$\beta'\beta'' - \dot\beta\dot\beta' + \gamma'\left(\dot\beta^2 - \beta'^2\right) + \beta\beta'\left(\dot\psi^2 + \psi'^2\right) - 2\beta\dot\beta\dot\psi\psi' = 0$$

Constraint equations

$$\ddot\psi + \frac{\dot\beta}{\beta}\dot\psi - \psi'' - \frac{\beta'}{\beta}\psi' = 0$$

$$\ddot\beta - \beta'' = 0$$

$$\ddot\gamma - \gamma'' = \dot\psi^2 - \psi'^2$$

Evolution equation

Liner Perturbations in Static AT-shell

Line element

$$ds^2 = e^{2(\gamma - \psi)}(-dt^2 + dr^2) + e^{-2\psi}\beta^2 d\varphi^2 + e^{2\psi} dz^2$$

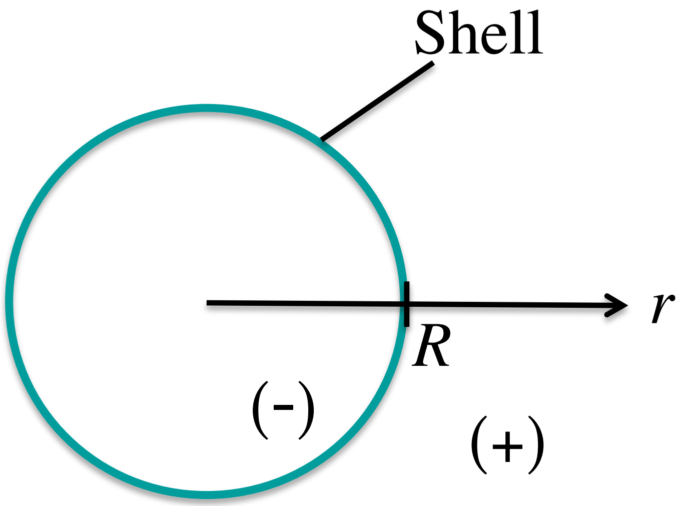
Three metric variables; $\gamma = \gamma(t,r)$, $\psi = \psi(t,r)$, $\beta = \beta(t,r)$

Perturbations

$$\beta^{(\pm)} = r + \beta_1^{(\pm)}$$

$$\begin{cases} \psi^{(-)} = \psi_s + \psi_1^{(-)} \\ \psi^{(+)} = \psi_s - \kappa \ln\left(\frac{r}{R}\right) + \psi_1^{(+)} \end{cases}$$

$$\begin{cases} \gamma^{(+)} = 4u_0^4 \ln\left(\frac{r}{R}\right) + 2\ln\left(1 + 2u_0^2\right) + \gamma_1 \\ \gamma^{(-)} = \gamma_1 \end{cases}$$



ψ_s =定数 $\qquad u_0 = \frac{\ell}{e^{-\psi_s} R} \qquad \kappa = 2u_0^2$

Liner Perturbations in Static AT-shell

$$\beta_1'' - \gamma_0'\beta_1' + \psi_0'^2\beta_1 + 2\beta_0\psi_0'\psi_1' - \gamma_1' = 0$$

$$\gamma_0'\dot{\beta}_1 - \dot{\beta}_1' + \dot{\gamma}_1 - 2\beta_0\psi_0'\dot{\psi}_1 = 0$$

Constraint equations

$$\beta_0\ddot{\psi}_1 - \beta_0\psi_1'' - \beta_0'\psi_1' - \psi_0'\beta_1' - \psi_0''\beta_1 = 0$$

$$\ddot{\beta}_1 - \beta_1'' = 0$$

$$\ddot{\gamma}_1 - \gamma_1'' = 2\psi_0'\psi_1'$$

Evolution equation

Quantity with a subscript 0 ＝ background one

Solutions

$$\beta_1^{(-)} = \int_{-\infty}^{+\infty} d\omega e^{-i\omega t_-} A_\beta^{(-)}(\omega)\left(e^{i\omega r} - e^{-i\omega r}\right)$$

$$\psi_1^{(-)} = \int_0^{+\infty} d\omega e^{-i\omega t_-} A_\psi^{(-)}(\omega) J_0(\omega r)$$

$$\gamma_1^{(-)} = \partial_r \beta_1^{(-)}$$

Solutions inside the shell

Time coordinate within the shell $t_- = e^{\psi_s}\tau$

Outgoing wave condition is imposed.

$$\beta_1^{(+)} = \int_{-\infty}^{+\infty} d\omega e^{-i\omega t_+} A_\beta^{(+)}(\omega) e^{i\omega r}$$

$$\psi_1^{(+)} = \int_0^{+\infty} d\omega e^{-i\omega t_+} A_\psi^{(+)}(\omega) H_0^{(1)}(\omega r) + \frac{\kappa}{r}\beta_1^{(+)}$$

$$\gamma_1^{(+)} = \partial_r \beta_1^{(+)} + \frac{\kappa(\kappa+1)}{r}\beta_1^{(+)} + 2\kappa\psi_1^{(+)} + C$$

Solutions outside the shell

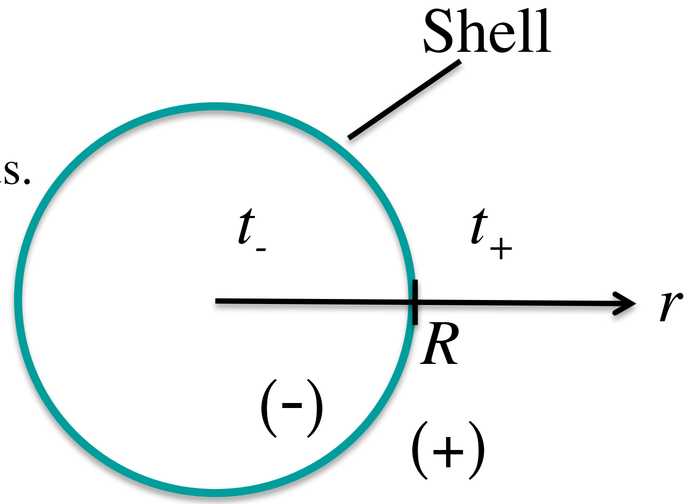
Time coordinate outside the shell $t_+ = e^{\psi_s - \gamma_0}\tau = \left(1 + 2\kappa\right)e^{\psi_s}\tau$

τ: proper time of the shell

Junction condition at the Shell

$$K^{+}_{\mu\nu} - K^{-}_{\mu\nu} = 8\pi G\left(S_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}S^{\alpha}_{\alpha}\right) \quad \text{at } r=R$$

We adopt the radial coordinate comoving to the shell: Shell always stays at $r=R$.



$$\beta_1^{(-)}\left(t_-(\tau),R\right) = \beta_1^{(+)}\left(t_+(\tau),R\right)$$
$$\psi_1^{(-)}\left(t_-(\tau),R\right) = \psi_1^{(+)}\left(t_+(\tau),R\right)$$

β and ψ are continuous.

$$\left[\psi_1' - \psi_0'\gamma_1\right]_-^+ = \frac{2\sqrt{2}e^{\gamma_0}\Lambda}{R\sqrt{2+\kappa}}\left(\frac{2}{2+\kappa}\frac{\beta_1}{R} + \frac{2(1+\kappa)}{2+\kappa}\psi_1\right)$$

$$\left[\gamma_1' - \gamma_0'\gamma_1\right]_-^+ = \frac{2\sqrt{2}e^{\gamma_0}\kappa}{R\sqrt{2+\kappa}}\left(\frac{2}{2+\kappa}\psi_1 + \frac{2(3+\kappa)}{2+\kappa}\frac{\beta_1}{R}\right)$$

$$\left[\beta_1' - \beta_0'\gamma_1\right]_-^+ = 2e^{\gamma_0}\Lambda\sqrt{2(2+\kappa)}\left(\frac{2}{2+\kappa}\psi_1 + \frac{\kappa}{2+\kappa}\frac{\beta_1}{R}\right)$$

The value of γ and the derivatives of β and ψ may be discontinuous.

Junction condition at the Shell

$$K^{+}_{\mu\nu} - K^{-}_{\mu\nu} = 8\pi G\left(S_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}S^{\alpha}_{\alpha}\right) \qquad \text{at } r=R$$

$$\begin{pmatrix} 0 & M_{12}(\omega) & 0 & M_{14}(\omega) \\ M_{21}(\omega) & M_{22}(\omega) & M_{23}(\omega) & 0 \\ M_{31}(\omega) & M_{32}(\omega) & M_{33}(\omega) & M_{34}(\omega) \\ M_{41}(\omega) & M_{42}(\omega) & M_{43}(\omega) & M_{44}(\omega) \end{pmatrix} \begin{pmatrix} A^{(+)}_{\psi}(\omega) \\ A^{(+)}_{\beta}(\omega) \\ A^{(-)}_{\psi}(\omega) \\ A^{(-)}_{b}(\omega) \end{pmatrix} = 0$$

By introducing a new variable $\hat{R} = (1+\kappa)^2 R$

$$M_{12}(\omega) = (1+\kappa)^2 e^{i\omega\hat{R}} \qquad M_{14}(\omega) = -2i\sin(\omega R) \qquad M_{43}(\omega) = -\frac{4\kappa^2}{2+\kappa} J_0(\omega R)$$

$$M_{21}(\omega) = (1+\kappa)^2 H_0^{(1)}(\omega\hat{R}) \qquad M_{22}(\omega) = -\kappa(1+\kappa)^2 e^{i\omega\hat{R}} \qquad M_{23}(\omega) = -2J_0(\omega R)$$

$$M_{31}(\omega) = -\omega R(1+\kappa)^4 H_1^{(1)}(\omega\hat{R}) - 2\kappa^2(1+\kappa)^2 H_0^{(1)}(\omega\hat{R}) \qquad M_{32}(\omega) = \kappa(1+\kappa^2)(1+\kappa)^2 e^{i\omega\hat{R}}$$

$$M_{33}(\omega) = -(1+\kappa)^4 \omega R J_1(\omega R) - \frac{2\kappa(1+\kappa)}{2+\kappa} J_0(\omega R) \qquad M_{34}(\omega) = -\frac{4i\kappa\sin(\omega R)}{2+\kappa}$$

$$M_{41}(\omega) = -2\omega R\kappa(1+\kappa)^4 H_1^{(1)}(\omega\hat{R}) + 2\kappa^3(1+\kappa)^2 H_0^{(1)}(\omega\hat{R})$$

$$M_{42}(\omega) = -(1+\kappa)^2\left[(\omega R)^2(1+\kappa)^4 + \kappa^2(1+\kappa^2)\right] e^{i\omega\hat{R}} \qquad M_{43}(\omega) = -\frac{4\kappa^2}{2+\kappa} J_0(\omega R)$$

$$M_{44}(\omega) = 2i\left[(\omega R)^2(1+\kappa)^2 + \frac{2\kappa(3+\kappa)}{2+\kappa}\right]\sin(\omega R)$$

$$\begin{pmatrix} 0 & M_{12}(\omega) & 0 & M_{14}(\omega) \\ M_{21}(\omega) & M_{22}(\omega) & M_{23}(\omega) & 0 \\ M_{31}(\omega) & M_{32}(\omega) & M_{33}(\omega) & M_{34}(\omega) \\ M_{41}(\omega) & M_{42}(\omega) & M_{43}(\omega) & M_{44}(\omega) \end{pmatrix} \begin{pmatrix} A_{\psi}^{(+)}(\omega) \\ A_{\beta}^{(+)}(\omega) \\ A_{\psi}^{(-)}(\omega) \\ A_{b}^{(-)}(\omega) \end{pmatrix} = 0$$

In order that the above equation has a non-trivial solution,

$$\det \begin{vmatrix} 0 & M_{12}(\omega) & 0 & M_{14}(\omega) \\ M_{21}(\omega) & M_{22}(\omega) & M_{23}(\omega) & 0 \\ M_{31}(\omega) & M_{32}(\omega) & M_{33}(\omega) & M_{34}(\omega) \\ M_{41}(\omega) & M_{42}(\omega) & M_{43}(\omega) & M_{44}(\omega) \end{vmatrix} = 0 \quad \Rightarrow \quad \omega \text{ is determined.}$$

Unstable modes exist.

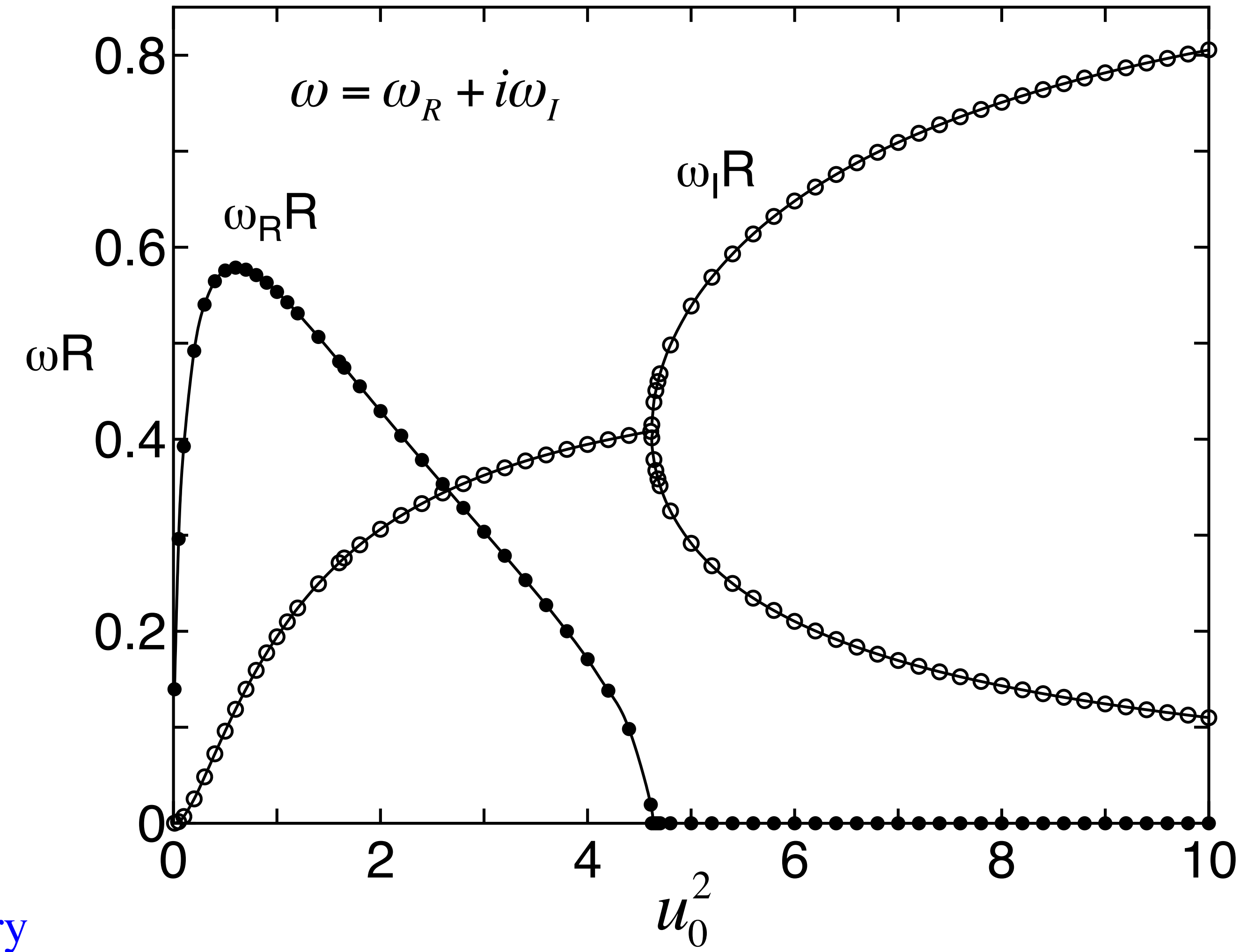
For example,

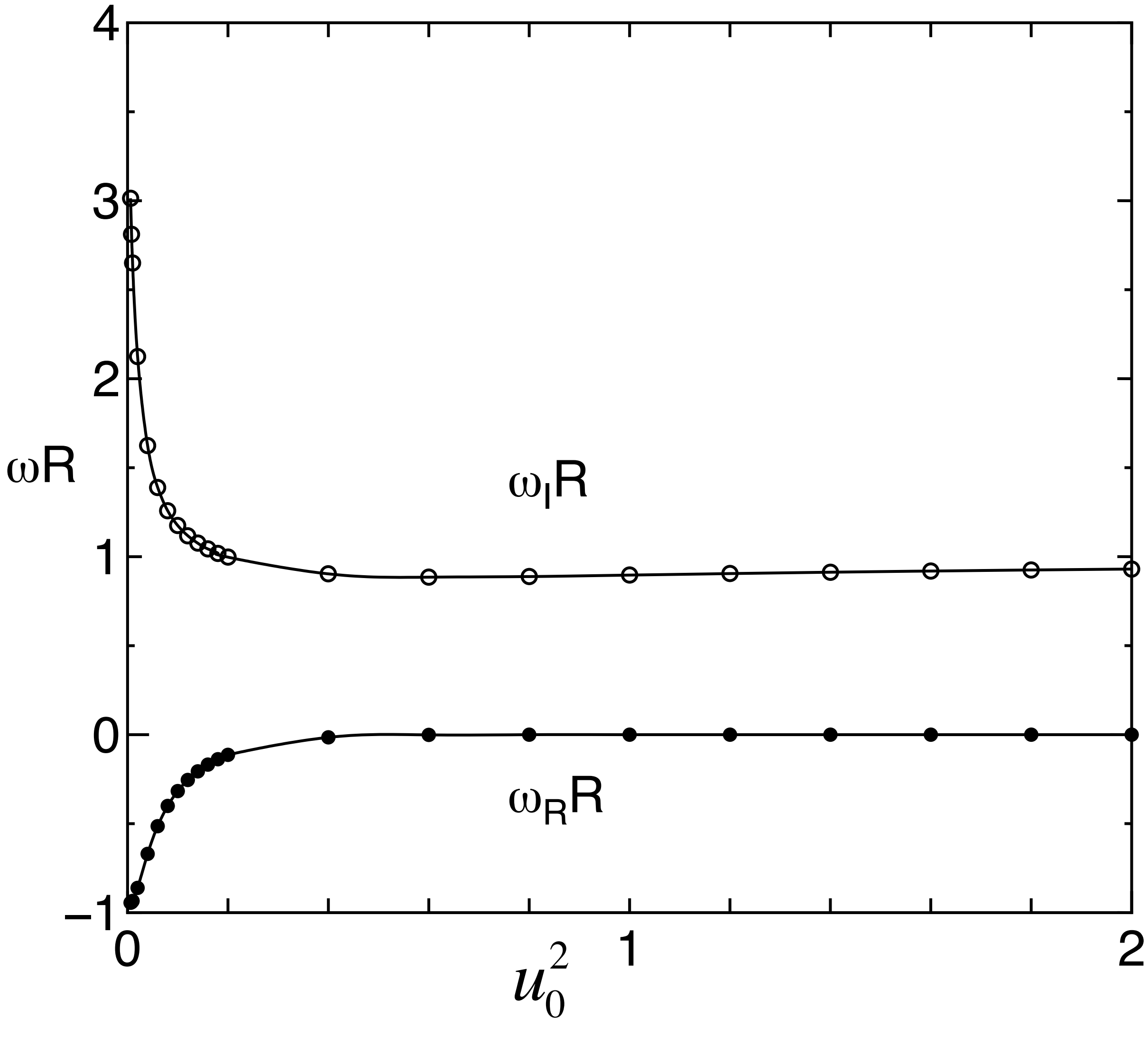
R : radius of the shell of the background

$$u_0 = \frac{v_0}{\sqrt{1 - v_0^2}}$$

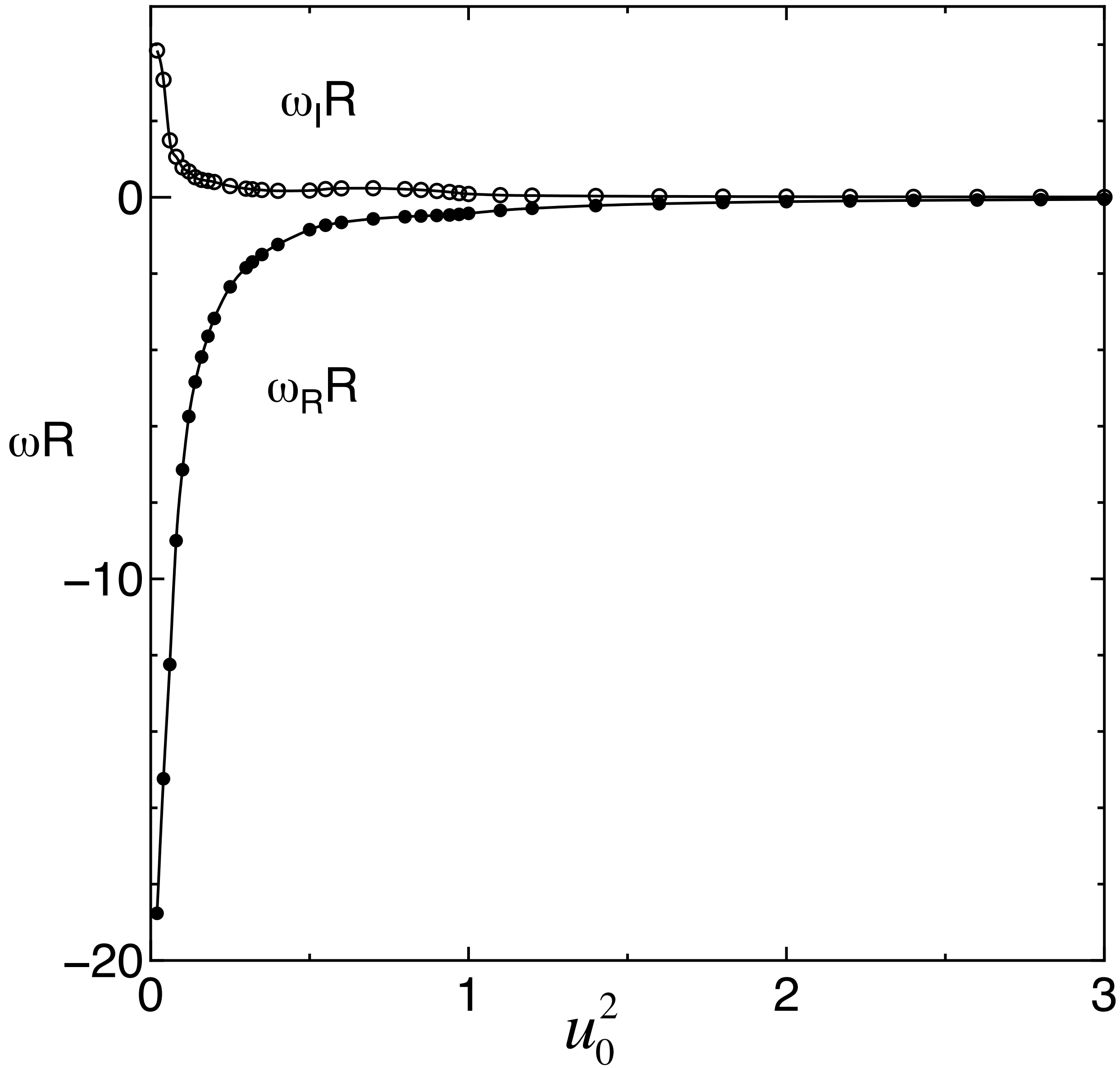
Rotational velocity of a constituent particle

$\omega_R = 0$, i.e.,
ω is pure imaginary
ω for $v_0 \geq 0.76$





4
3
2
1
0
−1
ωR
$\omega_I R$
$\omega_R R$
0
1
2
u_0^2



ωIR
ωRR
ωR
0
−10
−20
0
1
2
3
u02

Why is this system unstable?

Final configuration?

Future Work