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Black holes in the brane world: Time symmetric initial data

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We numerically construct time-symmetric initial data sets of a black hole in the Randall-Sundrum brane world model, assuming that the black hole is spherical on the brane. We find that the apparent horizon is cigar shaped in 5D spacetime.

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I. INTRODUCTION

Motivated by the Horava-Witten model [1], the so called brane world model has been actively investigated [2]. Among several models, a simple, but very attractive model was recently proposed by Randall and Sundrum [3,4]. According to their scenario, we are living in a 4D domain wall in 5D bulk spacetime. The noteworthy features of their model are that in the linearized theory conventional gravity can be recovered on the brane [3–7] and that a homogeneous, isotropic universe can be simply described if we consider a 4D domain wall moving in the 5D Schwarzschild–anti-de Sitter spacetime [8].

One of the most nonlinear objects in the theory of gravity is a black hole, which should be also investigated to understand the nature of the models in strong fields. However, because of the complexity of the equations, any realistic, exact solutions for black holes have not been discovered in the brane world model, even with help of numerical computation so far. We only know that the effective 4D gravitational equation on the brane is different from the Einstein equation [9] (see Appendix A), so that the static solution for a nonrotating black hole should not be identical with the 4D Schwarzschild solution. Indeed, a linear perturbation analysis [5,7] shows that a solution of gravitational field outside self-gravitating bodies on the brane is slightly different from the 4D Schwarzschild solution. Chamblin *et al.* [10] conjecture that the topology of black hole event horizons would be spherical with the cigar-shaped surface in the 5D spacetime. However, nothing has been clarified substantially.

In this paper, as a first step toward self-consistent studies for black holes in the brane world, we numerically compute a black hole space using a time symmetric initial value formulation; namely we solve the 5D Einstein equation only on a spacelike 4D hypersurface. Thus, the black hole obtained here is not static nor the exact solution for the 5D Einstein equation, implying that we cannot identify the event horizon. However, we can investigate the property of the horizon determining the apparent horizon which could give us an insight on the black hole in the brane world. We focus on the Randall-Sundrum's second model [4], and assume that the black hole is spherical on the brane, but the shape of the horizon is nontrivial in the bulk. We will determine the apparent horizon on the brane and show that the black hole is cigar-shaped as conjectured in [10].

II. FORMULATION AND RESULTS

We consider time symmetric, spacelike hypersurfaces, Σ_t, in the brane world model assuming the vanishing extrinsic curvature; i.e.,

$$H_{\mu\nu}:={}^{(4)}\nabla_\mu t_\nu=0 \tag{2.1}$$

where t^μ is the unit normal timelike vector to Σ_t and ${}^{(4)}\nabla_\mu$ is the covariant derivative with respect to the 4D metric on Σ_t. In this case, the momentum constraint is satisfied trivially, and the equation of the Hamiltonian constraint becomes

$${}^{(4)}R=16\pi G_5(\Lambda+{}^{(5)}T_{\mu\nu}t^\mu t^\nu), \tag{2.2}$$

where ${}^{(4)}R$ is the Ricci scalar on Σ_t, and $G_5(=\kappa_5^2/8\pi)$, Λ and ${}^{(5)}T_{\mu\nu}$ denote the gravitational constant, negative cosmological constant, and energy-momentum tensor in 5D spacetime [cf. Eq. (A1)]. We choose the line element on Σ_t in the form

$$dl^2=\frac{1}{z^2}[l^2dz^2+\psi^4(dr^2+r^2d\Omega)], \tag{2.3}$$

where $l=\sqrt{-\kappa_5^2\Lambda/6}$, z $(\geqslant 1)$ denotes the coordinate orthogonal to the brane and r $(\geqslant 0)$ is the radial coordinate on the brane. We assume that the brane is located at $z=1$. Note that we simply choose this line element for convenience of the analysis. In this paper, we focus on a black hole which is spherical on the brane, i.e., $\psi=\psi(r,z)$. Then, the explicit form of the Hamiltonian constraint in the bulk (for $z>1$) is written in the form

$$\psi''+\frac{2}{r}\psi'+\frac{3}{2l^2}\left[\left(\partial_z^2\psi-\frac{3}{z}\partial_z\psi\right)\psi^4+3(\partial_z\psi)^2\psi^3\right]$$
$$=-\frac{\kappa_5^2}{4}{}^{(5)}\tau_{\mu\nu}t^\mu t^\nu, \tag{2.4}$$

where $' = \partial/\partial r$, and ${}^{(5)}\tau_{\mu\nu}$ is the energy-momentum tensor in the bulk, which is introduced for numerical convenience.

Equation (2.4) is an elliptic type equation and should be solved imposing boundary conditions at $z=1$, $z \gg 1$, $r=0$, and $r \gg l$. The boundary condition at $z=1$ is derived from Israel's junction condition [11] as (see Appendix A for the derivation)

$$\partial_z \psi|_{z=1} = 0. \tag{2.5}$$

The boundary conditions at $z \gg 1$ and $r \gg l$ are obtained from the linear perturbation analysis (see Appendix B). For $r \gg l$ and $r > lz$, it becomes

$$\psi \simeq 1 + \frac{MG_4}{2r}\left[1 + \frac{1}{2}\left(\frac{R}{r}\right)^2 + O((l/r)^4)\right], \tag{2.6}$$

where $G_4 = G_5/l$, M is the gravitational mass on the brane, and $R = (2/3)^{1/2} l$. For $z \gg 1$,

$$\psi \simeq 1 + \frac{3}{4}\frac{G_4 M}{Rz}\left(1 + \frac{r^2}{z^2 R^2}\right)^{-3/2}. \tag{2.7}$$

To determine the existence of a black hole, we search for the apparent horizon. Here, we determine two horizons [12]. One is defined to be the spherical two-surface on the brane on which the expansion of the null geodesic congruence confined on the brane is zero [13], i.e.,

$$\theta_3 = \frac{2}{\psi^3}\left(2\psi' + \frac{1}{r}\psi\right) = 0. \tag{2.8}$$

The other is the apparent horizon in full 4D space, which is defined with respect to the null geodesic congruence in full 5D spacetime and satisfies [13]

$$\theta_4 = {}^{(4)}\nabla_i s^i = 0, \tag{2.9}$$

where s^i is a unit normal to the surface of the apparent horizon. Explicit equation for determining this apparent horizon is shown in Appendix C.

The procedure of numerical analysis is as follows. First, we artificially put the matter of $\rho_h \equiv {}^{(5)}\tau_{\mu\nu} t^\mu t^\nu > 0$ in the bulk. This method is employed because we do not have to consider the inner boundary condition of black holes with this treatment. As long as ρ_h is confined around the brane and inside the horizon, it does not significantly affect the geometry outside the horizon. Then, we solve Eq. (2.4), and try to find the apparent horizon both on the brane and in the bulk. When the distribution of ρ_h is sufficiently compact, the apparent horizons exist. It should be noted that two horizons do not coincidently appear. In some cases, the apparent horizon on the brane exists although that in the bulk does not.

Here, we show one example of numerical results. We set $G_4 = 1$. In this example, an artificial matter is put for $0 \leq r \leq 0.2R$ and $1 \leq z \leq 1.2$. Equation (2.4) is solved using a uniform grid with grid size 1200×1200 for r and z directions, which covers a domain with $0 \leq r/R \leq 17.1$ and $1 \leq z \leq 18.1$. In this case, the gravitational mass on the brane is $M \simeq 0.29R$, and both apparent horizons on the brane and in the bulk exist. We note that the results are essentially the same for $0.25 \leq M/R \leq 0.5$. In Fig. 1, we show the location of apparent horizons in the bulk and on the brane. The apparent horizon in the bulk is apparently cigar-shaped. Due to this cigar-shape the circumferential radius of the apparent horizon is different depending on the choice of the circumference in the bulk. In Fig. 2, we show that the profile of $\psi - 1$ on the brane. For $r \gg R$, $\psi - 1$ behaves as $M/2r$, implying that the solution approximately agrees with that in the 4D Einstein gravity, i.e., the bulk effect is small. However, the existence of the bulk is significant for $r \sim R$ as expected. Indeed, $\psi - 1$ deviates from $M/2r$ with decreasing r. This effect is in particular important for the location and area of the apparent horizon on the brane: In the case of 4D gravity without bulk, the apparent horizon is located at $r_{\mathrm{AH}} = M/2$ with the area $A_{\mathrm{AH}} = 16\pi M^2$. However, in the brane world model, they take different values in general. [In this example, $r_{\mathrm{AH}} \simeq 0.9M$ and $A_{\mathrm{AH}} \simeq 88.6M^2$, and the coefficients converge to well-know 4D values (0.5 and 16π) with increasing M, implying that the effect of the existence of the bulk becomes less important.]

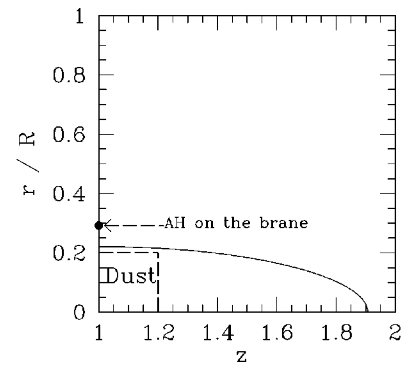

FIG. 1. Location of the apparent horizons on the brane (filled circle) and in the 4D space (solid line). Artificial matter is confined in the region shown by the dashed line.

III. SUMMARY

We numerically computed time symmetric initial data sets of a black hole in the brane world model, assuming that the

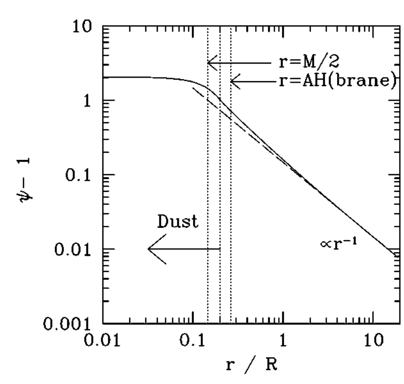

FIG. 2. Profile of $\psi - 1$ on the brane (solid line). Location of the apparent horizon on the brane is shown. The dashed line denotes $\psi - 1 = M/2r$.

black hole is spherical on the brane. As has been expected, the black hole (apparent horizon) is cigar-shaped in the bulk [10].

We remind that we only present time symmetric initial data of a black hole space. This implies that the black hole is not static and will evolve to other state with time evolution. The quantitative features of the final fate could be different from the present result. Self-consistent analysis for static black holes should be carried out for future to obtain a definite answer with regard to black holes in the brane world. However, we believe that the present result provides us a guideline for such future works.

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APPENDIX A: THE ESSENCE OF THE BRANE WORLD

We briefly review the covariant formalism of the brane world [9]. For the matter source of the 5D Einstein equation, ${}^{(5)}G_{\mu\nu}=\kappa_5^2({}^{(5)}T_{\mu\nu}-\Lambda{}^{(5)}g_{\mu\nu})$, we choose the energy-momentum tensor as

$$ {}^{(5)}T_{\mu\nu}=\delta(\chi)[-\lambda q_{\mu\nu}+{}^{(4)}T_{\mu\nu}]+{}^{(5)}\tau_{\mu\nu}, \qquad \text{(A1)} $$

where $\chi=l\ln z$, λ is the tension of the brane, $q_{\mu\nu}$ is the induced metric on the brane, and ${}^{(4)}T_{\mu\nu}$ is the energy momentum tensor on the brane. Due to the singular source at $\chi=0$ and the Z_2 symmetry, we can derive the Israel's junction condition at $\chi=0$ as

$$ K_{\mu\nu}=-\frac{1}{6}\kappa_5^2\lambda q_{\mu\nu}-\frac{1}{2}\kappa_5^2\left({}^{(4)}T_{\mu\nu}-\frac{1}{3}q_{\mu\nu}^{(4)}T_\sigma^\sigma\right), \qquad \text{(A2)} $$

where $K_{\mu\nu}=D_\mu n_\nu$, and D_σ and n^μ are the covariant derivative with respect to $q_{\mu\nu}$ and the unit spacelike normal vector to the brane. In the text, we consider the cases in which ${}^{(4)}T_{\mu\nu}=0$. Using (4+1) formalism, the effective 4D equation on the brane has the form

$$ {}^{(4)}G_{\mu\nu}=-\Lambda_4 q_{\mu\nu}-E_{\mu\nu}, \qquad \text{(A3)} $$

where ${}^{(4)}G_{\mu\nu}$ is the 4D Einstein tensor on the brane,

$$ \Lambda_4=\frac{1}{2}\kappa_5^2\left(\Lambda+\frac{1}{6}\kappa_5^2\lambda^2\right) \quad \text{and} \quad E_{\mu\nu}={}^{(5)}C_{\mu\rho\nu\sigma}n^\rho n^\sigma, \qquad \text{(A4)} $$

where ${}^{(5)}C_{\mu\rho\nu\sigma}$ is 5D Weyl tensor. In the above, for simplicity, we set ${}^{(5)}\tau_{\mu\nu}=0$. Equation (A3) implies that we can consider $E_{\mu\nu}$ as the effective source term of the 4D Einstein equation on the brane, and as long as $E_{\mu\nu}$ is not vanishing, the geometry on the brane is different from that in the 4D gravity even in the vacuum case. Only for very special case such as for the black string solution [10,14], $E_{\mu\nu}=0$ holds.

From Eq. (A3), we find that the Minkowski spacetime is realized on the brane when $E_{\mu\nu}=0$ and $\Lambda_4=0$. In this paper, we set $\Lambda_4=0$ to focus on asymptotically flat brane. Then, the junction condition at $\chi=0$ is rewritten to $K_{\mu\nu}=-(1/l)q_{\mu\nu}$. In the case when we choose the line element as Eq. (2.3), the junction condition reduces to Eq. (2.5).

APPENDIX B: ASYMPTOTIC BOUNDARY CONDITIONS

To specify the boundary condition at infinities, we investigate the linearized equation of Eq. (2.4):

$$ \varphi''+\frac{2}{r}\varphi'+\frac{1}{R^2}\left(\partial_z^2\varphi-\frac{3}{z}\partial_z\varphi\right)=-\frac{\kappa_5^2}{4}\rho_h, \qquad \text{(B1)} $$

where $\psi=1+\varphi$ and $\varphi\ll1$. We can obtain the formal solution with aid of the Green function $G(x,z;x',z')$ as

$$ \varphi\simeq-2\pi G_4 l\int d^3x'dz'G(x,z;x',z')\rho_h(x',z'). \qquad \text{(B2)} $$

Assuming that ρ_h is nonzero only in the small region around the brane, we can derive the relevant Green function as [5]

$$ \begin{aligned} G(x,z;x',z')&=-\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\\ &\quad\times\left[\frac{1}{l\mathbf{k}^2}+\int_0^\infty dm\frac{u_m(z)u_m(1)}{\mathbf{k}^2+m^2}\right]\\ &=G_0+G_{\mathrm{KK}}, \end{aligned} \qquad \text{(B3)} $$

where $u_m(z)$ is the mode function given from the Bessel functions J_n and N_n as

$$ u_m(z)=z^2\sqrt{\frac{mR^2}{2l}}\,\frac{J_1(mR)N_2(mRz)-N_1(mR)J_2(mRz)}{\sqrt{[J_1(mR)]^2+[N_1(mR)]^2}}, \qquad \text{(B4)} $$

where $R=(2/3)^{1/2}l$. G_0 and G_{KK} are the Green function of zero and KK modes, respectively. From Eq. (B2) we can derive the asymptotic boundary conditions shown in the text.

APPENDIX C: APPARENT HORIZON IN THE BULK

We derive the equation for the apparent horizon in the bulk. After we perform the coordinate transformation from (r,z) to (x,θ) as $z=1+x|\cos\theta|$ and $r=lx\sin\theta$, the surface of the apparent horizon is denoted by $x=h(\theta)$. Then, the nonzero components of s_i are written as

$$ s_x=C \quad \text{and} \quad s_\theta=-Ch_{,\theta}, \qquad \text{(C1)} $$

where $C[\equiv\psi^2\hat{C}/(1+x|\cos\theta|)]$ is a normalization constant calculated from $s^is_i=1$, and $h_{,\theta}=dh/d\theta$. Then, the equation for h can be written to the following ordinary differential equation of second order:

$$\frac{d^2h}{d\theta^2}=\frac{h^2}{\psi^4\hat{C}^2}\Bigg[\left(4\frac{\partial_x\psi}{\psi}+\frac{3}{h(1+h|\cos\theta|)}+\frac{\partial_x\hat{C}}{\hat{C}}\right)\left(\sin^2\theta+\psi^4\cos^2\theta-(1-\psi^4)\sin\theta\cos\theta\frac{h_{,\theta}}{h}\right)$$
$$+h^{-1}\left(4\frac{\partial_\theta\psi}{\psi}+3\frac{h\sin\theta}{1+h|\cos\theta|}+2\cot\theta+D\right)\left((1-\psi^4)\sin\theta\cos\theta-(\cos^2\theta+\psi^4\sin^2\theta)\frac{h_{,\theta}}{h}\right)$$
$$+4\psi^3\partial_x\psi(\cos^2\theta+h^{-1}\sin\theta\cos\theta h_{,\theta})+h^{-2}(1-\psi^4)\sin\theta\cos\theta h_{,\theta}+h^{-1}(1-\psi^4)\cos(2\theta)-4h^{-1}\sin\theta\cos\theta\psi^3\partial_\theta\psi$$
$$+\{(1-\psi^4)\sin(2\theta)-4\sin^2\theta\psi^3\partial_\theta\psi\}\frac{h_{,\theta}}{h^2}\Bigg], \qquad \text{(C2)}$$

where

$$D=-\hat{C}^2[(1-\psi^4)\{1-h^{-2}h_{,\theta}^2\}\sin\theta\cos\theta-h^{-1}(1-\psi^4)\cos(2\theta)h_{,\theta}+2\psi^3\partial_\theta\psi(\cos\theta+h^{-1}\sin\theta h_{,\theta})^2]. \qquad \text{(C3)}$$

Equation (C2) is solved imposing boundary conditions at $\theta=0$ and $\pi/2$. In the limit $\theta\to 0$, we impose the following boundary condition:

$$h=h_0+h_2\theta^2+O(\theta^3), \qquad \text{(C4)}$$

where h_2 is evaluated at $x=h_0$ and $\theta=0$ from the following equation:

$$h_2=\frac{h_0^2}{6}\left[\frac{8\partial_x\psi}{\psi}+\frac{3}{h_0(1+h_0)}+\frac{\partial_x\hat{C}}{\hat{C}}+\frac{3}{h_0}(1-\psi^4)\right]. \qquad \text{(C5)}$$

At $\theta=\pi/2$, the boundary condition is imposed as $h_{,\theta}=0$.

Note that in the limit $\theta\to\pi/2$ (i.e., on the brane), Eq. (C2) is written in the form

$$\frac{d^2h}{d\theta^2}=h+\frac{l^2h^2}{\psi^4}\left(\frac{4\partial_x\psi}{\psi}+\frac{2}{h}\right), \qquad \text{(C6)}$$

where we use $h_{,\theta}=0$ and the relation $\partial_\theta\psi=D=\partial_x\hat{C}=0$. Note that the equation which the apparent horizon on the brane satisfies is $4\partial_x\psi/\psi+2/h=0$ [cf. Eq. (2.8)]. Thus, unless $d^2h/d\theta^2=h$ at $\theta=\pi/2$, the apparent horizon on the brane cannot coincide with that in 4D space. Note that the black string solution [10,14] exceptionally satisfies $d^2h/d\theta^2=h$ at $\theta=\pi/2$.

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