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Ref: MH, ``Commuting Flows and Conservation Laws for NC Lax Hierarchies,''[hep-th/0311206]

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1. Introduction

Successful points on NC theories

- Appearance of new physical objects

- Description of real physics

- Various successful applications
  to D-brane dynamics etc.

NC Solitons play important roles
(Integrable!)

Final goal: NC extension of all soliton theroies

Ward's observation:
Almost all integrable equations are reductions of the ASDYM eqs.

Anti-Self-Dual Yang-Mills eq.

↓ Reductions

KP eq.   BCS eq.

KdV eq.   Boussinesq eq.

NLS eq. mKdV eq.

sine-Gordon eq.   Burgers eq. ...

(Almost all !)

e.g. [Mason&Woodhouse]

NC Ward's observation: Almost all NC integrable equations are reductions of the NC ASDYM eqs.

MH&K.Toda, PLA316('03)77[hep-th/0211148]

NC ASDYM eq.

↓ Reductions

Successful

Reductions

NC KP eq. NC BCS eq.
NC KdV eq. NC Boussinesq eq.
NC NLS eq. NC mKdV eq.
NC sine-Gordon eq. NC Burgers eq. …
(Almost all !?)

Successful?

A general framework is needed

2. NC Sato's Theory

- Sato's Theory : one of the most beautiful theory of solitons
    - Based on the exsitence of

        hierarchies and tau-functions

- Sato's theory reveals essential

    aspects of solitons:
    - Construction of exact solutions
    - Structure of solution spaces
    - Infinite conserved quantities
    - Hidden infinite-dim. symmetry

    Let's discuss NC extension of Sato's theory

Derivation of soliton equations

- Give a Lax operator which is a pseudo-differential operator

$$L := \partial_x + u_2 \partial_x^{-1} + u_3 \partial_x^{-2} + u_4 \partial_x^{-3} + \cdots$$

$$u_k = u_k(x^1, x^2, x^3, \cdots)$$

Noncommutativity is introduced here:

$$[x^i, x^j] = i\,\theta^{ij}$$

- Define a differential operator

$$B_m := (L * \cdots * L)_{\geq 0}$$
$$m\ times$$

- Introduce NC (KP) hierarchy equation:

$$\boxed{\frac{\partial L}{\partial x^m} = [B_m, L]_*}$$

Star product:

$$f(x) * g(x) := f(x) \exp\left(\frac{i}{2} \theta^{ij} \overleftarrow{\partial_i} \overrightarrow{\partial_j} \right) g(x)$$

which realizes the noncommutativity

$$\partial_m u_2 \partial_x^{-1} +$$
$$\partial_m u_3 \partial_x^{-2} +$$
$$\partial_m u_4 \partial_x^{-3} + \cdots$$

$$F_{m2}(u)\partial_x^{-1} +$$
$$F_{m3}(u)\partial_x^{-2} +$$
$$F_{m4}(u)\partial_x^{-3} + \cdots$$

$\longrightarrow$ Each coefficient yields a differential equation.

Closer look at NC (KP) hierarchy

For m=2

$$\partial_x^{-1}) \quad \partial_2 u_2 = 2u_3' + u_2''$$

$$\partial_x^{-2}) \quad \partial_2 u_3 = 2u_4' + u_3'' + 2u_2 * u_2' + 2[u_2, u_3]_*$$

$$\partial_x^{-3}) \quad \partial_2 u_4 = 2u_5' + u_4'' + 4u_3 * u_2' - 2u_2 * u_2'' + 2[u_2, u_4]_*$$

$$\vdots$$

Infinite kind of fields are represented
in terms of one kind of field $u_2 \equiv u$

MH&K.Toda, [hep-th/0309265]

$$u_x := \frac{\partial u}{\partial x}$$

$$\partial_x^{-1} := \int^x dx'$$

etc.

For m=3

$$\partial_x^{-1}) \quad \partial_3 u_2 = u_2''' + 3u_3'' + 3u_4'' + 3u_2' * u_2 + 3u_2 * u_2'$$

$$\downarrow$$

$$u_t = \frac{1}{4} u_{xxx} + \frac{3}{4}(u_x * u + u * u_x) + \frac{3}{4}\partial_x^{-1} u_{yy} + \frac{3}{4}[u, \partial_x^{-1} u_{yy}]_*$$

(2+1)-dim.
NC KP equation

and other NC equations
(NC hierarchy equations)

$$u = u(x^1, x^2, x^3, \cdots)$$

$$\updownarrow \quad \updownarrow \quad \updownarrow$$

$$x \quad y \quad t$$

(KP hierarchy) $\xrightarrow{\text{reductions}}$ (various hierarchies.)

- (Ex.) KdV hierarchy

  Reduction condition

  $$L^2 = B_2 (=: \partial_x^2 + u) \qquad : \text{2-reduction}$$

  gives rise to NC KdV hierarchy

  which includes (1+1)-dim. NC KdV eq.:

  $$u_t = \frac{1}{4} u_{xxx} + \frac{3}{4}(u_x * u + u * u_x)$$

  Note $\dfrac{\partial u}{\partial x_{2N}} = 0$: dimensional reduction in x_{2N} directions

  $$\begin{array}{l} \textbf{KP}: \quad u(x^1, x^2, x^3, x^4, x^5, ...) \\ \qquad\qquad\quad x \quad y \quad t \qquad\qquad\qquad\qquad : \text{(2+1)-dim} \\ \\ \textbf{KdV}: \quad u(x^1, \quad x^3, \quad x^5, ...) \\ \qquad\qquad\quad x \qquad t \qquad\qquad\qquad\qquad : \text{(1+1)-dim} \end{array}$$

l-reduction of NC KP hierarchy yields
wide class of other NC hierarchies

- No-reduction → NC KP $(x, y, t) = (x^1, x^2, x^3)$

- 2-reduction → NC KdV $(x, t) = (x^1, x^3)$

- 3-reduction → NC Boussinesq $(x, t) = (x^1, x^2)$

- 4-reduction → NC Coupled KdV $\ldots$

- 5-reduction → ...

- 3-reduction of BKP → NC Sawada-Kotera

- 2-reduction of mKP → NC mKdV

- Special 1-reduction of mKP → NC Burgers

- ...

Noncommutativity should be introduced into space-time coords

4. Conservation Laws

- **Conservation laws:** $\partial_t \sigma = \partial_i J^i$ σ : Conserved density

time ↗   ↖ **space**

Then $\quad Q := \int_{space} dx\, \sigma \quad$ is a conserved quantity.

$$\because \partial_t Q = \int_{space} dx\, \partial_t \sigma = \int_{\substack{spatial \\ \inf inity}} dS_i J^i = 0$$

Conservation laws for the hierarchies

$$\partial_m res_{-1} L^n = \partial_x J + \theta^{ij} \partial_j \Xi_i$$

time ↗   **space** ↗   ↑

I have found the explicit form !

$res_{-r} L^n$: coefficient of ∂_x^{-r} in L^n

Noncommutativity should be introduced in space-time directions only. →

$t \equiv x^m$

∂_j **should be space or time derivative**
→ ordinary conservation laws !

Hot Results

Infinite conserved densities for NC hierarchy eqs. (n=1,2,…)

$$\sigma = res_{-1}L^n + \theta^{mi}\sum_{k=1}^{m}\sum_{l=n+1}^{n+1+m-k}(-1)^{m+n-k-l}\binom{m-k}{l-n-1}a_{n-l}\Diamond\partial_i b_{m-k}^{(m+n-k-l+1)}$$

$$t \equiv x^m \qquad L^n = \partial_x^n + \sum_{l=1}^{\infty}a_{n-l}\partial_x^{n-l}, \quad B_m = \partial_x^m + \sum_{k=1}^{m}b_{m-k}\partial_x^{m-k}$$

$\Diamond$: Strachan's product

$$f(x)\Diamond g(x) := f(x)\left(\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s+1)!}\left(\frac{1}{2}\theta^{ij}\overleftarrow{\partial_i}\overrightarrow{\partial_j}\right)^{2s}\right)g(x)$$

MH, [hep-th/0311206]

We can calculate the explicit forms of conserved densities for wide class of NC soliton equations.

- Space-Space noncommutativity:

  NC deformation is slight: $\sigma = res_{-1}L^n$

- Space-time noncommutativity

  NC deformation is drastical:

  – Example: NC KP and KdV equations $([t, x] = i\theta)$

$$\sigma = res_{-1}L^n - 3\theta((res_{-1}L^n)\Diamond u_3' + (res_{-2}L^n)\Diamond u_2')$$

meaningful ?

4. Conclusion and Discussion

- We proved the existence of infinite conserved quantities for wide class of NC Lax hierarchies.

- We gave the infinite conserved densities explicitly from the viewpoint of Sato's theory, which suggests that infinite-dim. symmetry would be hidden in the NC (soliton) equations.

- The results show that NC soliton eqs. have very special properties though they include non-linear terms and infinite (time-!) derivatives.

- Of course, there are still many things to be seen in order to confirm the integrability.

Further directions

- Completion of NC Sato's theory
  - Hirota's bilinearization and tau-functions $u = \partial_x^2 \log \tau$
    → hidden symmetry (deformed affine Lie algebras?)

    Cf. Dimakis&Mueller-Hoissen, Wang&Wadati…
  - Geometrical descriptions from NC extension of the theories of Krichever, Mulase and Segal-Wilson and so on.

- Confirmation of NC Ward's conjecture
  - NC twistor theory

    e.g. Kapustin&Kuznetsov&Orlov, Hannabuss, Hannover group,…
  - D-brane interpretations → physical meanings

- Foundation of Hamiltonian formalism with space-time noncommutativity
  - Initial value problems, Liouville's theorem, Noether's thm,…