

可積分系の非可換化 の現状について

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Ref: MH, JMP46(2005)052701 [hep-th/0311206]

MH, [hep-th/0507nn] (to appear)

cf. MH, “Non-Commutative Solitons and
Integrable Systems” [hep-th/0504001]

URL: <http://www2.yukawa.kyoto-u.ac.jp/~hamanaka>

1. Introduction

Successful points on NC theories

- Appearance of new physical objects
- Description of real physics
- Various successful applications to D-brane dynamics etc.

**NC Solitons play important roles
(Integrable!)**

Final goal: NC extension of all soliton theories

Integrable equations in diverse dimensions

4	Anti-Self-Dual Yang-Mills eq. (instantons) $F_{\mu\nu} = -\tilde{F}_{\mu\nu}$	NC extension (Successful)
3	Bogomol'nyi eq. (monopoles)	NC extension (Successful)
2 (+1)	KP eq. BCS eq. DS eq. ...	NC extension (This talk)
1 (+1)	KdV eq. Boussinesq eq. NLS eq. Burgers eq. sine-Gordon eq. (affine) Toda field eq. ...	NC extension (This talk)

↑
Dim. of space

Ward's observation: Almost all
integrable equations are
reductions of the ASDYM eqs.

R.Ward, Phil.Trans.Roy.Soc.Lond.A315('85)451

ASDYM eq.



Reductions

KP eq. BCS eq. Ward's chiral model
KdV eq. Boussinesq eq.
NLS eq. mKdV eq.
sine-Gordon eq. Burgers eq. ...

(Almost all !?)

e.g. [The book of Mason&Woodhouse]

NC Ward's observation: Almost all
NC integrable equations are
reductions of the NC ASDYM eqs.

MH&K.Toda, PLA316('03)77[hep-th/0211148]

NC ASDYM eq.

Successful

↓ NC Reductions Reductions

NC KP eq. NC BCS eq. NC Ward's chiral model

NC KdV eq. NC Boussinesq eq.

NC NLS eq. NC mKdV eq.

NC sine-Gordon eq. NC Burgers eq. ...

(Almost all !?)

Successful?

A general framework is needed

Program of NC extension of soliton theories

- (i) Confirmation of NC Ward's conjecture
 - NC twistor theory → geometrical origin
 - D-brane interpretations → applications to physics
- (ii) Completion of NC Sato's theory
 - Existence of “hierarchies” → various soliton eqs.
 - Existence of infinite conserved quantities
 - infinite-dim. hidden symmetry
 - Construction of multi-soliton solutions
 - Theory of tau-functions → structure of the solution spaces and the symmetry

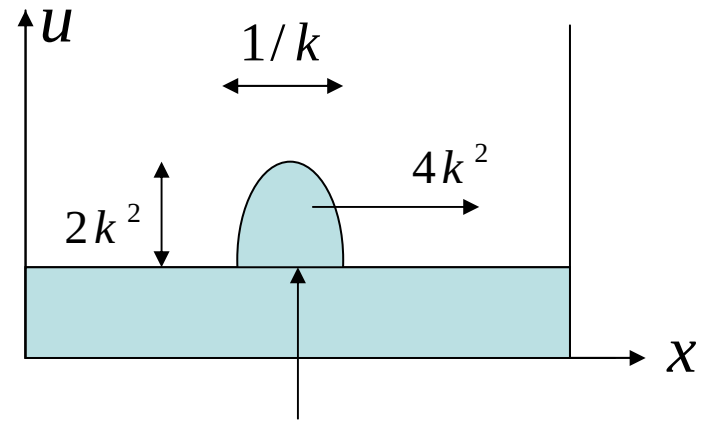
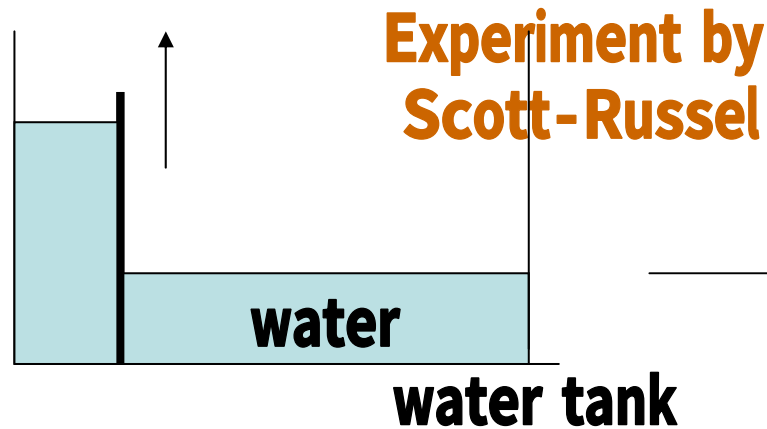
(i),(ii) → complete understanding of the NC soliton theories

Plan of this talk

1. Introduction
2. Review of soliton theories
3. NC Sato's theory
4. Conservation Laws
5. Exact Solutions and Ward's conjecture
6. Conclusion and Discussion

2. Review of Soliton Theories

- KdV equation : describe shallow water waves



solitary wave = soliton

This configuration satisfies

$$u = 2k^2 \cosh^{-2}(kx - 4k^3t)$$

$$\dot{u} + u''' + 6u'u = 0 \quad : \text{KdV eq. [Korteweg-de Vries, 1895]}$$

This is a typical integrable equation.

Let's solve it now !

- Hirota's method [PRL27(1971)1192]

$$\dot{u} + u''' + 6u'u = 0 \quad : \text{naively hard to solve}$$

$$\downarrow \quad u = 2\partial_x^2 \log \tau$$

$$\tau\dot{\tau}' - \tau'\dot{\tau} + 3\tau''\tau'' - 4\tau'\tau''' + \tau\tau'''' = 0$$

Hirota's bilinear relation : more complicated ?

A solution: $\tau = 1 + e^{2(kx - \omega t)}, \quad \omega = 4k^3$

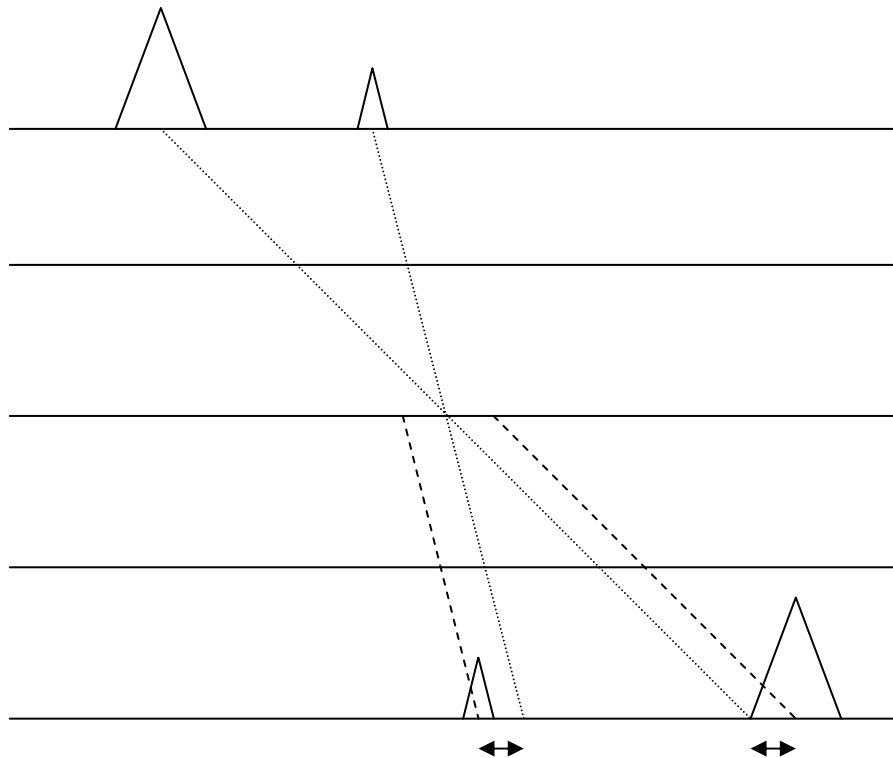
$\rightarrow u = 2k^2 \cosh^{-2}(kx - 4k^3 t) : \text{The solitary wave !}$
(1-soliton solution)

- 2-soliton solution

$$\tau = 1 + A_1 e^{2\theta_1} + A_2 e^{2\theta_2} + BA_1 A_2 e^{2(\theta_1 + \theta_2)},$$

$$\theta_i = k_i x - 4k_i^3 t, \quad B = \frac{k_1 - k_2}{k_1 + k_2}$$

Scattering process



The shape and velocity
is preserved ! (stable)

The positions are shifted ! (Phase shift)

There are many other soliton eqs.
with (similar) interesting properties

- KP equation (2-dim. KdV equation)
: describe 2-dim shallow water waves

$$u_t + u_{xxx} + 6u_x u + \partial_x^{-1} u_{yy} = 0 \quad \textbf{KP} \quad \xrightarrow{\partial_y = 0} \quad \textbf{KdV}$$
$$u_x := \frac{\partial u}{\partial x} \quad \partial_x^{-1} := \int^x dx' \quad \textbf{etc.}$$

Sato's theorem: [M.Sato&Y.Sato, 1981]

The solution space of KP eq. is an infinite-dim.

Grassmann mfd. (determined by tau –fcns.)

Many other soliton eqs. are obtained from KP.

[See e.g. The book of Miwa-Jimbo-Date (Cambridge UP, 2000)]

3. NC Sato's Theory

- Sato's Theory : one of the most beautiful theory of solitons
 - Based on the existence of hierarchies and tau-functions
- Sato's theory reveals essential aspects of solitons:
 - Construction of exact solutions
 - Structure of solution spaces
 - Infinite conserved quantities
 - Hidden infinite-dim. symmetry

Let's discuss NC extension of Sato's theory

Derivation of soliton equations

- Prepare a Lax operator which is a **pseudo-differential operator**

$$L := \partial_x + u_2 \partial_x^{-1} + u_3 \partial_x^{-2} + u_4 \partial_x^{-3} + \dots$$

$$u_k = u_k(\underbrace{x^1, x^2, x^3, \dots}_{\text{Noncommutativity is introduced here:}})$$

- Introduce a differential operator

$$B_m := (L * \dots * L)_{\geq 0}$$

m times

Noncommutativity
is introduced here:

$$[x^i, x^j] = i\theta^{ij}$$

- Define NC (KP) hierarchy:

$$\frac{\partial L}{\partial x^m} = [B_m, L]_*$$

Here all products are
star product:

$$\begin{aligned} &\partial_m u_2 \partial_x^{-1} + \\ &\partial_m u_3 \partial_x^{-2} + \\ &\partial_m u_4 \partial_x^{-3} + \dots \end{aligned}$$

$$\begin{aligned} &f_{m2}(u) \partial_x^{-1} + \\ &f_{m3}(u) \partial_x^{-2} + \\ &f_{m4}(u) \partial_x^{-3} + \dots \end{aligned}$$



**Each coefficient yields
a differential equation.**

Negative powers of differential operators

$$\partial_x^n \circ f := \sum_{j=0}^{\infty} \underbrace{\binom{n}{j}}_{\text{binomial coefficient}}$$

$$\frac{n(n-1)(n-2)\cdots(n-(j-1))}{j(j-1)(j-2)\cdots 1} (\partial_x^j f) \partial_x^{n-j}$$

: binomial coefficient
which can be extended
to negative n
→ negative power of
differential operator
(well-defined !)

$$\partial_x^3 \circ f = f \partial_x^3 + 3f \partial_x^2 + 3f \partial_x + f'''$$

$$\partial_x^2 \circ f = f \partial_x^2 + 2f \partial_x + f''$$

$$\partial_x^{-1} \circ f = f \partial_x^{-1} - f \partial_x^{-2} + f \partial_x^{-3} - \dots$$

$$\partial_x^{-2} \circ f = f \partial_x^{-2} - 2f \partial_x^{-3} + 3f \partial_x^{-4} - \dots$$

Star product:
$$f(x) * g(x) := f(x) \exp\left(\frac{i}{2} \theta^{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j\right) g(x)$$

which makes theories ``noncommutative``:

$$[x^i, x^j]_* := x^i * x^j - x^j * x^i = i \theta^{ij}$$

Notes

- Noncommutativity violates Lorentz symmetry.
 - U(1) part of gauge groups play important roles:
 - It behaves as non-abelian
 - SU(N) gauge theories are basically forbidden
- $$g_1, g_2 \in SU(N) \Rightarrow g_1 * g_2 \notin SU(N)$$
- Star-products contain infinite derivatives
 - Non-local ($\rightarrow$ Quantum treatment is hard)
 - Impossible to solve??? (solvable !!!)

Today's discussions are all classical.

Closer look at NC KP hierarchy

For $m=2$

$$\partial_x^{-1}) \quad \partial_2 u_2 = \underline{2u_3'} + u_2''$$

$$\partial_x^{-2}) \quad \partial_2 u_3 = \underline{2u_4'} + u_3'' + 2u_2 * u_2' + 2[u_2, u_3]_*$$

$$\partial_x^{-3}) \quad \partial_2 u_4 = \underline{2u_5'} + u_4'' + 4u_3 * u_2' - 2u_2 * u_2'' + 2[u_2, u_4]_*$$

$\vdots$

Infinite kind of fields are represented
in terms of one kind of field $u_2 \equiv u$

MH&K.Toda, [hep-th/0309265]

$$u_x := \frac{\partial u}{\partial x}$$

$$\partial_x^{-1} := \int^x dx'$$

For $m=3$

$$\partial_x^{-1}) \quad \partial_3 u_2 = u_2''' + 3u_3'' + 3u_4'' + 3u_2' * u_2 + 3u_2 * u_2'$$

$\vdots$



$$u_t = \frac{1}{4} u_{xxx} + \frac{3}{4} (u_x * u + u * u_x) + \frac{3}{4} \partial_x^{-1} u_{yy} + \frac{3}{4} [u, \partial_x^{-1} u_{yy}]_*$$

(2+1)-dim.
NC KP equation

and other NC equations
(NC KP hierarchy equations)

$$u = u(x^1, x^2, x^3, \dots)$$

$\updownarrow \quad \updownarrow \quad \updownarrow$
 $x \quad y \quad t$

etc.

(KP hierarchy) ^{reductions} $\rightarrow$ (various hierarchies.)

- (Ex.) KdV hierarchy

Reduction condition

$$L^2 = B_2 (=:\partial_x^2 + u) \quad : \text{2-reduction}$$

gives rise to NC KdV hierarchy

which includes (1+1)-dim. NC KdV eq.:

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{4}(u_x * u + u * u_x)$$

Note $\frac{\partial u}{\partial x_{2N}} = 0$: dimensional reduction in x_{2N} directions

KP : $u(x^1, x^2, x^3, x^4, x^5, \dots)$: (2+1)-dim.
 $\downarrow$
KdV : $u(x^1, x^3, x^5, \dots)$: (1+1)-dim.
 $\downarrow$

/-reduction of NC KP hierarchy yields wide class of other NC hierarchies

- No-reduction $\rightarrow$ NC KP $(x, y, t) = (x^1, x^2, x^3)$
 - 2-reduction $\rightarrow$ NC KdV $(x, t) = (x^1, x^3)$
 - 3-reduction $\rightarrow$ NC Boussinesq $(x, t) = (x^1, x^2)$
 - 4-reduction $\rightarrow$ NC Coupled KdV ...
 - 5-reduction $\rightarrow$...
 - 3-reduction of BKP $\rightarrow$ NC Sawada-Kotera
 - 2-reduction of mKP $\rightarrow$ NC mKdV
 - Special 1-reduction of mKP $\rightarrow$ NC Burgers
 - ...
- Noncommutativity should be introduced into space-time coords**
- 

4. Conservation Laws

- Conservation laws: $\overset{\text{time}}{\partial_t} \sigma = \overset{\text{space}}{\partial_i} J^i$ σ : Conserved density

Then $Q := \int_{\text{space}} dx \sigma$ is a conserved quantity.

$$\because \partial_t Q = \int_{\text{space}} dx \partial_t \sigma = \int_{\text{spatial infinity}} dS_i J^i = 0$$

Conservation laws for the hierarchies

$$\overset{\text{time}}{\partial_m} \text{res}_{-1} L^n = \overset{\text{space}}{\partial_x} J + \theta^{ij} \partial_j \Xi_i$$

I have succeeded in the evaluation explicitly !

$\text{res}_{-r} L^n$: coefficient
of ∂_x^{-r} in L^n

Noncommutativity should be introduced
in space-time directions only. $\rightarrow$

$$t \equiv x^m$$

∂_j should be space or time derivative
 $\rightarrow$ ordinary conservation laws !

Infinite conserved densities for the NC soliton eqs. ($n=1,2,\dots,\infty$)

$$\sigma_n = \text{res}_{-1} L^n + \theta^{im} \sum_{k=0}^{m-1} \sum_{l=0}^k \binom{k}{l} (\partial_x^{k-l} \text{res}_{-(l+1)} L^n) \diamond (\partial_i \text{res}_k L^m)$$

$t \equiv x^m$ $\text{res}_r L^n$: coefficient of ∂_x^r in L^n

$\diamond$: Strachan's product (commutative and non-associative)

$$f(x) \diamond g(x) := f(x) \left(\sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)!} \left(\frac{1}{2} \theta^{ij} \bar{\partial}_i \bar{\partial}_j \right)^{2s} \right) g(x)$$

MH, JMP46 (2005)
[hep-th/0311206]

This suggests infinite-dimensional symmetries would be hidden.

We can calculate the explicit forms of conserved densities for the wide class of NC soliton equations.

- Space-Space noncommutativity:

NC deformation is slight: $\sigma_n = \text{res}_{-1} L^n$

involutive (integrable in Liouville's sense)

- Space-time noncommutativity

NC deformation is drastical: $([t, x] = i\theta)$

– Example: NC KP and KdV equations

$$\sigma_n = \text{res}_{-1} L^n - 3\theta((\text{res}_{-1} L^n) \diamond u'_3 + (\text{res}_{-2} L^n) \diamond u'_2)$$

meaningful ?

5. Exact Solutions and Ward's conjecture

- We have found exact N-soliton solutions for the wide class of NC hierarchies.
- 1-soliton solutions are all the same as commutative ones because of

$$f(x-vt) * g(x-vt) = f(x-vt)g(x-vt)$$

- Multi-soliton solutions behave in almost the same way as commutative ones except for phase shifts.
- Noncommutativity affects the phase shifts

Exact N-soliton solutions of the NC KP hierarchy

$L = \Phi \partial_x \Phi^{-1}$ **solves the NC KP hierarchy !**

$\Phi f := \left| W(y_1, \dots, y_N, f) \right|_{N+1, N+1}$ **quasi-determinant
of Wronski matrix**

$y_i = \exp \xi(x, \alpha_i) + a_i \exp \xi(x, \beta_i)$ **Etingof-Gelfand-Retakh,**
 $\xi(x, \alpha) = x_1 \alpha + x_2 \alpha^2 + x_3 \alpha^3 + \dots$ **[q-alg/9701008]**

The exact solutions are actually N-soliton solutions !
Noncommutativity might affect the phase shift by $\theta^{ij} \omega_i k_j$

$\therefore \exp(\omega_i t - k_i x) * \exp(\omega_j t - k_j x)$ **[MH, work in progress]**

$\cong \exp([t, x]_* \omega_{[i} k_{j]}) \exp((\omega_i + \omega_j)t - (k_i + k_j)x)$

Exactly solvable!

NC Burgers hierarchy

MH&K.Toda,JPA36('03)11981[hep-th/0301213]

- NC (1+1)-dim. Burgers equation:

$$\dot{u} = u'' + 2u * u' : \text{Non-linear \&}$$

Infinite order diff. eq. w.r.t. time ! (Integrable?)

NC Cole-Hopf transformation

$$u = \tau^{-1} * \tau' \quad (\xrightarrow{\theta \rightarrow 0} \partial_x \log \tau)$$

(NC) Diffusion equation:

$$\dot{\tau} = \tau'' : \text{Linear \& first order diff. eq. w.r.t. time}$$

(Integrable !)

A solution : $\tau = 1 + \sum_{l=1}^N e^{k_l^2 t} * e^{\pm k_l x} = 1 + \sum_{l=1}^N \underline{e^{\frac{i}{2} k_l^3 \theta}} e^{k_l^2 t \pm k_l x}$ **Deformed!**

NC Ward's conjecture (NC NLS eq.)

- Reduced ASDYM eq.: $x^\mu \rightarrow (t, x)$

Legare,
[hep-th/0012077]

$$(i) \quad B' = 0$$

$$(ii) \quad C' - \dot{A} + [A, C]_* = 0$$

$$(iii) \quad A' - \dot{B} + [C, B]_* = 0$$

**A, B, C: 2 times 2
matrices (gauge fields)**



**Further
Reduction:**

$$A = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}, B = \frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, C = i \begin{pmatrix} q * \bar{q} & \bar{q}' \\ q' & -\bar{q} * q \end{pmatrix}$$

$$(ii) \Rightarrow \begin{pmatrix} 0 & i\dot{q} - q'' - 2q * \bar{q} * q \\ i\dot{\bar{q}} + \bar{q}'' - 2\bar{q} * q * \bar{q} & 0 \end{pmatrix} = 0$$

NOT traceless

$$i\dot{q} = q'' + 2q * \bar{q} * q \quad : \text{NC NLS eq. !!!}$$

Note: $A, B, C \in u(2) \xrightarrow{\theta \rightarrow 0} su(2)$

[MH, to appear]
U(1) part is necessary!

NC Ward's conjecture (NC Burgers eq.)

- Reduced ASDYM eq.: $x^\mu \rightarrow (t, x)$ MH & K.Toda, JPA36 [hep-th/0301213 v2]

$$(i) \quad \dot{A} + \underline{[B, A]_*} = 0$$

$$(ii) \quad \dot{C} - B' + \underline{[B, C]_*} = 0, \quad A, B, C \in \underline{u(1)}$$

$$\cong \underline{u(\infty)}$$

should remain

Further

Reduction: $A = 0, B = u' - u^2, C = u$

$$(ii) \Rightarrow \dot{u} = u'' + 2u * u'$$

: NC Burgers eq. !!!

Note: Without the commutators $[,]$, (ii) yields:

$$\dot{u} = u'' + \underline{u' * u + u * u'} : \text{neither linearizable nor Lax form}$$

Symmetric

NC Ward's conjecture (NC KdV eq.)

- Reduced ASDYM eq.: $x^\mu \rightarrow (t, x)$ MH, (to appear)
[hep-th/05070nn]

(i) $B' = 0$

(ii) $C' + \dot{A} + [A, C]_* = 0$

(iii) $A' - \dot{B} + [C, B]_* = 0$

**A, B, C: 2 times 2
matrices (gauge fields)**

**Further
Reduction:**

$$A = \begin{pmatrix} q & -1 \\ q' + q^2 & -q \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} \frac{1}{2}q'' + q' * q & -q' \\ \underline{f(q, q', q'', q''')} & -\frac{1}{2}q'' - q * q' \end{pmatrix}$$

**NOT
Traceless !**

(ii) $\Rightarrow \begin{pmatrix} 0 & 0 \\ \otimes & 0 \end{pmatrix} = 0 \Rightarrow \dot{q} = \frac{1}{4}q''' + \frac{3}{4}q' * q' : \text{NC pKdV eq. !!!}$
 $u = q' \rightarrow \text{NC KdV}$

Note: $A, B, C \in u(2) \xrightarrow{\theta \rightarrow 0} su(2)$ **U(1) part is necessary !**

6. Conclusion and Discussion

- Confirmation of NC Ward's conjecture  Going well
 - NC twistor theory → geometrical origin
 - D-brane interpretations → applications to physics
 -  Work in progress → [NC book of Mason&Woodhouse ?]
- Completion of NC Sato's theory
 - Existence of ``hierarchies'' →  Solved!
 - Existence of infinite conserved quantities  Successful
 - infinite-dim. hidden symmetry
 - Construction of multi-soliton solutions  Successful
 - Theory of tau-functions → description of the symmetry and the soliton solutions  Work in progress