A Backlund Transformation for Noncommutative Anti-Self-Dual Yang-Mills (ASDYM) Equations

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Note: In **my** poster, the word ``noncommutative (=NC)'' means noncommutative spaces but most of results can be extended to more general situation.

This work would be related to twistor rathar than twistor string…

$$[x^{\mu}, x^{\nu}] = i\theta^{\mu\nu}$$

1. Introduction

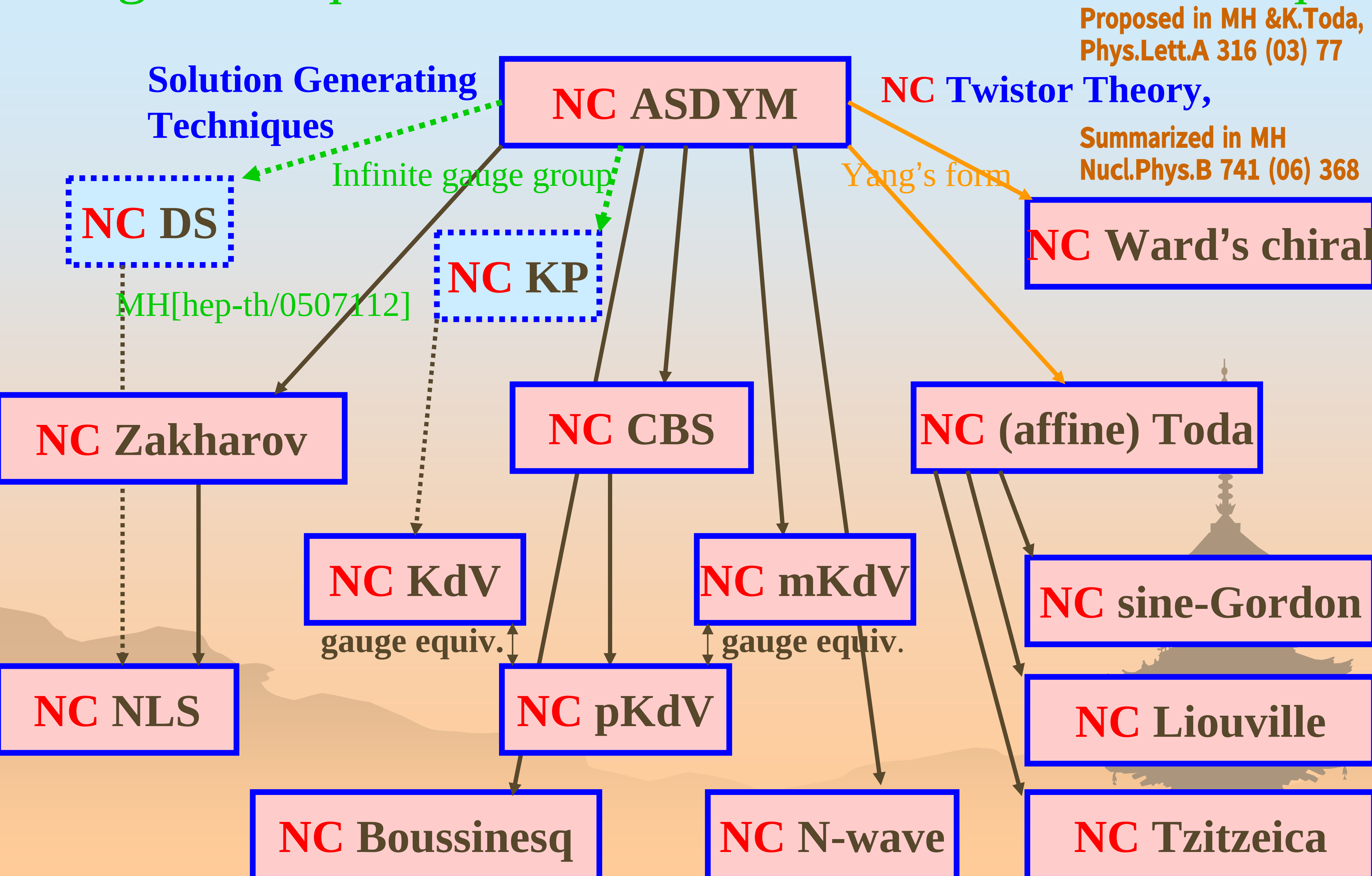
Successful points in NC theories

- Appearance of new physical objects
- Description of real physics (in gauge theory)
- Various successful applications
  to D-brane dynamics etc.

Construction of exact solitons are important.
(partially due to their integrablity)

Final goal: NC extension of all soliton theories
(Soliton eqs. can be embedded in gauge theories
via Ward's conjecture ! [R. Ward, 1985])

NC Ward's conjecture: Many (perhaps all?) NC integrable eqs are reductions of the NC ASDYM eqs.



Program of NC extension of soliton theories

- **(i) Confirmation of NC Ward's conjecture**
  - **NC twistor theory → geometrical origin**
  - **D-brane interpretations → applications to physics**
- **(ii) Completion of NC Sato's theory**
  - **Existence of ``hierarchies'' → various soliton eqs.**
  - **Existence of infinite conserved quantities**
    → infinite-dim. hidden symmetry
  - **Construction of multi-soliton solutions**
  - **Theory of tau-functions → structure of the solution spaces and the symmetry**

(i),(ii) → complete understanding of the NC soliton theories

2. Backlund transform for NC ASDYM eqs.

- In this section, we derive (NC) ASDYM eq. from the viewpoint of linear systems, which is suitable for discussion on integrable aspects.
- We define NC Yang's equations which is equivalent to NC ASDYM eq. and give a Backlund transformation for it.
- The generated solutions would contain not only finite-action solutions (NC instantons) but also infinite-action solutions (non-linear plane waves and so on.)
- This Backlund transformation could be applicable for lower-dimensional integrable eqs. via Ward's conjecture.

(Q) How we get NC version of the theories?

(A) We have only to replace all products of fields in ordinary commutative gauge theories

with star-products: $f(x)g(x) \to f(x) * g(x)$

- **The star product: (NC and associative)**

$$f(x) * g(x) := f(x) \exp\left(\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu\right) g(x) = f(x)g(x) + i\frac{\theta^{\mu\nu}}{2}\partial_\mu f(x) \partial_\nu g(x) + O(\theta^2)$$

A deformed product

Presence of background magnetic fields

Note: coordinates and fields themselves are usual c-number functions. But commutator of coordinates becomes…

$$[x^\mu, x^\nu]_* := x^\mu * x^\nu - x^\nu * x^\mu = i\theta^{\mu\nu}$$

NC !

Here we discuss G=GL(N) NC ASDYM eq. from the viewpoint of linear systems with a spectral parameter ζ .

(All products are star-products.)

- **Linear systems (NC case):**

$$L*\psi=(D_w-\zeta D_{\tilde{z}})*\psi=0,$$
$$M*\psi=(D_z-\zeta D_{\tilde{w}})*\psi=0.$$

e.g. $\begin{pmatrix}\tilde{z} & w\\ \tilde{w} & z\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}x^0+ix^1 & x^2-ix^3\\ x^2+ix^3 & x^0-ix^1\end{pmatrix}$

- **Compatibility condition of the linear system:**

$$[L,M]_*=[D_w,D_z]_*+\zeta([D_z,D_{\tilde{z}}]_*-[D_w,D_{\tilde{w}}]_*)+\zeta^2[D_{\tilde{z}},D_{\tilde{w}}]_*=0$$

$$\Leftrightarrow\begin{cases}F_{zw}=[D_z,D_w]_*=0,\\ F_{\tilde{z}\tilde{w}}=[D_{\tilde{z}},D_{\tilde{w}}]_*=0,\\ F_{z\tilde{z}}-F_{w\tilde{w}}=[D_z,D_{\tilde{z}}]_*-[D_w,D_{\tilde{w}}]_*=0\end{cases}$$

:NC ASDYM equation

$$(F_{\mu\nu}:=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu]_*)$$

$$\theta^{\mu\nu}=\begin{bmatrix}0 & \theta^1 & & O\\ -\theta^1 & 0 & & \\ & & 0 & \theta^2\\ O & & -\theta^2 & 0\end{bmatrix}$$

Yang's form and NC Yang's equation

- **NC ASDYM eq. can be rewritten as follows**

$$\begin{cases} F_{zw} = [D_z, D_w]_* = 0, \quad \Rightarrow \quad \exists h,\ D_z * h = 0,\ \ D_w * h = 0 \\ F_{\tilde{z}\tilde{w}} = [D_{\tilde{z}}, D_{\tilde{w}}]_* = 0, \quad \Rightarrow \quad \exists \tilde{h},\ D_{\tilde{z}} * \tilde{h} = 0,\ \ D_{\tilde{w}} * \tilde{h} = 0 \\ F_{z\tilde{z}} - F_{w\tilde{w}} = [D_z, D_{\tilde{z}}]_* - [D_w, D_{\tilde{w}}]_* = 0 \end{cases}$$

If we define Yang's matrix: $J := \tilde{h}^{-1} * h$

then we obtain from the third eq.:

$$\partial_z (J^{-1} * \partial_{\tilde{z}} J) - \partial_w (J^{-1} * \partial_{\tilde{w}} J) = 0$$ **:NC Yang's eq.**

The solution J reproduces the gauge fields as

$$A_z = -h_z * h^{-1},\ \ A_w = h_w * h^{-1},\ A_{\tilde{z}} = -\tilde{h}_{\tilde{z}} * \tilde{h}^{-1},\ \ A_{\tilde{w}} = \tilde{h}_{\tilde{w}} * \tilde{h}^{-1}$$

Backlund transformation for NC Yang's eq.

- **Yang's J matrix can be decomposed as follows**

$$J = \begin{pmatrix} f - g * b^{-1} * e & -g * b^{-1} \\ b^{-1} * e & b^{-1} \end{pmatrix}$$

- **Then NC Yang's eq. becomes**

$$\partial_z(f^{-1} * g_{\tilde{z}} * b^{-1}) - \partial_w(f^{-1} * g_{\tilde{w}} * b^{-1}) = 0, \quad \partial_{\tilde{z}}(b^{-1} * e_z * f^{-1}) - \partial_{\tilde{w}}(b^{-1} * e_w * f^{-1}) = 0,$$

$$\partial_z(b_{\tilde{z}} * b^{-1}) - \partial_w(b_{\tilde{w}} * b^{-1}) + e_z * f^{-1} * g_{\tilde{z}} * b^{-1} - e_w * f^{-1} * g_{\tilde{w}} * b^{-1} = 0,$$

$$\partial_z(f^{-1} * f_{\tilde{z}}) - * \partial_w(f^{-1} * f_{\tilde{w}}) + f^{-1} * g_{\tilde{z}} * b^{-1} * e_z - f^{-1} * g_{\tilde{w}} * b^{-1} * e_w = 0.$$

- **The following trf. leaves NC Yang's eq. as it is:**

$$\beta : \begin{cases} \partial_z e^{new} = f^{-1} * g_{\tilde{w}} * b^{-1}, \ \partial_w e^{new} = f^{-1} * g_{\tilde{z}} * b^{-1}, \\ \partial_{\tilde{z}} g^{new} = b^{-1} * e_w * f^{-1}, \ \partial_{\tilde{w}} g^{new} = b^{-1} * e_z * f^{-1}, \\ f^{new} = b^{-1}, \ b^{new} = f^{-1} \end{cases}$$

We could generate various (non-trivial) solutions of NC Yang's eq. from a (trivial) seed solution by using the previous Backlund trf. together with a simple trf.

$$\gamma_0 : J^{new} = C^{-1} J C, \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Leftrightarrow \gamma_0 : \begin{pmatrix} f^{new} & g^{new} \\ e^{new} & b^{new} \end{pmatrix} = \begin{pmatrix} b & e \\ g & f \end{pmatrix}^{-1}$$

This combined trf. would generate a group of hidden symmetry of NC Yang's eq., which would be also applied to lower-dimension.

For G=GL(2), we can present the transforms more explicitly and give an explicit form of a class of solutions (NC Atiyah-Ward ansatz).

Backlund trf. for NC ASDYM eq.

- **Let's consider the combined Backlund trf.**

$$J_{[1]} \xrightarrow{\alpha=\gamma_0\circ\beta} J_{[2]} \xrightarrow{\alpha} J_{[3]} \xrightarrow{\alpha} \cdots$$

$$J_{[n]} = \begin{pmatrix} f_{[n]} - g_{[n]} * b_{[n]}^{-1} * e_{[n]} & -g_{[n]} * b_{[n]}^{-1} \\ b_{[n]}^{-1} * e_{[n]} & b_{[n]}^{-1} \end{pmatrix}$$

- **If we take a seed sol.,** $f_{[1]} = b_{[1]} = e_{[1]} = g_{[1]} = \Delta_0^{-1},\ \partial^2 \Delta_0 = 0$

the generated solutions are :

$$f_{[n]} = \left|D_{[n]}\right|_{11}^{-1}, b_{[n]} = \left|D_{[n]}\right|_{nn}^{-1}, e_{[n]} = \left|D_{[n]}\right|_{1n}^{-1}, g_{[n]} = \left|D_{[n]}\right|_{n1}^{-1}$$

$$\frac{\partial \Delta_r}{\partial z} = -\frac{\partial \Delta_{r+1}}{\partial \tilde{w}}, \quad \frac{\partial \Delta_r}{\partial w} = -\frac{\partial \Delta_{r+1}}{\partial \tilde{z}}$$

$$D_{[n]} = \begin{pmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-(n-1)} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{-(n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{n-1} & \Delta_{n-2} & \cdots & \Delta_0 \end{pmatrix}$$

NC Atiyah-Ward ansatz sols.

Quasideterminants !
(a kind of NC determinants)

[Gelfand-Retakh]

Quasi-determinants

- **Quasi-determinants are not just a NC generalization of commutative determinants, but rather related to inverse matrices.**
- **For an n by n matrix** $X = (x_{ij})$ **and the inverse** $Y = (y_{ij})$ **of X, quasi-determinant of X is directly defined by**

$$|X|_{ij} = y_{ji}^{-1} \qquad \left(\xrightarrow{\theta\to 0} \frac{(-1)^{i+j}}{\underline{\det X^{ij}}} \det X \right)$$

some factor

X^{ij}**: the matrix obtained from X deleting i-th row and j-th column**

- **Recall that**

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \Rightarrow$$

$$Y = X^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix}$$

→ We can also define quasi-determinants recursively

Quasi-determinants

[For a review, see Gelfand et al., math.QA/0208146]

- **Defined inductively as follows**

$$\left|X\right|_{ij} = x_{ij} - \sum_{i',j'} x_{ii'}\left(\left(X^{ij}\right)^{-1}\right)_{i'j'} x_{j'j} = x_{ij} - \sum_{i',j'} x_{ii'}\left(\left|X^{ij}\right|_{j'i'}\right)^{-1} x_{j'j} = \begin{vmatrix} & \vdots & \\ \cdots & \boxed{x_{ij}} & \cdots \\ & \vdots & \end{vmatrix}$$

convenient notation

$$n=1: \quad \left|X\right|_{ij} = x_{ij}$$

$$n=2: \quad \left|X\right|_{11} = \begin{vmatrix} \boxed{x_{11}} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} = x_{11} - x_{12}\cdot x_{22}^{-1}\cdot x_{21}, \quad \left|X\right|_{12} = \begin{vmatrix} x_{11} & \boxed{x_{12}} \\ x_{21} & x_{22} \end{vmatrix} = x_{12} - x_{11}\cdot x_{21}^{-1}\cdot x_{22},$$

$$\left|X\right|_{21} = \begin{vmatrix} x_{11} & x_{12} \\ \boxed{x_{21}} & x_{22} \end{vmatrix} = x_{21} - x_{22}\cdot x_{12}^{-1}\cdot x_{11}, \quad \left|X\right|_{22} = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & \boxed{x_{22}} \end{vmatrix} = x_{22} - x_{21}\cdot x_{11}^{-1}\cdot x_{12},$$

$$n=3: \quad \left|X\right|_{11} = \begin{vmatrix} \boxed{x_{11}} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} = x_{11} - (x_{12}, x_{13}) \begin{pmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{pmatrix}^{-1} \begin{pmatrix} x_{21} \\ x_{31} \end{pmatrix}$$

$$= x_{11} - x_{12}\cdot(x_{22} - x_{23}\cdot x_{33}^{-1}\cdot x_{32})^{-1}\cdot x_{21} - x_{13}\cdot(x_{32} - x_{33}\cdot x_{23}^{-1}\cdot x_{22})^{-1}\cdot x_{21}$$

$$- x_{12}\cdot(x_{23} - x_{22}\cdot x_{32}^{-1}\cdot x_{33})^{-1}\cdot x_{31} - x_{13}\cdot(x_{33} - x_{32}\cdot x_{22}^{-1}\cdot x_{23})^{-1}\cdot x_{31}$$

$$\cdots$$

Explicit Atiyah-Ward ansatz solutions of NC Yang's eq. G=GL(2)

$$f_{[1]} = b_{[1]} = e_{[1]} = g_{[1]} = \Delta_0^{-1},\ \ \partial^2 \Delta_0 = 0$$

$$f_{[2]} = \begin{vmatrix} \boxed{\Delta_0} & \Delta_{-1} \\ \Delta_1 & \Delta_0 \end{vmatrix}^{-1},\ b_{[2]} = \begin{vmatrix} \Delta_0 & \Delta_{-1} \\ \Delta_1 & \boxed{\Delta_0} \end{vmatrix}^{-1},\ e_{[1]} = \begin{vmatrix} \Delta_0 & \boxed{\Delta_{-1}} \\ \Delta_1 & \Delta_0 \end{vmatrix}^{-1},\ g_{[1]} = \begin{vmatrix} \Delta_0 & \Delta_{-1} \\ \boxed{\Delta_1} & \Delta_0 \end{vmatrix}^{-1},$$

$$\partial_z \Delta_0 = -\partial_{\tilde{w}} \Delta_1,\ \partial_z \Delta_{-1} = -\partial_{\tilde{w}} \Delta_0,\ \partial_w \Delta_0 = -\partial_{\tilde{z}} \Delta_1,\ \partial_w \Delta_{-1} = -\partial_{\tilde{z}} \Delta_0$$

$$f_{[n]} = \begin{vmatrix} \boxed{\Delta_0} & \cdots & \Delta_{-(n-1)} \\ \vdots & \ddots & \vdots \\ \Delta_{n-1} & \cdots & \Delta_0 \end{vmatrix}^{-1},\ b_{[n]} = \begin{vmatrix} \Delta_0 & \cdots & \Delta_{-(n-1)} \\ \vdots & \ddots & \vdots \\ \Delta_{n-1} & \cdots & \boxed{\Delta_0} \end{vmatrix}^{-1},$$

$$e_{[n]} = \begin{vmatrix} \Delta_0 & \cdots & \boxed{\Delta_{-(n-1)}} \\ \vdots & \ddots & \vdots \\ \Delta_{n-1} & \cdots & \Delta_0 \end{vmatrix}^{-1},\ g_{[n]} = \begin{vmatrix} \Delta_0 & \cdots & \Delta_{-(n-1)} \\ \vdots & \ddots & \vdots \\ \boxed{\Delta_{n-1}} & \cdots & \Delta_0 \end{vmatrix}^{-1}$$

$$\frac{\partial \Delta_r}{\partial z} = -\frac{\partial \Delta_{r+1}}{\partial \tilde{w}},\ \ \frac{\partial \Delta_r}{\partial w} = -\frac{\partial \Delta_{r+1}}{\partial \tilde{z}}$$

Quasideterminants naturally appear in this framework. The proof is done direcly by NC Jacobi identity etc. and indirectly by twistor.

[Gilson-MH-Nimmo. forthcoming paper]

We could generate various solutions of NC ASDYM eq. from a simple seed solution Δ_0 by using the previous Backlund trf. $\alpha = \gamma_0 \circ \beta$

A seed solution:

[Lechtenfeld-Popov]

$$\Delta_0 = 1 + ``\frac{1}{z\tilde{z} - w\tilde{w}}"$$

→ NC instantons

$$\Delta_0 = \exp(\, linear\ \ of\ \ z, \tilde{z}, w, \tilde{w})$$

→ NC Non-Linear plane-waves

NC twistor could give an origin of NC CFYG trf.

$$\beta : F^{new} = B^{-1} F B, \gamma_0 : F^{new} = C^{-1} F C, B = \begin{pmatrix} 0 & 1 \\ \zeta^{-1} & 0 \end{pmatrix}, F = \begin{pmatrix} \zeta^{n} & \delta(x,\zeta) \\ 0 & \zeta^{-n} \end{pmatrix}$$

Patching matrix of holomorphic vec. bdl over NC twistor sp.

NC CFYG trf. would generate all solutions in some sense and could be applied to reveal the hidden symmetry of NC ASDYM eq. and the reductions.