Deconstructing Exact Results in Supersymmetric Gauge Theories

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References:

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- M.H., PRD94, 025039 (2016) (arXiv:1604.08653 [hep-th])
- M.H., arXiv:1710.05010 [hep-th]
- Russo, JHEP 1206 (2012) 038 (arXiv:1203.5061 [hep-th])
- Aniceto-Russo-Schiappa, JHEP 1503 (2015) 172 (arXiv:1410.5834 [hep-th])

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$\exists$ Many exact results in SUSY QFT

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Typically, for supersymmetric quantities,

(path integral) $\longrightarrow$ $\int d^{|G|}x \; f(x)$

(|G|: rank of gauge group G)

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Typically, for supersymmetric quantities,

$$\text{(path integral)} \longrightarrow \int d^{|G|}x \ f(x)$$

(|G|: rank of gauge group G)

In this talk, I will discuss

these exact results are useful for understanding properties of perturbative series in QFT

Perturbative series of exact results in QFT

This talk:

1. Reinterpret exact results in terms of Borel resum.
2. Study analytic property of Borel trans. in detail
3. Get some lessons for more nontrivial cases

Setup

[cf. some low rank cases: Russo, Aniceto-Russo-Schiappa, Gerchkovitz-Gomis-Ishtiaque-Karashik-Komargodski-Pufu]

- 4d N=2 (& 5d N=1) SUSY theories on spheres

  —— expansion by g_{YM} around instanton backgrounds

- 3d N=2 Chern-Simons theories on S^3 (& lens sp.)

  —— expansion by inverse CS levels

Summary of main results

(～10 minutes)

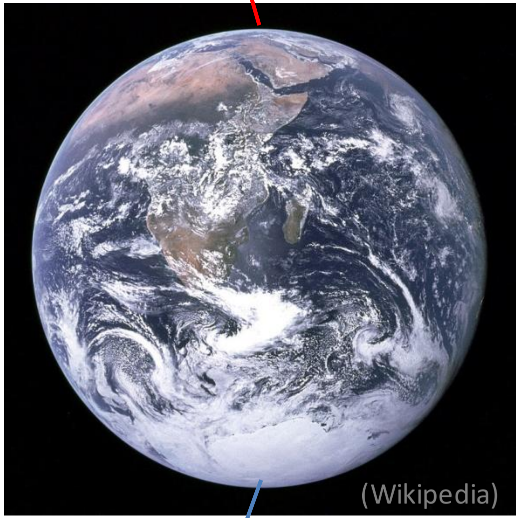
Results on 4d N=2 SUSY theories (w/ 8 SUSY)

[M.H. '16]

Set up:

- Theories w/ $\beta \leqq 0$ and Lagrangians $(Z_{S^4} < \infty)$
- Perturbative expansion by g_{YM} around fixed # of instanton/anti-inst.

inst.



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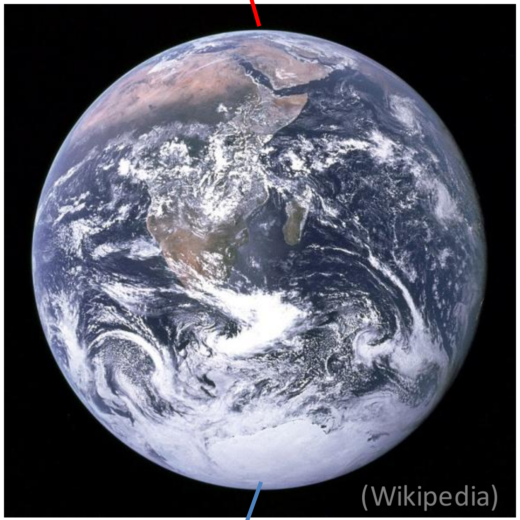

anti-inst.

Result:

(similar for 5d N=1 case)

[cf. some SU(2) theories: Russo, Aniceto-Russo-Schiappa]

- Find explicit finite dimensional integral rep. of Borel trans.
  for various observables
- $^{\exists}$ Singularities only along R- → Borel summable along R+
- (Exact) = $\sum_{\text{instantons}}$ (Borel resum)

Typical case: SU(2) w/ fundamentals

Borel trans. around trivial b.g. : $\mathcal{B}Z_{S^4}^{(0,0)}(t) \propto \sqrt{t} \prod_{n=1}^{\infty} \frac{\left(1+\frac{4t}{n^2}\right)^{2n}}{\left(1+\frac{t}{n^2}\right)^{2N_f n}}$

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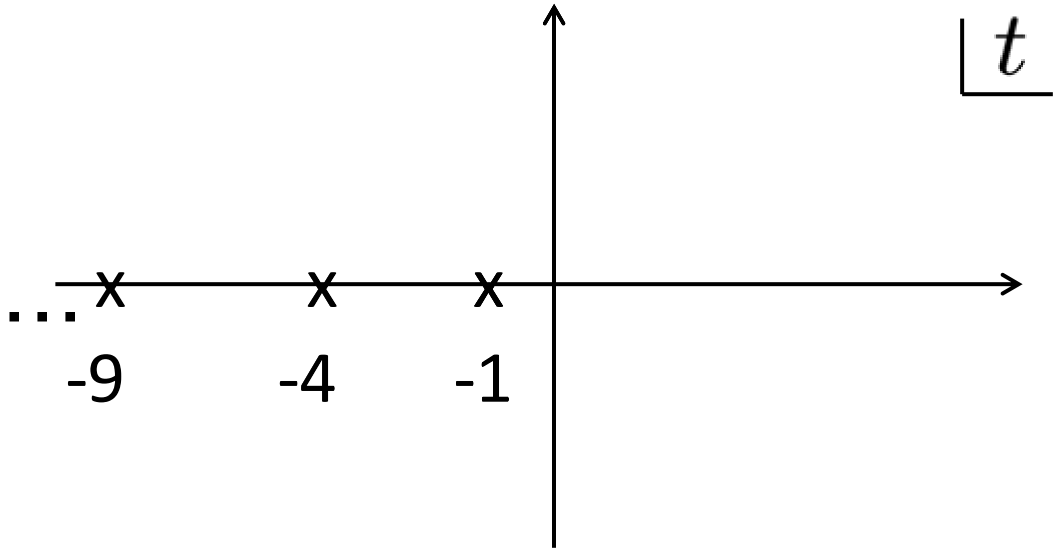

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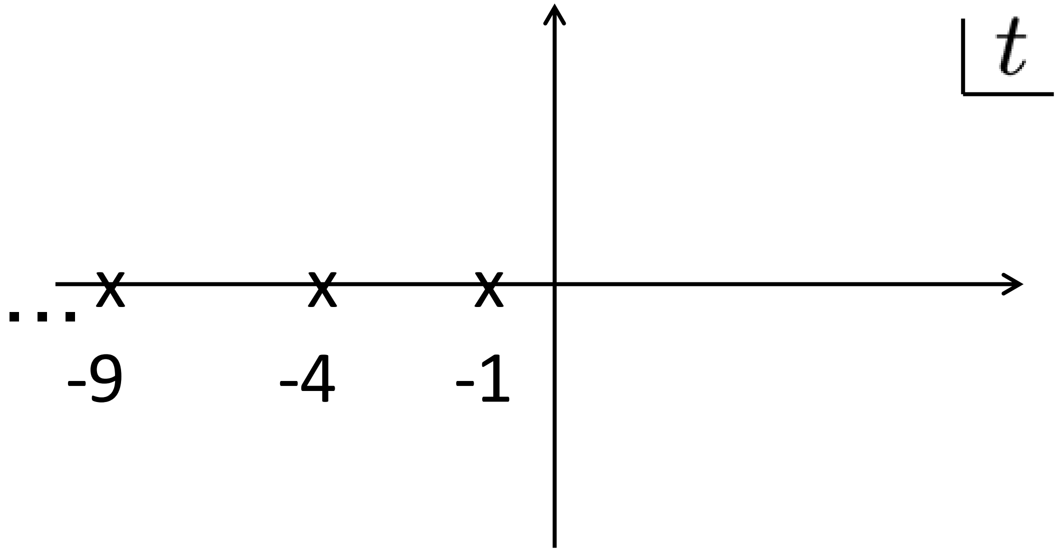
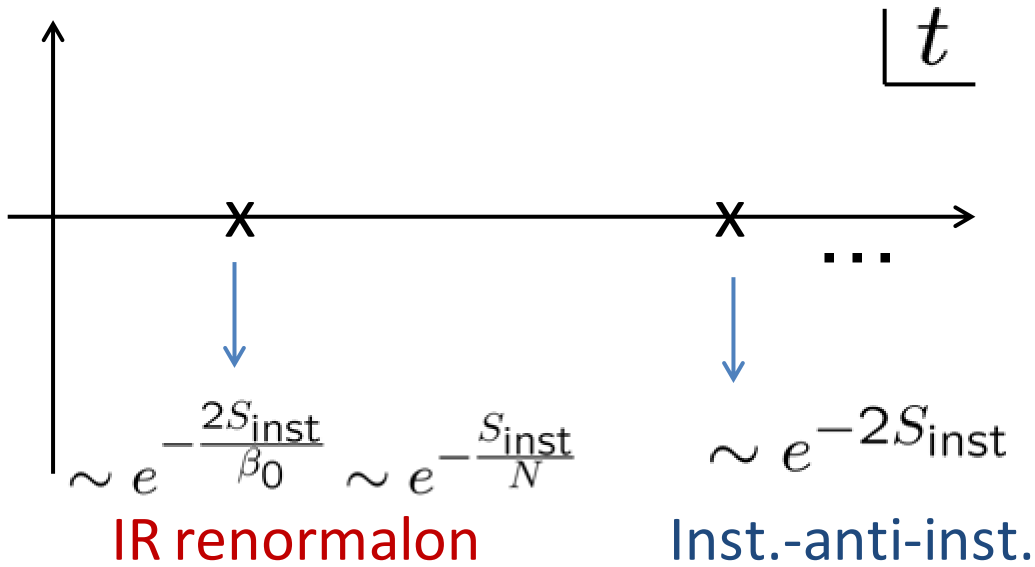


- $^\exists \infty$ singularities along R-
- All singularities are NOT instantons & IR/UV renormalons
- No qualitative difference between CFT and non-CFT

Nontrivial consistency w/ a conjecture on QCD

Borel plane in typical gauge theory (?) :



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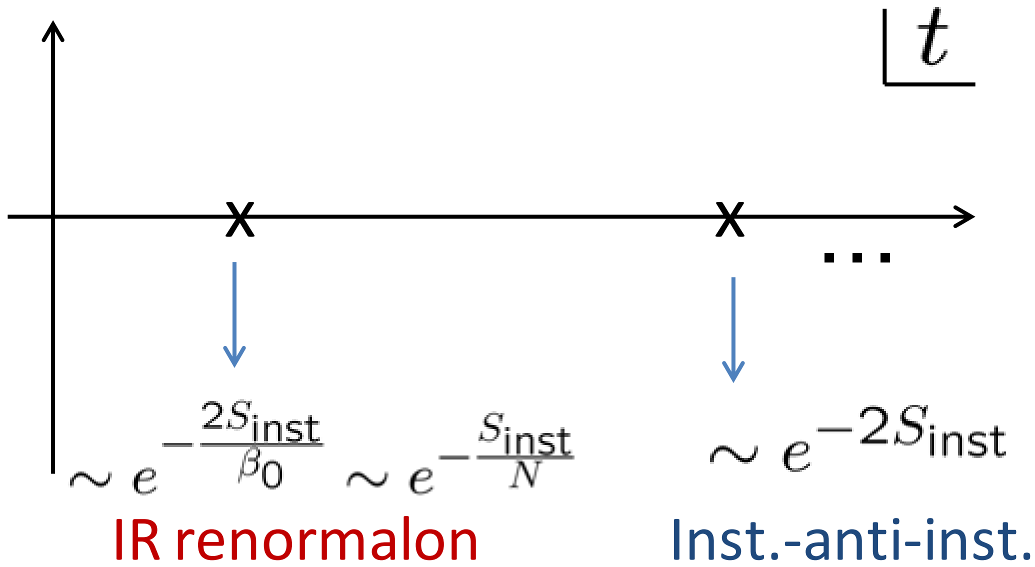

Conjecture: (IR renormalon) = (Combination of monopoles)

[Argyres-Unsal'12]

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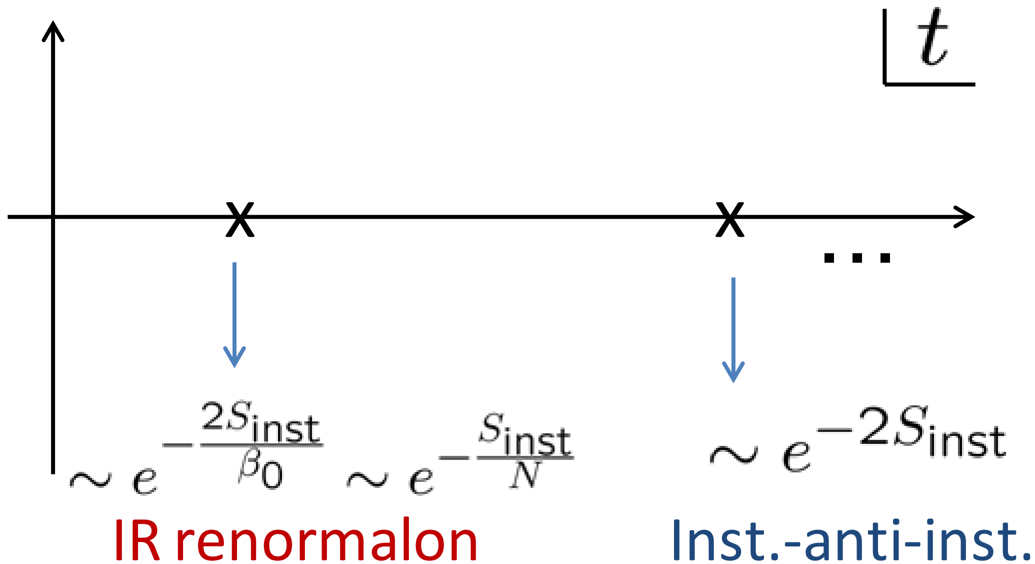

Conjecture: (IR renormalon) = (Combination of monopoles)

[Argyres-Unsal'12]

But we don't have such solution for $\mathcal{N}=2$ [Popitz-Unsal]

$\longrightarrow$ No IR renormalon singularities for $\mathcal{N}=2$?

Confusion?

Usually Borel singularities come from nontrivial saddles w/ the same topological numbers [cf. Lipatov '77]

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For example, around trivial saddle, we expect

$\exists$ Borel singularities from $k = \bar{k}$ (namely, at t=2k)

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But we do not have such singularities.

Differences from (inst.)-(anti-inst.) in QM:

- SUSY configuration (non-interacting)
- finitely separated

(I'm looking for more precise understanding)

Results on 3d N=2 SUSY Chern-Simons theories

(w/ 4 SUSY)

[M.H. '16]

Set up:

- General Chern-Simons (CS) theories coupled to matters $(Z_{S^3} < \infty)$
- Perturbative expansion by inverse CS levels

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  $(Z_{S^3} < \infty)$

- Perturbative expansion by inverse CS levels

Result:

$$S_\theta I(g) = \int_0^{e^{i\theta}\infty} dt\ e^{-\frac{t}{g}}\ \mathcal{B}I(t)$$

- Find finite dimensional integral rep. of Borel trans.

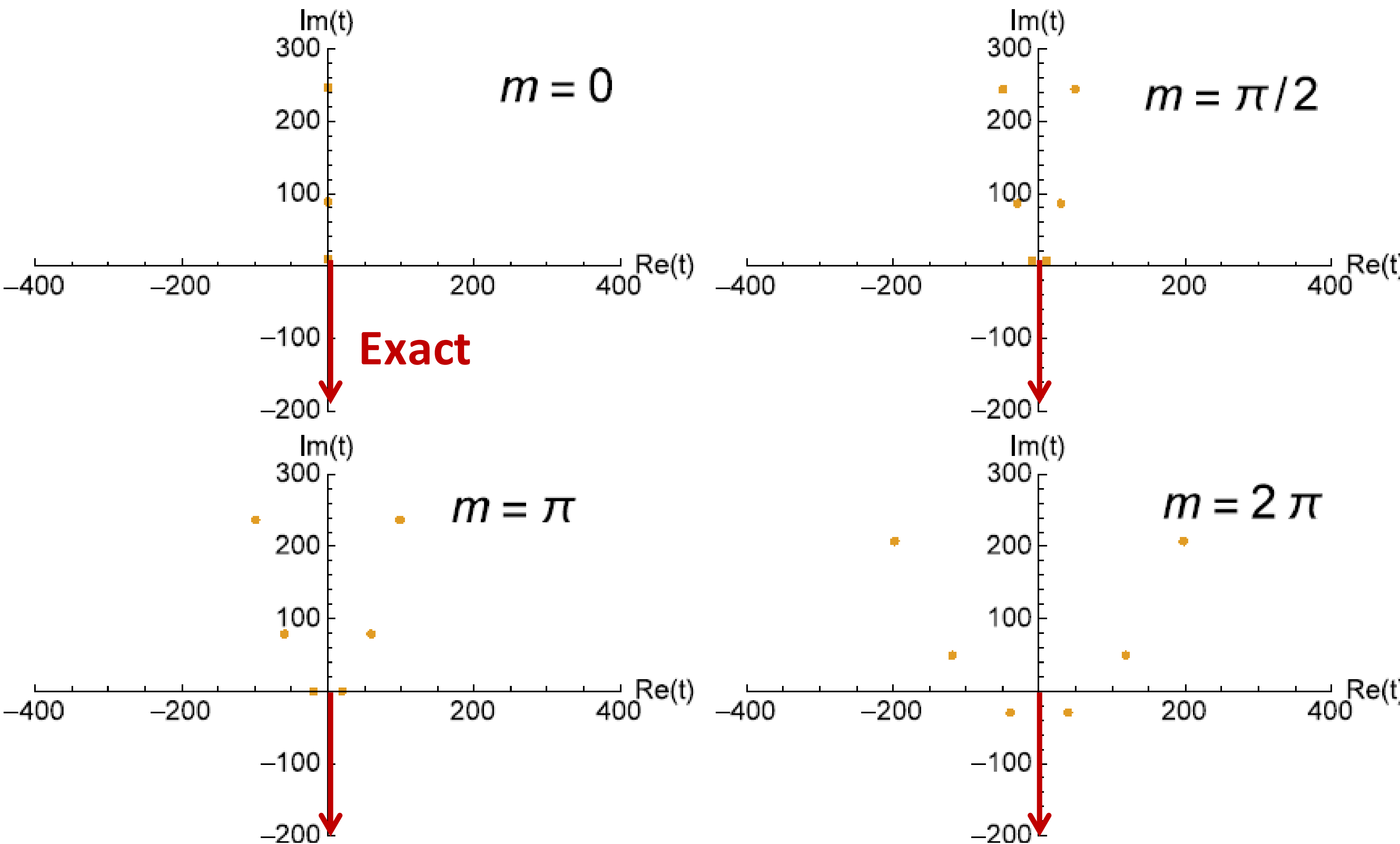
- Usually non-Borel summable along R+

- But always Borel summable along (half-)imaginary axis

- (Borel resum. w/ θ=±π/2) = (exact result)

Ex.) SU(2) SQCD w/ hypers and real mass

$$\mathcal{B}Z_{S^3}(t) \propto \frac{\sinh^2(\sigma)}{\sigma \cdot \left(\cosh\frac{\sigma-m}{2}\cosh\frac{\sigma+m}{2}\right)^{N_f}}\Bigg|_{\sigma=\sqrt{i\mathrm{sgn}(k)t}}$$



Interpretation of Borel singularities (3d)

[M.H. '17]

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All the singularities can be explained by

Complexified SUSY Solutions

which are not on original contour of path integral
but formally satisfy SUSY conditions: $Q(\text{fields}) = 0$

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but formally satisfy SUSY conditions: $Q(\text{fields}) = 0$

Proposal:

If there are n_B bosonic & n_F fermionic solutions with action $S=S_c/g$, then

$$(\text{Borel trans.}) \supset \prod_{\text{solutions}} \frac{1}{(t - S_c)^{n_B - n_F}}$$

Contents

Partition function of SU(N) theory on S^4 ($\beta \leqq 0$)

Exact result:

[Pestun '07]

$$Z_{S^4}(g,\theta) = \int_{-\infty}^{\infty} d^N a \; e^{-\frac{1}{g}\sum_{j=1}^{N} a_j^2} \tilde{Z}(a) Z_{\text{inst}}(g,\theta;a)$$

($\tilde{Z}(a)$: 1-loop determinant w/ traceless constraint)

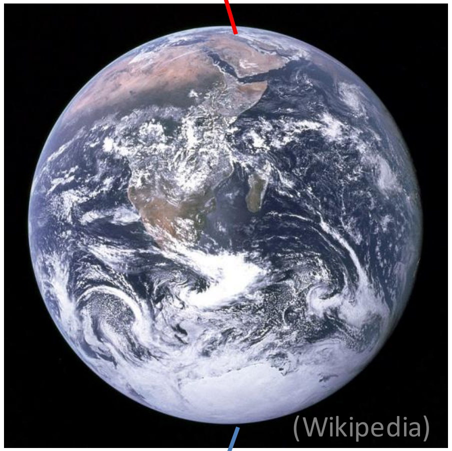
$g \propto$ 1 − loop effective g_{YM}^2 at scale $R_{S^4}^{-1}$

$$Z_{\text{inst}}(g,\theta;a) = \sum_{k,\bar{k}=0}^{\infty} e^{-\frac{k+\bar{k}}{g}+i(k-\bar{k})\theta} Z_{\text{inst}}^{(k,\bar{k})}(a)$$

$$Z_{S^4}^{(k,\bar{k})}(g) = \int_{-\infty}^{\infty} d^N a \; e^{-\frac{1}{g}\sum_{j=1}^{N} a_j^2} \tilde{Z}(a) Z_{\text{inst}}^{(k,\bar{k})}(a)$$

We are interested in small-g expansion of this

inst.



anti-inst.

Borel trans. hidden in localization formula

$$Z_{S^4}^{(k,\bar{k})}(g) = \int_{-\infty}^{\infty} d^N a \ e^{-\frac{1}{g}\sum_{j=1}^{N} a_j^2} \tilde{Z}(a) Z_{\text{inst}}^{(k,\bar{k})}(a)$$

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Taking polar coordinate $a_i = \sqrt{t}\widehat{x}_i$ w/ $(\widehat{x}^i)^2 = 1,$

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$$Z_{S^4}^{(k,\bar{k})}(g) = \int_0^{\infty} dt \; e^{-\frac{t}{g}} f^{(k,\bar{k})}(t)$$

similar to Borel resummation formula?

$$\left(f^{(k,\bar{k})}(t) = \int_{S^{N-1}} d^{N-1}\hat{x} \; h^{(k,\bar{k})}(t,\hat{x}), \; h^{(k,\bar{k})}(t,\hat{x}) = \tilde{Z}(a) Z_{\text{inst}}^{(k,\bar{k})}\Big|_{a^i=\sqrt{t}\hat{x}^i} \right)$$

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We can actually prove

$$f^{(k,\bar{k})}(t) = \mathcal{B} Z_{S^4}^{(k,\bar{k})}(t)$$

Analytic property of Borel trans.

$$\mathcal{B}Z_{S^4}^{(k,\bar{k})}(t) = \int_{S^{N-1}} d^{N-1}\hat{x}\ h^{(k,\bar{k})}(t,\hat{x})$$

For example, in SU(N) w/ fundamentals around trivial b.g.,

$$h^{(0,0)}(t,\hat{x}) = \delta\left(\sum_j \hat{x}_j\right) \prod_{i<j} (\hat{x}_i - \hat{x}_j)^2 \prod_{n=1}^{\infty} \frac{\prod_{i<j}\left(1+\frac{t(\hat{x}_i-\hat{x}_j)^2}{n^2}\right)^{2n}}{\prod_j \left(1+\frac{t(\hat{x}_j)^2}{n^2}\right)^{N_f n}}$$

Singularities only along R_- ⇒ Borel summable along R_+!!

true also for non-zero inst. b.g. & other theories

(Exact result)

||

$$\sum_{k,\bar{k}} \text{(Borel resummation along } R_+\text{)}$$

(up to resummation of instanton expansion)

More general cases

[M.H. ’16]

Other N=2 theories:

Similar results hold as long as β≦0 & Lagrangians

Other observables:

- SUSY Wilson loop on S^4
- Bremsstrahrung function in SCFT on R^4 [cf. Fiol-Gerchkovitz-Komargodski ’15]

$$(\text{Energy of quark}) = B \int dt\ \dot{a}^2$$

- Extremal correlator in SCFT on R^4 [cf. Gerchkovitz-Gomis-Ishtiaque -Karasik-Komargodski-Pufu ’16]

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \bar{\mathcal{O}} \rangle$$

- Partition function on squashed S^4～SUSY Renyi entrory

[cf. Hama-Hosomichi, Nosaka-Terashima Nishioka-Yaakov ’13, Crossley-Dyer-Sonner, Huang-Zhou]

3d N=2 SUSY CS matter theory

Partition function of U(N) CS theory on S^3

Exact result: $\left(g \propto 1/k, \quad \text{k>0: CS level} \right)$

[Kapustin-Willett-Yaakov, Jafferis, Hama-Hosomichi-Lee]

$$Z_{S^3}(g) = \int_{-\infty}^{\infty} d^N\sigma \; e^{\frac{i}{g}\sum_{j=1}^{N}\sigma_j^2}\,\tilde{Z}(\sigma)$$

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$$Z_{S^3}(g) = i\int_0^{-i\infty} dt\; e^{-\frac{t}{g}} f(it),\;\; if(\tau) = \mathcal{B}Z_{S^3}(-i\tau)$$

$$\left(f(\tau) = \int_{S^{N-1}} d^{N-1}\hat{x}\; h(\tau,\hat{x}),\;\; h(\tau,\hat{x}) = \tilde{Z}(\sigma)\Big|_{\sigma^i=\sqrt{\tau}\hat{x}^i} \right)$$

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Namely,

$$Z_{S^3}(g) = \int_0^{-i\infty} dt\; e^{-\frac{t}{g}}\mathcal{B}Z_{S^3}(t)$$

(exact result) = (Borel resum. along θ=-π/2)

Ex.) U(N) w/ (anti-)fundamentals & adjoints

$$Z_{\mathsf{SQCD}}(g) = \int_0^{-i\infty} dt\; e^{-\frac{t}{g}} \mathcal{B}Z_{\mathsf{SQCD}}(t), \qquad \mathcal{B}Z_{\mathsf{SQCD}}(t) = \int_{S^{N-1}} d^{N-1}\hat{x}\;\; \tilde{Z}\left(\sigma = \sqrt{it}\hat{x}\right)$$

$$\tilde{Z}(\sigma) = \prod_{j=1}^{N} \frac{s_1^{\bar{N}_f}\left(\sigma_j + i(1-\bar{\Delta}_f)\right)}{s_1^{N_f}\left(\sigma_j - i(1-\Delta_f)\right)} \frac{\prod_{i<j} 4\sinh^2\left(\pi(\sigma_i - \sigma_j)\right)}{\prod_{i,j} s_1^{N_a}\left(\sigma_i - \sigma_j - i(1-\Delta_a)\right)}, \qquad s_1(z) = \prod_{n=1}^{\infty} \left(\frac{n - iz}{n + iz}\right)^n$$

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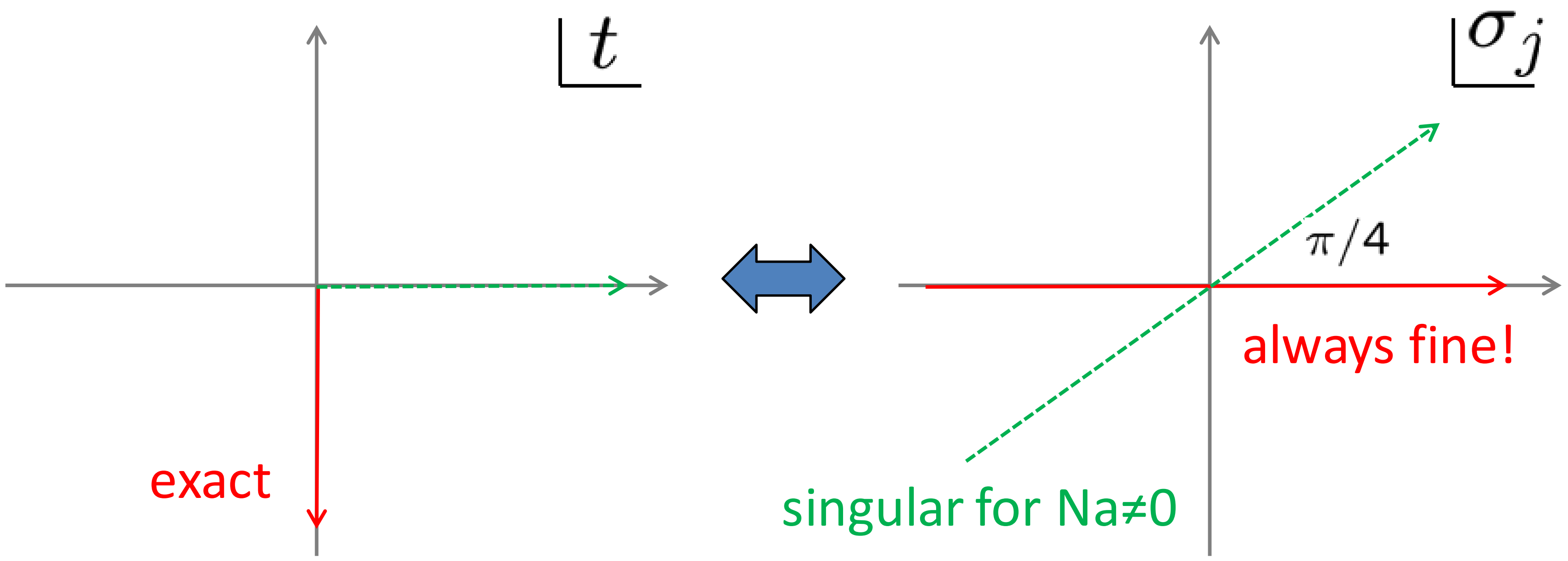

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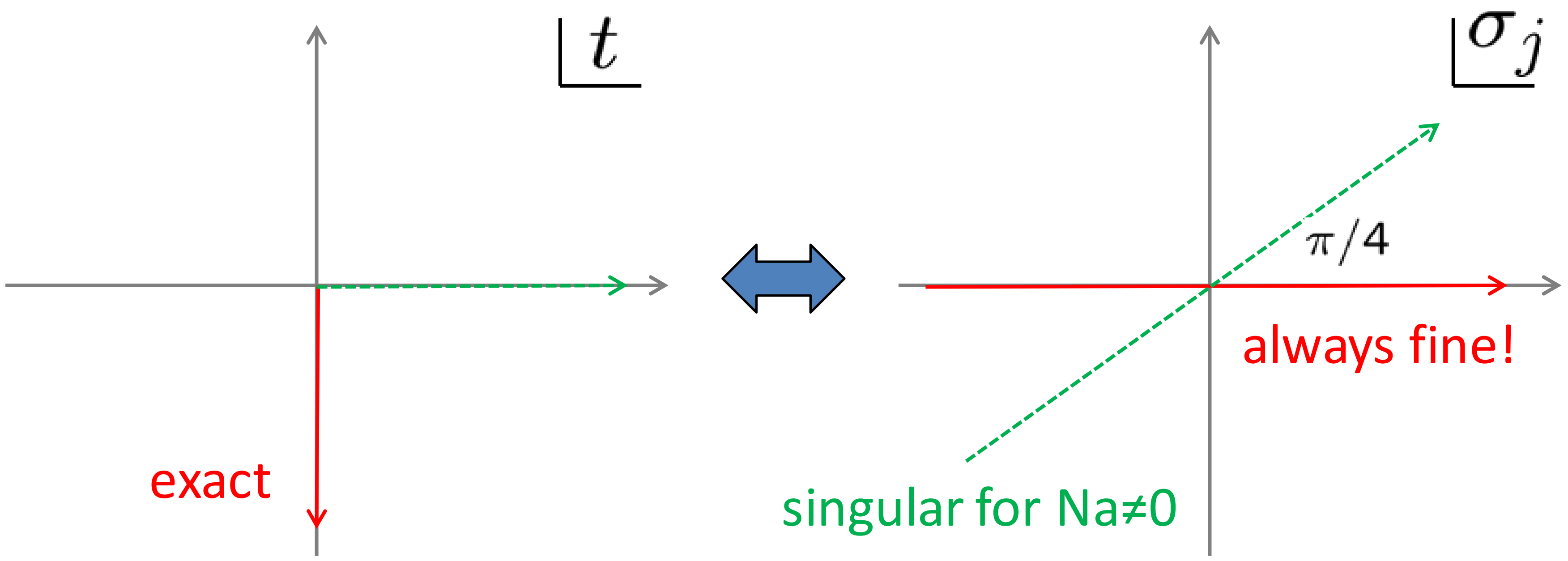


- When we have adjoint matters, would be non-Borel summable along R+
- But always Borel summable along θ=-π/2

More general cases

[M.H. '16]

Other theories:

Similar results hold as long as $Z_{S^3} < \infty$

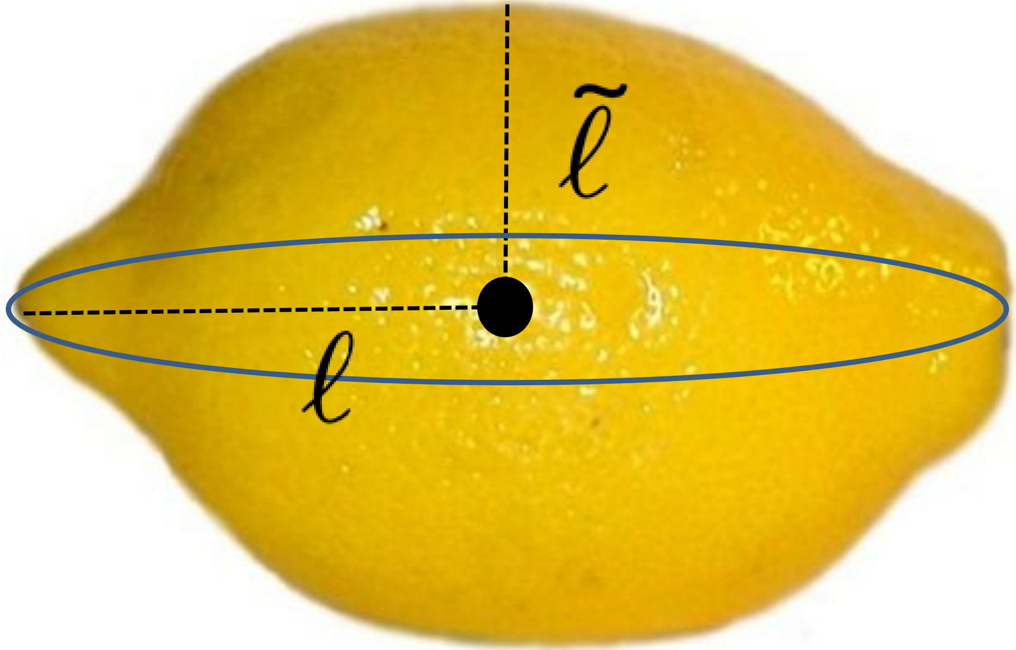
Other quantities:

- SUSY Wilson loop on S^3
- Bremsstrahrung function in SCFT on R^3 [cf. Lewkowycz-Maldacena '13]
- 2-pt. function of U(1) flavor current in SCFT
- 2-pt. function of stress tensor in SCFT
- Partition function on squashed S^3 ～ SUSY Renyi entropy
- Partition function on squashed lens space

Interpretation of singularities (3d)

= Complexified Supersymmetric Solutions

[M.H. '17]



S_b^3

$$b = \sqrt{\tilde{\ell}/\ell}$$

For a technical convenience,

we consider 3d N=2 theories on ellipsoid

(Round sphere corresponds to b=1)

Bosonic Complexified SUSY Solutions

Under the Coulomb branch solution (constant σ),

we look for solutions w/ $\psi = \bar{\psi} = F = \bar{F} = 0$

Nontrivial condition for scalar: $0 = Q\psi = -\gamma^{\mu}\epsilon D_{\mu}\phi - \epsilon\sigma\phi - \frac{i\Delta}{f(\vartheta)}\epsilon\phi$

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Useful eigenvalue problem:

[already solved in Hama-Hosomichi-Lee]

$$\begin{cases} \gamma^\mu \epsilon D_\mu \Phi + \epsilon\sigma\Phi + \frac{i\Delta}{f(\vartheta)}\epsilon\Phi = M\epsilon\Phi \\ M = M_{m,n} = \sigma + i\left(mb + nb^{-1} + \frac{b+b^{-1}}{2}\Delta\right), \qquad m, n \in \mathbf{Z}_{\geq 0} \end{cases}$$

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$$
\begin{cases}
\gamma^{\mu}\epsilon D_{\mu}\Phi+\epsilon\sigma\Phi+\frac{i\Delta}{f(\vartheta)}\epsilon\Phi=M\epsilon\Phi \\
M=M_{m,n}=\sigma+i\left(mb+nb^{-1}+\frac{b+b^{-1}}{2}\Delta\right), \qquad m,n\in\mathbf{Z}_{\geq 0}
\end{cases}
$$

SUSY condition is M=0 but this cannot be realized for $\sigma\in\mathbf{R}$

(=original path)

If we relax this, we have

$$
\boxed{\sigma=-i\left(mb+nb^{-1}+\frac{b+b^{-1}}{2}\Delta\right), \ \ \phi=\Phi_{m,n}}
$$

Fermionic Complexified SUSY Solutions

We look for solutions w/ $\phi = \bar{\phi} = F = \bar{F} = 0$

Nontrivial condition for fermion: $\epsilon(-\gamma^\mu D_\mu + \sigma)\psi + \frac{i(2\Delta - 1)}{2f(\vartheta)}\epsilon\psi = 0.$

Fermionic Complexified SUSY Solutions

We look for solutions w/ $\phi = \bar{\phi} = F = \bar{F} = 0$

Nontrivial condition for fermion: $\epsilon(-\gamma^\mu D_\mu + \sigma)\psi + \frac{i(2\Delta - 1)}{2f(\vartheta)}\epsilon\psi = 0.$

Useful eigenvalue problem:

[already solved in Hama-Hosomichi-Lee]

$$\begin{cases} \epsilon(-\gamma^\mu D_\mu \Psi + \sigma\Psi) + \frac{i(2\Delta - 1)}{2f(\vartheta)}\epsilon\Psi = M\epsilon\Psi \\ M = M_{m,n} = \sigma - i\left(mb + nb^{-1} - \frac{(b+b^{-1})(\Delta - 2)}{2}\right), \qquad m, n \in \mathbf{Z}_{\geq 0} \end{cases}$$

SUSY condition is M=0 but this cannot be realized for $\sigma \in \mathbf{R}$

If we relax this,

$$\boxed{\sigma = i\left(mb + nb^{-1} - \frac{(b+b^{-1})(\Delta - 2)}{2}\right), \ \psi = \Psi_{m,n}}$$

Comparison w/ Borel trans.

For U(1) theory w/ charge q_a chiral multiplets,

$$\mathcal{B}Z_{S_b^3}(t) = \frac{1}{2\sqrt{-it}\prod_{a=1}^{N_f} s_b\left(q_a\sqrt{it} - \frac{iQ(1-\Delta_a)}{2}\right)}$$

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Locations of poles & zeroes:

$$t_{\text{pole}}^{m,n} = -\frac{i}{q_a^2}\left(mb + nb^{-1} + \frac{b+b^{-1}}{2}\Delta_a\right)^2,$$

$$t_{\text{zero}}^{m,n} = -\frac{i}{q_a^2}\left(mb + nb^{-1} - \frac{(b+b^{-1})(\Delta_a - 2)}{2}\right)^2$$

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Actions of the solutions:

$$S_{\text{bos}} = \frac{i\pi k}{q_a^2}\left(mb + nb^{-1} + \frac{b+b^{-1}}{2}\Delta_a\right)^2 = \frac{t_{\text{pole}}^{m,n}}{g}$$

$$S_{\text{fer}} = \frac{i\pi k}{q_a^2}\left(mb + nb^{-1} - \frac{(b+b^{-1})(\Delta_a - 2)}{2}\right)^2 = \frac{t_{\text{zero}}^{m,n}}{g}$$

Remarks

- Degeneration of poles & zeroes in round sphere limit:

$$s_b(z) = \prod_{m=0}^{\infty}\prod_{n=0}^{\infty}\frac{mb+nb^{-1}+\frac{b+b^{-1}}{2}-iz}{mb+nb^{-1}+\frac{b+b^{-1}}{2}+iz} \quad \Rightarrow \quad s_1(z) = \prod_{n=1}^{\infty}\left(\frac{n-iz}{n+iz}\right)^n$$

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- Contribution from hyper multiplet:

$$\frac{1}{s_1(z - i/2)\, s_1(-z - i/2)} = \frac{1}{2\cosh(\pi z)}$$

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- In the planar limit: N→∞, gN=fixed,

(actions)→∞ $\Rightarrow$ Borel singularities →∞

consistent w/ expected convergence in the planar limit

Summary & Outlook

Summary

Deconstructing exact results in SUSY gauge theories

4d N=2 theories:

- ∃ Singularities only along R- → Borel summable along R+
- (Exact) = $\sum_{\text{instantons}}$ (Borel resum)

3d N=2 CS matter theories:

- Usually non-Borel summable along R+
- Always Borel summable along (half-)imaginary axis
- (Exact result) =(Borel resummation along the direction)
- (Poles/zeroes) = (Complexified SUSY solutions)

List of interesting points for SUSY guys

- Parameter t in Borel trans.
  = radial direction of Coulomb branch parameter
- Borel trans. $\sim$ Integrand of localization
- Borel singularities
  = poles of 1-loop det.& Nekrasov partition function
- Real mass affects Borel summability?
- Complexified SUSY solutions determine
  analytic structure of "effective potential"?

Open questions

- Less SUSY case?
- Other observables? [For 't Hooft loop, M.H.-D.Yokoyama, in preparation]
- Implication of Borel zeroes??
- Expansion by other parameters? (such as 1/N)

4d N=2 theories:

- Physical interpretation of poles in complex plane?

—— Probably similar but we have to interpret poles of Nekrasov partition function

3d N=2 CS matter theories:

[Fujimori-M.H.-Kamata-Misumi-Nitta-Sakai, work in progress]

- Understanding the resurgence structure

—— Connection to resurgence in complex CS? [cf. Gukov-Marino-Putrov]

Thanks!

Appendix

Some details on S^4 partition function

$$Z_{S^4} = \int_{-\infty}^{\infty} d^{|G|}a \; Z_{\mathrm{VdM}} Z_{\mathrm{cl}} Z_{\mathrm{1loop}} Z_{\mathrm{inst}}$$

$$Z_{\mathrm{VdM}}(a) = \prod_{\alpha \in \mathrm{root}_+} (\alpha \cdot a)^2 \qquad Z_{\mathrm{cl}}(a) = \exp\left[-\sum_{p=1}^{n} \frac{1}{g_p} \mathrm{tr}(a^{(p)})^2\right]$$

$$Z_{\mathrm{1loop}}(a) = \frac{\prod_{\alpha \in \mathrm{root}_+} H^2(\alpha \cdot a)}{\prod_{m=1}^{N_f} \prod_{\rho_m \in \mathbf{R_m}} H(\rho_m \cdot a)}$$

$$H(x) = e^{-(1+\gamma)x^2} G(1+ix) G(1-ix)$$

$$Z^{(k)}_{\mathrm{SQCD}} = \int_{-\infty}^{\infty} d^N a \, \delta(\sum_j a_j) \prod_{i<j} (a_i - a_j)^2 e^{-\frac{1}{g}\sum_j a_j^2} \frac{\prod_{i<j} H^2(a_i - a_j)}{\prod_j H^{2N}(a_j)} Z^{(k)}_{\mathrm{inst}}(a)$$

Outline of Proof

$$Z_{\rm SQCD}^{(k,\bar{k})}(g) = \int_0^\infty dt\ e^{-\frac{t}{g}} f^{(k,\bar{k})}(t)$$

$$f^{(k,\bar{k})}(t) = \sum_{\ell=0}^{\infty} \frac{c_\ell^{(k,\bar{k})}}{\Gamma(\sharp+\ell)} t^{\sharp+\ell-1} \ ??$$

(1) Show $f^{(k,\bar{k})}(t)$ purely consists of convergent power series:

$$f^{(k,\bar{k})}(t) = \sum_{\ell=0}^{\infty} f_\ell^{(k,\bar{k})} t^{\sharp+\ell-1}$$

(2) Laplace trans. guarantees $f_\ell^{(k,\bar{k})} = \frac{c_\ell^{(k,\bar{k})}}{\Gamma(\sharp+\ell)}$

Proof of (1):

$$f^{(k,\bar{k})}(t) = \int_{S^{N-1}} d^{N-1}\hat{x}\ h^{(k,\bar{k})}(t,\hat{x})$$

(a) Show $h^{(k,\bar{k})}(t,\hat{x})$ consists of convergent power series of t

(b) Small-t expansion of $h^{(k,\bar{k})}(t,\hat{x})$ commutes w/ the integral

(This is true if small-t expansion of $h^{(k,\bar{k})}(t,\hat{x})$ uniform convergent)

Non-zero instanton sector

$$Z^{(k,\bar{k})}_{\text{SQCD}}(g) = \int_0^\infty dt\ e^{-\frac{t}{g}} f^{(k,\bar{k})}(t), \quad f^{(k,\bar{k})}(t) = \int d^{N-1}\hat{x}\ h^{(k,\bar{k})}(t,\hat{x})$$

$$h^{(k,\bar{k})}(t,\hat{x}) = h^{(0,0)}(t,\hat{x}) Z^{(k,\bar{k})}_{\text{inst}}(a = \sqrt{t}\hat{x})$$

Rational function of a, whose poles are not in real axis

[cf. Nekrasov '03]

Thus,

Borel trans. is not singular for $t \in R_+$

⇒ Borel summable!!

General theory w/ Lagrangians (&β≦0)

Suppose a theory w/ gauge group: $G = G_1 \times \cdots \times G_n$

$$Z_{S^4}(g,\theta) = \int_{-\infty}^{\infty} d^{|G|}a \; Z_{\text{cl}}(g;a)\tilde{Z}(a)Z_{\text{inst}}(g,\theta;a)$$

$$Z_{\text{cl}}(g;a) = \exp\left[-\sum_{p=1}^{n}\frac{1}{g_p}\text{tr}(a^{(p)})^2\right]$$

Taking polar coordinate $a_i^{(p)} = \sqrt{t_p}\hat{x}_i^{(p)}$,

$$Z_{S^4}^{(\{k\},\{\bar{k}\})}(g) = \int_0^{\infty} d^n t \; e^{-\sum_p \frac{t_p}{g_p}} f^{(\{k\},\{\bar{k}\})}(t_1,\cdots,t_n)$$

Borel trans.

⇒ Borel summable!!

General 3d N=2 CS matter theory

Suppose a theory w/ gauge group: $G = G_1 \times \cdots \times G_n$

$$Z_{S^3}(g) = \int_{-\infty}^{\infty} d^{|G|}\sigma \; Z_{\rm cl}(g;\sigma)\tilde{Z}(\sigma)$$

$$Z_{\rm cl}(g;a) = \exp\left[\sum_{p=1}^{n} \frac{i \cdot {\rm sgn}(k_p)}{g_p} {\rm tr}(\sigma^{(p)})^2\right]$$

Taking polar coordinate $\sigma_i^{(p)} = \sqrt{\tau_p}\hat{x}_i^{(p)}$,

$$Z_{S^3}(g) = \left[\prod_{p=1}^{n} \int_0^{-i{\rm sgn}(k_p)\infty} d^n t \; e^{-\frac{t_p}{g_p}}\right] \mathcal{B}Z_{S^3}(t)$$

⇒ Borel summable along $\theta_p = -\frac{{\rm sgn}(k_p)\pi}{2}$