Electrodynamics problems

1. For simplicity, consider $(1+1)$-dimensional spacetime, and an inertial frame K with coordinates (x^0, x^1).

   (a) Draw a spacetime diagram with x^0 being the vertical axis, and draw light cones $s^2 = -(x^0)^2 + (x^1)^2 = 0$. Also, draw a curve $s^2 = -(x^0)^2 + (x^1)^2 =$const. in the region $x^0 > 0$ in the case of $s^2 < 0$ and in the region $x^1 > 0$ in the case of $s^2 > 0$. Let each constant be $s^2 = -c^2\tau^2$ for $s^2 < 0$ and $s^2 = \ell^2$ for $s^2 > 0$.

   (b) Consider another intertial frame $\overline{K}$ with coordinates $(\overline{x}^0, \overline{x}^1)$ which moves with the velocity v in the positive direction of x^1-axis of the inertial frame K. By making the origins of spacetime coordinates of these two frames coincide with each other, and putting $v = 0.5c$, draw the coordinate axises of the frame $\overline{K}$ in the diagram.

   (c) The world line $\overline{x}^1 = 0$ corresponds to $x^1 = vt = (v/c)x^0$ in the frame K. Show that, along this world line, a lapse of time τ in the frame $\overline{K}$ is measured in the frame K as a lapse of time,

   $$\tau_K = \frac{\tau}{\sqrt{1-(v/c)^2}}\,.$$

   Since $\tau_K > \tau$, this means "a moving clock runs more slowly than a stationary clock", known as the time dilatation effect.

   (d) Consider a bar with length ℓ being at rest in the frame $\overline{K}$. Assuming the left edge of this bar passes through the origin at time $x^0 = 0$, draw the world line of the right edge in the diagram. Show that its length measured in the frame K is given by

   $$\ell_K = \ell\sqrt{1-(v/c)^2}\,,$$

   which is known as the Lorentz contraction of a moving body.

2. Consider an inertial frame K with coordinates (x^0, x^1), and consider a second frame K_1 with (y^0, y^1) moving with velocity v^1 relative to the frame K in the positive direction of the x^1-axis, and a third frame K_2 moving with velocity v^2 relative to the frame K_1 in the positive direction of the y^1-axis. Show that the velocity of the frame K_2 relative to K is given by

   $$v = \frac{v_1 + v_2}{1 + v_1 v_2/c^2} = c\tanh(\psi_1 + \psi_2)$$

   where $v_1 = c\tanh\psi_1$, $v_2 = c\tanh\psi_2$.

3. For a Lorentz transformation $\overline{x}^\mu = \Lambda^\mu{}_\alpha x^\alpha$, let its inverse transformation be $x^\alpha = (\Lambda^{-1})^\alpha{}_\mu \overline{x}^\mu$. A quantity with n lower indices which transforms under the Lorentz transformation as

   $$\overline{T}_{\mu_1\mu_2\cdots\mu_n}(\overline{x}) = T_{\alpha_1\alpha_2\cdots\alpha_n}(x)(\Lambda^{-1})^{\alpha_1}{}_{\mu_1}(\Lambda^{-1})^{\alpha_2}{}_{\mu_2}\cdots(\Lambda^{-1})^{\alpha_n}{}_{\mu_n}$$

   is called a covariant tensor, and a quantity with n upper indices which transforms as

   $$\overline{T}^{\mu_1\mu_2\cdots\mu_n}(\overline{x}) = \Lambda^{\mu_1}{}_{\alpha_1}\Lambda^{\mu_2}{}_{\alpha_2}\cdots\Lambda^{\mu_n}{}_{\alpha_n}T^{\alpha_1\alpha_2\cdots\alpha_n}(x)$$

   is called a contravariant tensor. A Lorentz transformation is characterized by the property that the components of the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ remain unchanged, i.e.,

   $$\overline{\eta}_{\mu\nu} = \eta_{\alpha\beta}(\Lambda^{-1})^\alpha{}_\mu(\Lambda^{-1})^\beta{}_\nu = \eta_{\mu\nu}\,.$$

   (a) Let $\eta^{\mu\nu}$ be the components of the inverse matrix of $\eta_{\mu\nu}$, i.e., $\eta^{\mu\rho}\eta_{\rho\nu} = \delta^\mu_\nu$. Show that

   $$\overline{\eta}^{\mu\nu} = \eta^{\mu\nu} = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\eta^{\alpha\beta}\,.$$

Thus $\eta^{\mu\nu}$ is a Lorentz invariant contravariant tensor ($\eta^{\mu\nu}$ is called the contravariant metric).

(b) Show that the partial derivative operator $\dfrac{\partial}{\partial x^\mu}$ (often denoted by ∂_μ) behaves as a covariant tensor. Then show that $\eta^{\mu\nu}\partial_\mu\partial_\nu$ is a Lorentz invariant scalar operator. This operator is commonly denoted by $\Box$ and called the d'Alembertian.

4. Let a world line of a point mass be parametrized as $x^\mu(\lambda)$ with some parameter λ and let $\dot{x}^\mu = \dfrac{dx^\mu(\lambda)}{d\lambda}$. If one regards λ as 'time', the action functional S of a free particle with mass m can be expressed as

$$S = -mc\int L\, d\lambda\,; \qquad L \equiv \sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\,.$$

(a) Show that the action is invariant under a transformation $\lambda \to \overline{\lambda} = f(\lambda)$ where $f(\lambda)$ is an arbitrary monotonically increasing function of λ, i.e., $df(\lambda)/d\lambda > 0$ for $^\forall\lambda$.

(b) If one chooses $\lambda = \tau$ where τ is particle's proper time, one has $L = \sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} = c$ along any world line of the particle. Using this fact, show that another form of the action,

$$\tilde{S} = -\frac{m}{2}\int L^2 d\tau = \frac{m}{2}\int \eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\, d\tau\,,$$

is equivalent to S, i.e., $\tilde{S}$ and S gives the same equations of motion.

5. Using the proper time τ as a time parameter along a world line of a particle with mass m, let us consider a general form of the action with interaction. We assume the mass m to be constant. Let S_0 be the action of a free particle and let the total action be $S = S_0 + \int L_{int} d\tau$.

(a) Let $L_{int} = L_{int}(x^\mu, u^\mu)$ $\left(u^\nu \equiv \dfrac{dx^\nu}{d\tau}\right)$, and let the equations of motion be of the form, $m\dfrac{du_\mu}{d\tau} = F_\mu$. Express F_μ in terms of L_{int}.

(b) Assuming that the force F_μ contains only derivatives of $x^\mu(\tau)$ up to first order with respect to τ, show that L_{int} must have the form,

$$L_{int} = \phi(x) + A_\mu(x)u^\mu\,.$$

(c) Recalling the normalization condition of the four velocity $\eta_{\mu\nu}u^\mu u^\nu = -c^2$, show that the term $\phi(x)$ cannot give a physically meaningful force, and hence the only possibility is the form, $L_{int} = A_\mu(x)u^\mu$.

(d) Show that adding the total time derivative of an arbitrary function $f(x)$ to the Lagrangean is equivalent to the change of A_μ as $A_\mu(x) \to \tilde{A}_\mu = A_\mu(x) + \partial_\mu f(x)$. Also show explicitly that the force F_μ remains invariant under this transformation of A_μ.

6. Let $\epsilon_{\mu\nu\rho\sigma}$ be a totally antisymmetric tensor with $\epsilon_{0123} = +1$.

(a) Show the following equalities (remember that $\epsilon^{0123} = -1$).

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\beta} = -3!\,\delta^\sigma_\beta\,, \qquad \epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\alpha\beta} = -2\left(\delta^\rho_\alpha\delta^\sigma_\beta - \delta^\rho_\beta\delta^\sigma_\alpha\right)$$

(b) Show that $\epsilon_{\mu\nu\alpha\beta}$ remains invariant under an arbitrary Lorentz transformation, i.e., $\overline{\epsilon}_{\mu\nu\alpha\beta} = \epsilon_{\mu\nu\alpha\beta}$ for $x^\mu \to \overline{x}^{\mu'} = \Lambda^{\mu'}{}_\mu x^\mu$.

7. For an anti-symmetric second rank tensor $A_{\mu\nu}$, define ${}^*A_{\mu\nu}$ by

$$ {}^*A_{\mu\nu} := \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}A^{\alpha\beta} . $$

${}^*A_{\mu\nu}$ is called the tensor dual to $A_{\mu\nu}$.

(a) Show that ${}^{**}A_{\mu\nu} = {}^*({}^*A_{\mu\nu}) = -A_{\mu\nu}$.

(b) In terms of ${}^*F^{\mu\nu}$ and $F^{\mu\nu}$, the Maxwell equations in vacuum are expressed as

$$ \partial_\nu F^{\mu\nu} = 0, \qquad \partial_\nu {}^*F^{\mu\nu} = 0 . $$

Derive the second set of the equations above.

(c) Regarding $\tilde{F}_{\mu\nu} = {}^*F_{\mu\nu}$ as another electromagnetic field strength tensor, show that its electric field $\tilde{\boldsymbol{E}}$ and magnetic field $\tilde{\boldsymbol{B}}$ are expressed in terms of the original fields as

$$ \tilde{\boldsymbol{E}} = -\boldsymbol{B} , \qquad \tilde{\boldsymbol{B}} = \boldsymbol{E} . $$

8. In the Lorentz gauge, the source-free Maxwell equations expressed in terms of the four potential, $\partial^\nu\partial_\nu A_\mu = \Box A_\mu = 0$, contain a residual gauge degree of freedom $A_\mu \to \overline{A}_\mu = A_\mu + \partial_\mu f$ where f is an arbitrary function satisfying $\Box f = 0$. By expanding A_μ in the Fourier series, $A_\mu(x) = \int d^3k\, \tilde{a}_\mu(\boldsymbol{k})\, e^{ik_\mu x^\mu}$, $f(x) = \int d^3k\, \tilde{f}(\boldsymbol{k})\, e^{ik_\mu x^\mu}$ $(-k_0 = k^0 = |\boldsymbol{k}|)$, show that the residual gauge freedom can be used to choose a gauge in which $\overline{A}_0 = 0$ (called the Coulomb gauge).

9. An electromagnetic field with its amplitude slowly varying over a scale L sufficiently larger than its characteristic wavelength λ can be approximated by a plane wave. To do so, one chooses the Coulomb gauge ($\Box A^i = 0$, $\partial_i A^i = 0$), sets $A^i(x) = a^i(x)\, e^{iS(x)}$ and assumes

$$ \partial_\mu a^i = O\left(\frac{a^i}{L}\right), \quad \partial_\mu S = O\left(\frac{S}{\lambda}\right) = \frac{1}{\epsilon}\,O\left(\frac{S}{L}\right), \quad \partial_\mu\partial_\nu S = \frac{1}{\epsilon}\,O\left(\frac{S}{L^2}\right) $$

where $\epsilon = \lambda/L \ll 1$. One can then derive equations at each order of ϵ, which is called the geometric optics approximation.

(a) From the equations of $O(\epsilon^{-2})$ and $O(\epsilon^{-1})$, derive

$$ \partial_\mu S\partial^\mu S = 0, \quad a^i\partial_i S = 0, \tag{1} $$
$$ 2\partial_\mu a^i\partial^\mu S + a^i\Box S = 0 . \tag{2} $$

Eq. (1) shows that $k_\mu := \partial_\mu S$ gives a 4-dimensional wavenumber vector on scales much smaller than L, and has the property $a^i k_i = 0$, i.e., A^i is transverse.

(b) Let $|a|^2 = a_i^* a^i$. Show that Eq. (2) then gives

$$ \partial_\mu(|a|^2\partial^\mu S) = \partial_\mu(N^\mu) = 0\,; \quad N^\mu := |a|^2 k^\mu . \tag{3} $$

Also, define the 4-dimensional Poynting flux by $S^\mu := (\rho\, c, S^i)$. Noting $\boldsymbol{A} = \mathrm{Re}(\boldsymbol{a}e^{iS})$ and $\omega = k\, c$, show that the time average of S^μ is given by

$$ \langle S^\mu\rangle_{\mathrm{timeaverage}} = \frac{\omega\, k^\mu}{8\pi}|a|^2 \propto \omega\, N^\mu . $$

Recalling that $\hbar\omega$ gives the energy of a photon in quantum theory, Eq. (3) describes the photon number conservation.

10. Show that the retarded Green function satisfying $-\Box\, G_R(x-x') = \delta^4(x-x')$ can be concisely expressed as

$$G_R(x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot x}}{k^2 - i\epsilon k^0}\,,$$

where $k^2 = -(k^0)^2 + \boldsymbol{k}^2$, ϵ is an infinitesimal positive constant, and the integral is over all real values of (k^0, k^1, k^2, k^3). Similarly, show that the advanced Green function is expressed as

$$G_A(x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot x}}{k^2 + i\epsilon k^0}\,.$$

11. Solve the equations of motion of a charged particle,

$$\frac{du^\mu}{d\tau} = \frac{q}{mc} F^{\mu\nu} u_\nu\,; \qquad u^\mu = \frac{dx^\mu}{d\tau}$$

under the following situations, with $v^i(0) = \dfrac{dx^i}{dt}(0) = (v, 0, 0)$ as the initial condition at $t = 0$.

(a) Under the presence of a homogeneous magnetic field along the x^3-axis, $B^i = (0, 0, B)$.

(b) Under the presence of a homogeneous electric field along the x^1-axis, $E^i = (E, 0, 0)$.

(c) Calculate the rate of the radiated energy, $\dfrac{dE}{dt} = \dfrac{2q^2}{3c^3}\dot{u}^\alpha \dot{u}_\alpha \left(\dot{} = \dfrac{d}{d\tau}\right)$ for each of the cases (a), (b) above.

12. Under the slow motion approximation, the vector potential A^μ of the radiated field in the wave zone is expressed in the Lorentz gauge ($\partial_\mu A^\mu = 0$) in the series form as

$$\begin{aligned} A^\mu(x) &= \frac{1}{c\,r}\int J^\mu\left(t_R + \frac{\cdot r'}{c}, r'\right) d^3r' \qquad \left(t_R := t - \frac{r}{c}, \quad n^i = \frac{r^i}{r}\right) \\ &= \frac{1}{c\,r}\int \sum_{\ell=0}^{\infty} \frac{1}{\ell!}\left(\frac{n\cdot r'}{c}\right)^\ell \frac{\partial^\ell}{\partial t^\ell} J^\mu(t_R, r') d^3r'\,. \end{aligned}$$

We set

$$Q = \int \rho(t, r)\, d^3r\,, \quad d^i(t) = \int \rho(t, r)\, r^i\, d^3r\,,$$

where $J^0 = \rho\, c$. Q is the total charge of the source and d^i is the electric dipole moment.

(a) Express $A^0 = \phi$ to the order $\ell = 1$ in terms of Q and d^i.

(b) Using the charge conservation law $\partial_\mu J^\mu = 0$, express the order $\ell = 0$ term of A^i in terms of d^i.

(c) Show that there exists a gauge transformation that eliminates the order $\ell = 1$ term of A^0, and derive A^i in this gauge, say $\overline{A}^i$.

13. By emitting radiation, a reaction force acts on the particle. Assuming the effect of the reaction is small, it is known that the equations of motion is modified to be

$$m\frac{du^\mu}{d\tau} = qF^{\mu\nu}u_\nu + F^\mu_{rad}\,; \qquad F^\mu_{rad} = \frac{2q^2}{3c^3}\left(\ddot{u}^\mu - \frac{u^\mu}{c^2}\dot{u}^\alpha \dot{u}_\alpha\right).$$

The radiation reaction force F^μ_{rad} is known as the Abraham-Lorentz-Dirac force.

(a) Show $F^\mu_{rad} u_\mu = 0$.

(b) The rate of the energy-momentum radiated by the particle is given by $dP^\mu = \dfrac{2q^2}{3c^5}\dot{u}^\alpha \dot{u}_\alpha u^\mu d\tau$. Assuming the acceleration of the particle vanishes at $\tau = \pm\infty$, show

$$\int_{-\infty}^{\infty} F^\mu_{rad}\, d\tau = -\int_{-\infty}^{\infty} dP^\mu\,.$$