Cosmological Perturbations from Inflation

Misao Sasaki

Kyoto University

§1. Introduction

- Horizon problem

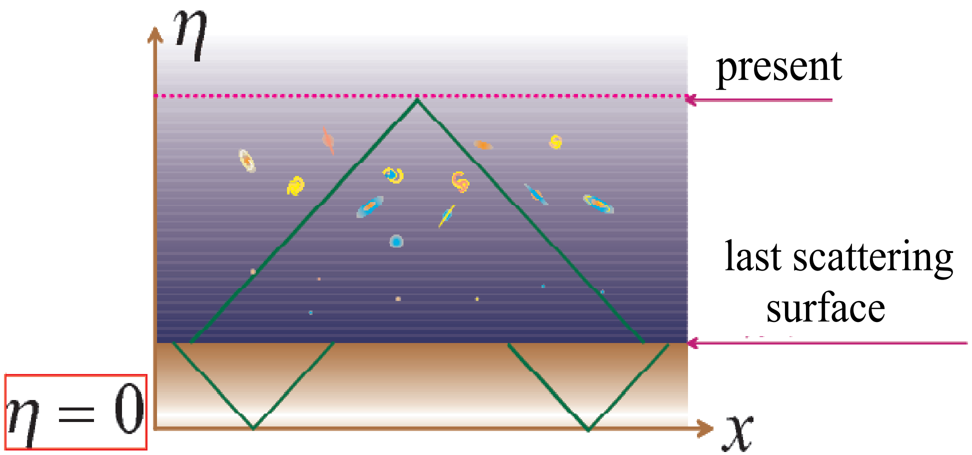
$$ds^2 = -dt^2 + a^2(t)d\vec{x}^2 \quad + \quad \text{Einstein eqs.}$$

$$\Rightarrow \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \qquad \boxed{\rho + 3p > 0 \quad \Leftrightarrow \quad \text{decelerated expansion}}$$

If $a \propto t^n$, then $n(n-1) < 0 \quad \Rightarrow \quad 0 < n < 1$

$$ds^2 = a^2(\eta)\left(-d\eta^2 + d\vec{x}^2\right), \quad d\eta = \frac{dt}{a}.$$

(η: conformal time $\cdots$ maintains causality)



$d\eta = \pm dx$: light ray

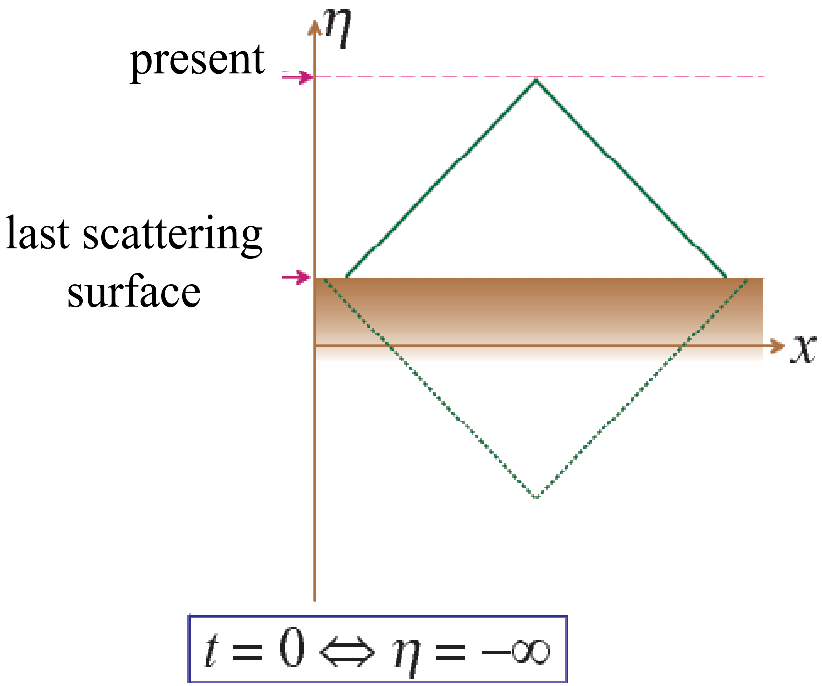
$$\eta = \int \frac{dt}{a} \longrightarrow 0 \quad \text{for } t \longrightarrow 0$$

- Solution to the horizon problem

Existence of a stage $a \propto t^n$ $\boxed{n > 1}$ in the early universe

$$\Leftrightarrow \quad \rho + 3p < 0$$

$$\Rightarrow \quad \int_0^t \frac{dt}{a} = \int d\eta = \infty \;!!$$



- Entropy problem (= flatness problem)

Entropy within the curvature radius: N_γ ~ conserved

$$N_\gamma = n_\gamma \left(\frac{a}{\sqrt{|K|}}\right)^3 \sim \left(\frac{T_0}{H_0}\right)^3 |1 - \Omega_0|^{-3/2} > \left(\frac{T_0}{H_0}\right)^3 \sim 10^{87}$$

$T_0 \sim 10^{-4}\text{eV} \quad H_0 \sim 10^{-33}\text{eV}$

Where does this big number come from?

"Huge entropy production in the early universe"

§2. Single-field slow-roll inflation

Universe dominated by a scalar field:

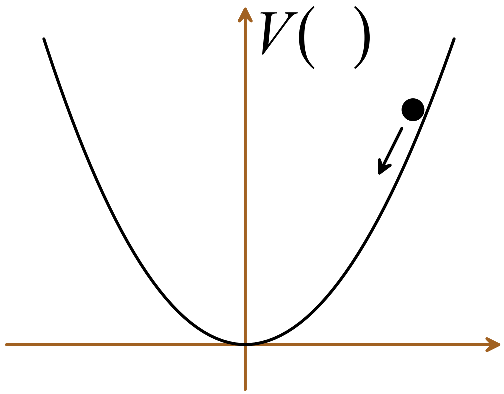
$$\begin{cases} \rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \\ p = \frac{1}{2}\dot{\phi}^2 - V(\phi) \end{cases} \Rightarrow \quad \rho + 3p = 2(\dot{\phi}^2 - V(\phi))$$

$$\text{if} \quad \dot{\phi}^2 < V(\phi) \quad \Longrightarrow \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) > 0$$

accelerated expansion

* **Chaotic inflation (or Creation of Universe from nothing)**

(Linde, Vilenkin, Hartle-Hawking, $\cdots$)



$$\rho_{\text{initial}} \lesssim m_{pl}^4 \approx \left(10^{19}\,\text{GeV}\right)^4$$

$\cdots$ quantum gravitational

$$\text{if} \;\; V''(\phi) \ll m_{pl}^2, \quad \text{then} \;\; \phi \gg m_{pl}$$

- Equations of motion:

$$\ddot{\phi} + \underbrace{3H\dot{\phi}}_{\text{friction}} + V'(\phi) = 0 \quad (H \lesssim m_{pl} \text{ initially in chaotic inflation})$$

$$\Rightarrow \quad \boxed{\dot{\phi} \approx -\frac{V'}{3H}} \quad \text{(slow roll (1))} \quad \Leftrightarrow \quad \left|\frac{\ddot{\phi}}{3H\dot{\phi}}\right| \ll 1$$

$$\begin{cases} \dot{H} = -4\pi G(\rho + p) = -4\pi G\dot{\phi}^2 \\ H^2 = \dfrac{8\pi G}{3}\left(\dfrac{1}{2}\dot{\phi}^2 + V(\phi)\right) \end{cases}$$

$$\Rightarrow \quad \boxed{H^2 \approx \frac{8\pi G}{3}V(\phi)} \quad \text{(potential dominated (2))} \quad \Leftrightarrow \quad \left|\frac{\dot{H}}{H^2}\right| \approx \frac{3\dot{\phi}^2}{2V(\phi)} \ll 1$$

The slow-roll condition (1) is satisfied, provided that

$$\frac{V''}{9H^2} \approx \frac{V''}{24\pi GV} \ll 1\,, \quad \frac{\dot{H}}{3H^2} \approx \frac{V'^2}{48\pi GV^2} \ll 1$$

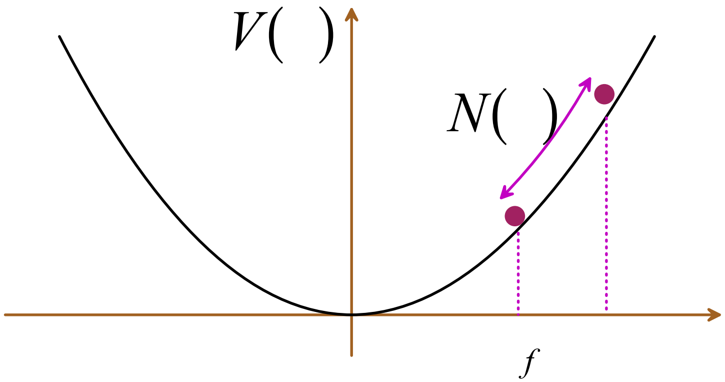
· Slow-roll inflation assumes that the above two are fulfilled.
(Note that these are not necessary but sufficient conditions.)

· There are models that violate either or both of the above two conditions.
(Need special care in the calculation of perturbations)

- e-folding number of inflation $\quad a \propto e^{-N}$

$$N(\phi) = \int_t^{t_f} H\,dt = \int_\phi^{\phi_f} \frac{H}{\dot{\phi}} d\phi \approx 8\pi G \int_{\phi_f}^{\phi} \frac{V}{V'} d\phi = 2\pi G(\phi^2 - \phi_f^2) \sim 2\pi \left(\frac{\phi}{m_{pl}}\right)^2$$

$\Uparrow$ slow roll $\qquad \Uparrow$ $V = \frac{1}{2} m^2 \phi^2$



For $V(\phi) \sim (10^{15}\mathrm{GeV})^4$, $N(\phi) \gtrsim 60$ solves horizon & flatness problems

$$N(\phi) \gtrsim 60 \quad \text{at} \quad \phi \gtrsim 3m_{pl} \quad \text{for} \quad V = \frac{1}{2} m^2 \phi^2$$

Slow roll ends at $\phi \lesssim 0.2 m_{pl}$ $\quad \Rightarrow \quad$ Reheating (entropy generation)

§3. Generation of cosmological perturbations

Action: $$S = \int d^4x \sqrt{-g}\left(\frac{1}{16\pi G}R - \frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V(\phi)\right).$$

Cosmological perturbations are generated from quantum (vacuum) fluctuations of the inflaton ϕ and the metric $g_{\mu\nu}$.

- Scalar-type (density) perturbations
  - $g_{\mu\nu}$ and ϕ:

$$ds^2 = a^2\Big[-(1+2A)d\eta^2 - 2\partial_i B\, d\eta dx^j + \Big((1+2\mathcal{R})\delta_{ij} + 2\partial_i\partial_j H_T\Big)dx^i dx^j\Big],$$

$$\phi(t,x^i) = \phi(t) + \chi(t,x^i)$$

A : Lapse function ($\sim$ time coordinate) perturbation

B : Shift vector ($\sim$ space coordinate) perturbation

Scalar perturbation has 2 degrees of coordinate gauge freedom.

$\mathcal{R}$: Spatial curvature (potential) perturbation $\left(\delta \overset{(3)}{R} = -\frac{4}{a^2}\overset{(3)}{\Delta}\mathcal{R}\right)$

H_T : Shear of the metric ($\sim$ traceless part of the extrinsic curvature)

No dynamical degree of freedom in the metric itself.

⋆ **Action expanded to 2nd order (in the Hamiltonian form)**

cf. Garriga, Montes, MS & Tanaka (1998)

$$S_2 = \int d\eta\, d^3x \left(\sum_a P_a Q_a' - \mathcal{H}_s - A\, C_A - B\, C_B \right)$$

$$\begin{aligned}
\mathcal{H}_s &= \frac{1}{2a^2} P_\chi^2 - 4\pi G \phi' P_{\mathcal{R}} \chi + \cdots\,, \quad {}' = d/d\eta\,, \\
C_A &= \phi' P_\chi + \cdots \quad \text{(Hamiltonian constraint)}\,, \\
C_B &= P_{H_T} \quad \text{(Momentum constraint)}\,, \\
Q_a &= \{\mathcal{R}, H_T, \chi\}, \quad P_a = \{\text{Momentum conjugate to } Q_a\}\,.
\end{aligned}$$

· **Gauge transformation** $[\xi^\mu = (T, \partial_i L)\,]$ **is generated by** C_A **and** C_B**:**

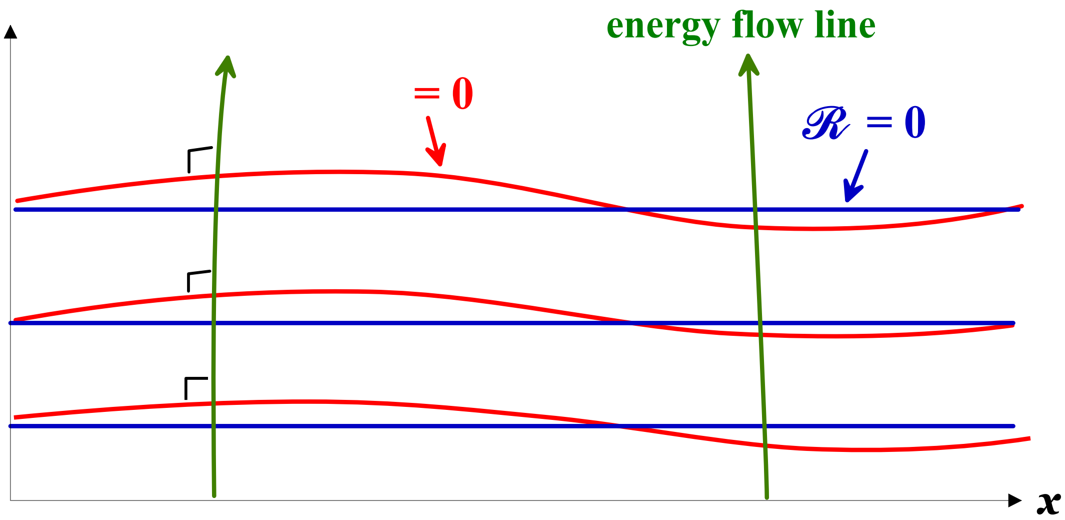
$$\delta_g Q = \left\{ Q, \int (T C_A + L C_B)\, d^3x \right\}_{P.B.} \qquad (\, Q = \{Q_a\,, P_a\}\,)$$

- **Reduction to unconstrained variables a lá Faddeev-Jackiw (1988)**

1. **Solve $C_A = \phi' P_\chi + \cdots = 0$ with respect to P_χ and insert it into S_2. Also, insert $C_B = P_{H_T} = 0$ into S_2.**
2. **The resulting S_2 is a functional of $\{P_\mathcal{R}, \mathcal{R}, \chi\}$: $\quad S_2^* = S_2^*[P_\mathcal{R}, \mathcal{R}, \chi]$**
3. **Since S_2 is gauge-invariant, S_2^* must be written solely in terms of gauge-invariant variables. Indeed, we find**

$$S_2^* = S_2^*[P_c, \mathcal{R}_c]\ ; \quad P_c \equiv P_\mathcal{R} + \frac{a^2}{4\pi G \phi'} \overset{(3)}{\Delta} \chi\,, \quad \mathcal{R}_c \equiv \mathcal{R} - \frac{\mathcal{H}}{\phi'}\chi$$

This is in fact the same as choosing $\chi = 0$ gauge (called 'comoving' slicing). i.e., $\mathcal{R}_c$ is the curvature perturbation on the comoving hypersurface.



- **Useful geometrical gauge-invariant variables:** $\mathcal{H} \equiv \frac{a'}{a} = aH$

$$\zeta := \mathcal{R} + \frac{\delta\rho}{3(\rho + P)}$$ **curvature perturbation on uniform density slices**

($\rho =$ const.)

$$\mathcal{R}_c := \mathcal{R} + \mathcal{H}(v + B)$$ **curvature perturbation on comoving slices**

(hypersurface normal $n^\mu = u^\mu$)

$$\boxed{\zeta = \mathcal{R}_c + O(k^2/H^2a^2)}$$

$$\Phi := \mathcal{R} + \mathcal{H}(B - H_T{}')$$ **curvature perturbation on Newtonian slices**

(hypersurface shear $= 0$)

$$\Psi := A + \frac{1}{a}\left[a(B - H_T{}')\right]'$$ **Newton potential in Newtonian slices**

⋆ **ζ expressed in terms of Φ and Ψ:**

$$\boxed{\zeta = \Phi - \frac{3H\dot{\Phi} - 3H^2\Psi - a^{-2}\nabla^2\Phi}{3\dot{H}}}$$

$$\zeta \approx \mathcal{R}_c \approx \Phi - \frac{3H\dot{\Phi} - 3H^2\Psi}{3\dot{H}}$$ **on superhorizon scales**

⋆ S_2^* in the Lagrangean form:

$$S_2^* = \int d\eta\, d^3x\, \frac{a^2\phi'^2}{2\mathcal{H}^2}\left(\mathcal{R}_c'^2 - (\nabla\mathcal{R}_c)^2\right); \quad \mathcal{H} \equiv \frac{a'}{a} = a\,H$$

Equation of motion (for Fourier modes: $\overset{(3)}{\Delta} \to -k^2$)

$$\mathcal{R}_c'' + 2\frac{z'}{z}\mathcal{R}_c' + k^2\mathcal{R}_c = 0\,; \quad z \equiv \frac{a\phi'}{\mathcal{H}} = \frac{a\dot\phi}{H} \;(\propto a \text{ for slow-roll inflation})\,.$$

For $k < \mathcal{H}$ ($\Leftrightarrow\ k/a < H$),

$$\mathcal{R}_c' \propto \begin{cases} z^{-1} & \sim \text{decaying mode} \\ 0 & \sim \text{growing mode} \end{cases}$$

· Growing mode of $\mathcal{R}_c$ stays constant on super-horizon scales.

· This holds for adiabatic perturbations in general cosmological models. (i.e., the existence of a constant mode)

> But this does not mean that $\mathcal{R}_c$ is constant on super-horizon scales.

- **Inflaton perturbation on flat slicing**

Alternatively, in terms of χ on $\mathcal{R}=0$ hypersurface (flat slicing),

$$\chi_F \equiv \chi - \frac{\phi'}{\mathcal{H}}\mathcal{R} = -\frac{\phi'}{\mathcal{H}}\mathcal{R}_c$$

$$S_2^* = S_2^*[\chi_F] = \int d\eta\, d^3x\, \frac{a^2}{2}\left(\chi_F'^{\,2} - (\nabla\chi_F)^2 - a^2 m_{eff}^2\, \chi_F^2\right);$$

$$m_{eff}^2 = -\frac{\left\{a^2\left(\phi'/\mathcal{H}\right)'\right\}'}{a^4\left(\phi'/\mathcal{H}\right)} = \partial_\phi^2 V + 16\pi G \frac{d}{dt}\left(\frac{V}{H}\right)$$

$\chi_F \sim$ minimally coupled almost massless scalar in de Sitter space

($\because$ $\partial_\phi^2 V \ll H^2$, $16\pi G (V/H)^{\dot{}} \approx 6\dot{H} \ll H^2$ for slow-roll inflation.)

N.B. the sufficient conditions for slow roll were $\partial_\phi^2 V \ll 3H^2$ and $\dot{H} \ll 3H^2$.

· de Sitter approximation for the background:

$$H = \text{const.}\,, \quad a(\eta) = \frac{1}{-H\eta} \quad (-\infty < \eta < 0)$$

This is a good approximation for $k > \mathcal{H}$ (sub-horizon scale) modes.

- **Canonical quantization**

$$\pi(\eta,\vec{x}) = \frac{\delta S_2^*[\chi_F]}{\delta \chi_F'(\eta,\vec{x})}\,, \qquad [\chi_F(\eta,\vec{x}),\pi(\eta,\vec{x}')] = i\delta(\vec{x}-\vec{x}')$$

$$\Rightarrow \quad \hat{\chi}_F = \int \frac{d^3k}{(2\pi)^{3/2}}\left(\hat{a}_{\vec{k}}\,\chi_k(\eta)\,e^{i\vec{k}\cdot\vec{x}} + \text{ h.c.}\right); \quad [\hat{a}_{\vec{k}},\,\hat{a}^\dagger_{\vec{k}'}] = \delta(\vec{k}-\vec{k}')$$

$$\chi_k'' + 2\mathcal{H}\chi_k' + \left(k^2 + m_{eff}^2 a^2\right)\chi_k = 0\,; \quad \chi_{\vec{k}}\,\bar{\chi}_{\vec{k}}' - \chi_{\vec{k}}'\,\bar{\chi}_{\vec{k}} = \frac{i}{a^2}$$

$$\Leftrightarrow \quad \ddot{\chi}_k + 3H\dot{\chi}_k + \left(\frac{k^2}{a^2} + m_{eff}^2\right)\chi_k = 0\,; \quad \chi_{\vec{k}}\,\dot{\bar{\chi}}_{\vec{k}} - \dot{\chi}_{\vec{k}}\,\bar{\chi}_{\vec{k}} = \frac{i}{a^3}$$

(in terms of the cosmic proper time t)

slow roll $\Rightarrow$ $m_{eff}^2 \ll H^2$ $\sim$ massless

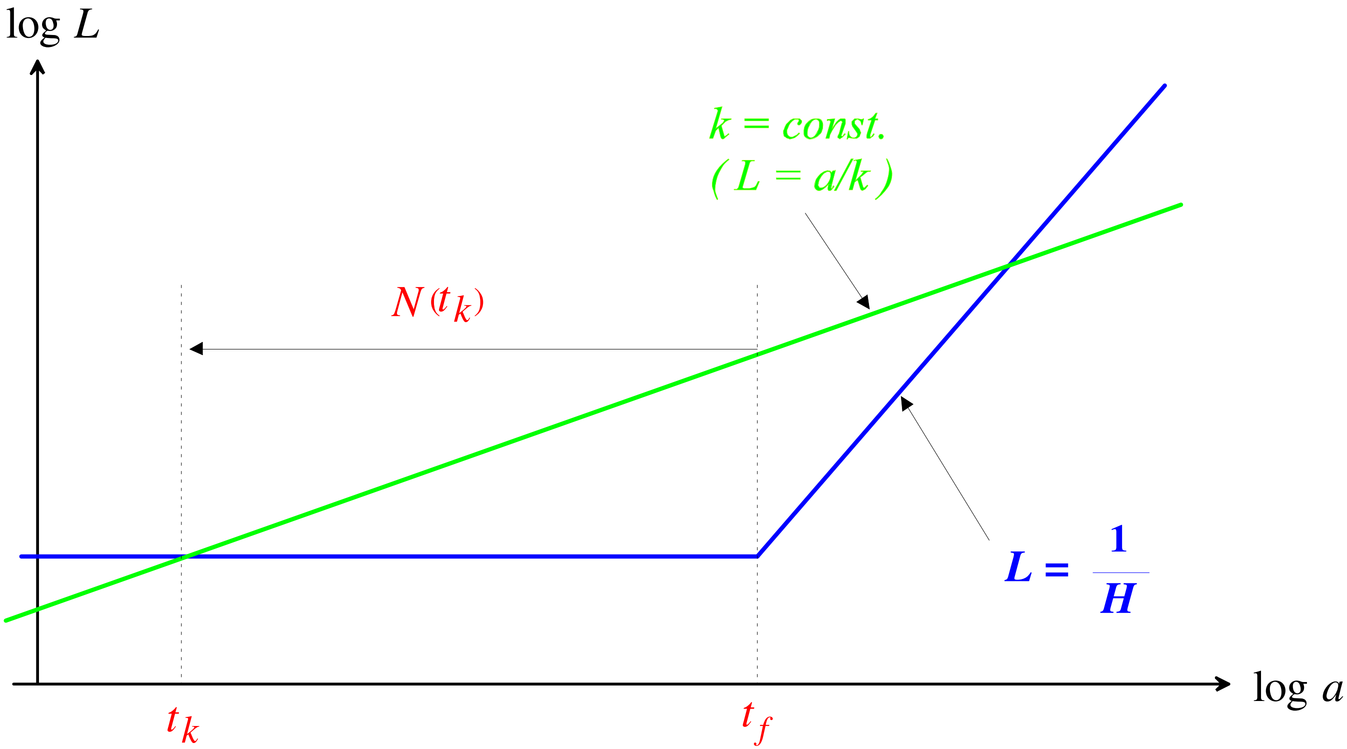
de Sitter approximation:

$$\Rightarrow \quad \chi_k \approx \frac{H}{(2k)^{3/2}}(i - k\eta)\,e^{-i\eta} \quad \begin{cases} \xrightarrow[k/\mathcal{H}\to\infty]{} \dfrac{1}{\sqrt{2k}a}e^{-ik\eta} \\[2ex] \xrightarrow[k/\mathcal{H}\to 0]{} \dfrac{H}{\sqrt{2k^3}}e^{-i\alpha_k} \end{cases}$$

$$\langle\delta\phi^2\rangle_k\big|_{\text{on flat slice}} = \langle\chi_F^2\rangle_k \equiv \frac{4\pi k^3}{(2\pi)^3}|\chi_k|^2 \rightarrow \left(\frac{H}{2\pi}\right)^2 \quad \text{for} \quad k \lesssim \mathcal{H}$$

- de Sitter approximation breaks down at $k \ll \mathcal{H}$.
  i.e., the time-variation of χ_k on super-horizon scales cannot be neglected.
- However, the corresponding k-mode of $\mathcal{R}_c$ becomes constant on super-horizon scales.

$$\Rightarrow \quad \mathcal{R}_{c,k}(\eta) \approx \mathcal{R}_{c,k}(\eta_k) = -\frac{\mathcal{H}}{\phi'}\chi_k(\eta_k) \approx \frac{H^2(t_k)}{\sqrt{2k^3}\,\dot{\phi}(t_k)}e^{-i\alpha_k}\,.$$



$$t = t_k \quad \Leftrightarrow \quad \eta = \eta_k \quad \Leftrightarrow \quad k = \mathcal{H}(\eta_k)$$ $\cdots$ horizon crossing time

- Curvature perturbation spectrum (say, at $\eta = \eta_f$)

$$\langle \mathcal{R}_c^2 \rangle_k \equiv \frac{4\pi k^3}{(2\pi)^3} P_{\mathcal{R}_c}(k;\eta) = \frac{4\pi k^3}{(2\pi)^3} |\mathcal{R}_{c,k}(\eta)|^2 = \left(\frac{H^2}{2\pi\dot{\phi}}\right)^2 \Bigg|_{t=t_k}$$

Since $dN = -Hdt$,

$$\frac{\partial N}{\partial \phi} = -\frac{H}{\dot{\phi}} \quad \Rightarrow \quad \langle \mathcal{R}_c^2 \rangle_k = \left(\frac{\partial N}{\partial \phi}\frac{H}{2\pi}\right)^2 \Bigg|_{t=t_k} = \left(\frac{\partial N}{\partial \phi}\delta\phi\right)^2 \Bigg|_{t=t_k} \text{ on flat slice}$$

That is, for single-field slow-roll inflation,

$$\mathcal{R}_c = \delta N|_{t=t_k} = \frac{\partial N}{\partial \phi}\delta\phi\Big|_{t=t_k} \quad \left(\delta\phi = \frac{H}{2\pi}\right) \quad \text{on flat slice}$$

Only the knowledge of the homogeneous background is sufficient to predict the perturbation spectrum. "δN-formula"

If $\langle \mathcal{R}_c^2 \rangle_k \propto k^{n-1}$

$n = 1$: scale-invariant (Harrison-Zeldovich) spectrum

$n = 1 - \epsilon$ ($\epsilon \ll 1$) for chaotic inflation ($V(\phi) \propto \phi^p$).

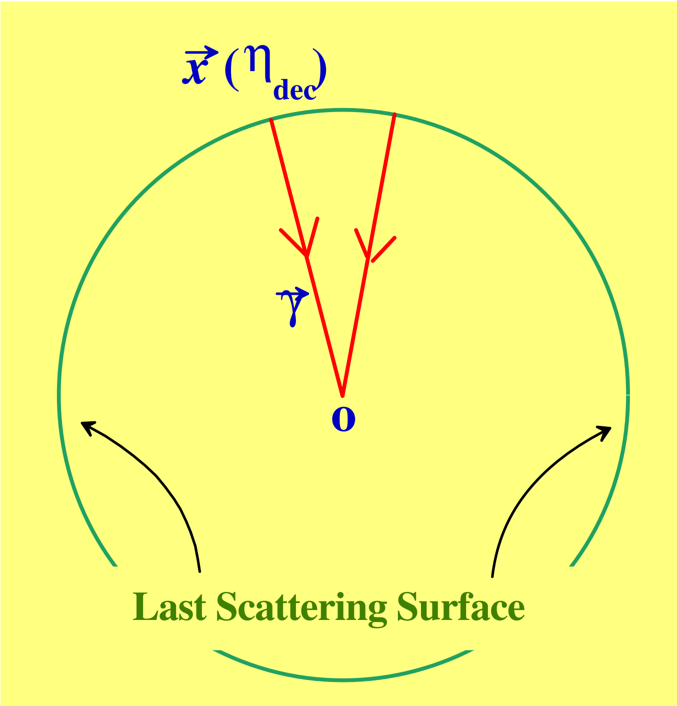
- **Large angle CMB anisotropy**

$$\left(\frac{\delta T}{T}\right)(\vec{\gamma},\eta_0) = (\zeta_{\rm r} + \Theta)\,(\eta_{\rm dec},\vec{x}(\eta_{\rm dec})) + \int_{\eta_{\rm dec}}^{\eta_0} d\eta\,\partial_\eta \Theta(\eta,\vec{x}(\eta))$$

(Sachs-Wolfe) (Integrated Sachs-Wolfe)

$\zeta_{\rm r} \sim$ curvature perturbation on $\rho_{\rm photon} =$ const. surfaces

$\Theta \equiv \Psi - \Phi$



$\int_{\eta_{\rm dec}}^{\eta_0} d\eta$ is along the photon path.

For a dust-dominated universe at decoupling,

$$\text{SW:} \quad \zeta_{\rm r} + \Theta \approx -\frac{1}{5}\zeta_* - \frac{2}{5} S_{\rm dr}$$

ζ_* $\sim$ (initial) adiabatic curvature perturbation

$$S_{\rm dr} = \frac{\delta\rho_{\rm d}}{\rho_{\rm d}} - \frac{3}{4}\frac{\delta\rho_{\rm r}}{\rho_{\rm r}} \quad \sim \text{ entropy perturbation}$$

For standard adiabatic perturbations,

$$\mathcal{R}_c (= \zeta_*) \approx -\frac{5\rho + 3p}{3(\rho + p)}\Psi \to -\frac{5}{3}\Psi\,, \quad \therefore \quad \zeta_{\rm r} + \Theta \approx \frac{1}{3}\Psi$$

- **CMB Observation vs Inflation Model**
  COBE-DMR: ApJ Lett. 464 (1996); WMAP: astro-ph/0306132

$$\left\langle \left(\frac{\delta T}{T}\right)^2 \right\rangle \sim 10^{-10} \quad \text{at } \theta \sim 10^\circ$$

$$\Downarrow$$

$$\langle \Psi^2 \rangle_k \sim 10^{-10} \quad \text{at} \quad \frac{k_0}{a_0} = H_0 \sim \frac{1}{3000\,\text{Mpc}} \sim \frac{1}{10^{28}\text{cm}}$$

For $V = \frac{1}{2}m^2\phi^2$,

$$\langle \Psi^2 \rangle_{k_0} \approx \left(\frac{3}{5}\right)^2 \langle \mathcal{R}_c^2 \rangle_{k_0} = \left(\frac{3}{5}\right)^2 \left(\frac{H^2}{2\pi\dot{\phi}}\right)^2 \Bigg|_{\frac{k_0}{a}=H} \approx \frac{m^2}{m_{pl}^2} N^2(\phi)\Bigg|_{\frac{k_0}{a}=H}$$

$$\Rightarrow \begin{cases} m \sim 10^{13}\text{GeV} \\ V \sim (10^{16}\text{GeV})^4 \end{cases}$$

- **power-law index:** $n_{\text{WMAP}} = 0.93 \pm 0.03$ **(for scalar perturbations)**

  Slight deviation from scale invariant spectrum ($n = 1$)

- **Tensor type perturbation**

$$ds^2 = -dt^2 + a^2(t)\,(\delta_{ij} + h_{ij})\,dx^i dx^j$$

$$h_{ij} \;\cdots\; \text{Transverse-Traceless}$$

$$\delta^2 S_G = \frac{1}{64\pi G}\int d^4x\, a^3 \left(\dot{h}_{ij} - \frac{1}{a^2}(\nabla h_{ij})^2\right)$$

$$= \frac{1}{2}\int d^4x\, a^3 \left(\dot{\varphi}_{ij}^2 - \frac{1}{a^2}(\nabla \varphi_{ij})^2\right) \;; \quad \varphi_{ij} := \frac{1}{\sqrt{32\pi G}} h_{ij}$$

$\varphi_{ij} \sim$ massless scalar (2 degrees of freedom)

$$\langle \varphi_{ij}^2 \rangle_k = 2 \times \left(\frac{H}{2\pi}\right)^2$$

$$\Rightarrow \quad \langle h_{ij}^2 \rangle_k = 2 \times 32\pi G \times \left(\frac{H}{2\pi}\right)^2 = \frac{8}{\pi}\frac{H^2}{m_{pl}^2}$$

$\uparrow$ contribute to CMB anisotropy

$$\frac{T}{S} = \frac{\text{tensor}}{\text{scalar}} \sim \frac{\langle h_{ij}^2 \rangle}{\langle \mathcal{R}_c^2 \rangle} = 24 \left.\frac{\dot{\phi}^2}{V}\right|_{k_0 = aH} \qquad \text{slow roll} \quad \Rightarrow \quad \frac{T}{S} \ll 1\,.$$

$$\frac{T}{S} \sim 0.13 \quad \text{for} \quad V = \frac{1}{2}m^2\phi^2$$ (small but non-negligible)

- **Model dependence**

  * **power-law inflation**

$$V(\phi) \propto \exp[\lambda\phi/m_{pl}] \;\longleftarrow\; \text{dilaton in string theories ?}$$

$$a \propto t^{\alpha} \quad (\alpha = \frac{16\pi}{\lambda^2})$$

$$\Rightarrow \quad n < 1\,, \qquad \frac{T}{S} \gtrsim 0.1$$

  * **hybrid inflation** $\longleftarrow$ **supergravity-motivated ?**

$$\text{e.g.,} \quad V(\phi,\psi) = \frac{1}{4\lambda}\left(M^2 - \lambda\psi^2\right)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{2}g^2\phi^2\psi^2$$

$$a \propto e^{Ht}\,, \quad H^2 \approx \frac{8\pi G}{3}V_0 \quad \text{when} \quad \psi = 0,\ \phi > M/g.$$

$$\Rightarrow \quad n > 1\,, \qquad \frac{T}{S} \ \text{can be large or small.}$$

§4. Perturbation spectrum in non-slow-roll inflation

Leach, MS, Wands & Liddle (2001)

Curvature perturbation on comoving slice:

$$\mathcal{R}_c|_{final} \approx \mathcal{R}_c(t_k) \approx \left(\frac{H^2}{2\pi\dot{\phi}}\right)_{k=aH} \qquad \text{for slow-roll inflation}$$

What if slow-roll condition is violated?

- Reconsideration of EOM for $\mathcal{R}_c$:

$$\mathcal{R}_c'' + 2\frac{z'}{z}\mathcal{R}_c' + k^2\mathcal{R}_c = 0\,; \qquad {}' = \frac{d}{d\eta}, \qquad z = \frac{a\dot{\phi}}{H} = \frac{a\phi'}{\mathcal{H}}, \qquad \mathcal{H} = \frac{a'}{a}\,.$$

Two independent solutions for $k^2 \to 0$:

$$u(\eta) \approx \text{const.} \qquad \cdots \qquad \text{“growing mode”}$$

$$v(\eta) \approx \int_\eta^{\eta_*} \frac{d\eta'}{z^2(\eta')} \qquad \cdots \qquad \text{“decaying mode”} \quad (\eta_* \ \cdots \ \text{end of inflation})$$

- $v \to 0$ as $\eta \to \eta_*$ by definition, but u is arbitrary.
- In slow-roll inflation, $v(\eta) \ll v(\eta_k)$ for $\eta > \eta_k$ $\quad (|k\eta| \sim k/\mathcal{H} \ll 1)$
- v may not decay right after horizon crossing in general.

- **Long-wavelength approximation**

$$u(\eta) = \sum_{n=0}^{\infty} u_n(\eta)\; k^{2n}\,, \quad u''_{n+1} + 2\frac{z'}{z}u'_{n+1} = -u_n\,, \quad u_0 = \text{const.}$$

To $O(k^2)$,

$$u \approx u_0 + [C_1 + C_2 D_0(\eta) + F(\eta)]\, u_0\,;$$

$$D_0(\eta) = 3\mathcal{H}(\eta_k) \int_{\eta}^{\eta_*} d\eta' \, \frac{z^2(\eta_k)}{z^2(\eta')} \quad \cdots \text{ lowest order decaying mode}$$

$$F(\eta) = k^2 \int_{\eta}^{\eta_*} \frac{d\eta'}{z^2(\eta')} \int_{\eta_k}^{\eta'} z^2(\eta'')d\eta''\,.$$

$F \to 0$ **as** $\eta \to \eta_*$ **(F behaves also like decaying mode)**

⋆ **Convenient choice for the "growing mode":**

$$C_1 = 0\,, \quad C_2 = -\frac{F_k}{D_k}\,; \quad F_k = F(\eta_k)\,, \quad D_k = D_0(\eta_k)$$

$$\Rightarrow \quad u(\eta) \approx \left[1 - F_k \frac{D_0(\eta)}{D_k} + F(\eta)\right] u_0\,;$$

$$u(\eta_k) = u(\eta_*)\,, \;\; u'(\eta_k) = 3\mathcal{H}_k \frac{F_k}{D_k}\, u(\eta_k) \quad \Big(\mathcal{H}_k = \mathcal{H}(\eta_k)\Big)$$

⋆ **Decaying mode accurate to $O(k^2)$:**

$$v(\eta) = u(\eta)\frac{\tilde{D}(\eta)}{\tilde{D}(\eta_k)}\,; \quad \tilde{D}(\eta) \equiv 3\mathcal{H}_k \int_\eta^{\eta_*} d\eta' \, \frac{z^2(\eta_k)u^2(\eta_k)}{z^2(\eta')u^2(\eta')}\,.$$

⋆ **General solution for $\mathcal{R}_c$:**

$$\mathcal{R}_c(\eta) \;=\; \alpha u(\eta) + \beta v(\eta)\,; \quad \alpha + \beta = 1\,, \quad \mathcal{R}_c(\eta_k) = u(\eta_k)\,.$$

$$\Rightarrow \mathcal{R}_c(\eta_*) = \alpha\, u(\eta_*) = \alpha\, u(\eta_k) = \alpha\, \mathcal{R}_c(\eta_k)$$

$$\mathcal{R}_c'(\eta_k) = u'(\eta_k) - \frac{3(1-\alpha)\mathcal{H}_k u(\eta_k)}{\tilde{D}(\eta_k)}$$

$$\Rightarrow \quad \alpha \approx 1 + \frac{D_k}{3\mathcal{H}_k}\left[\frac{\mathcal{R}_c'}{\mathcal{R}_c} - \frac{u'}{u}\right]_{\eta=\eta_k}.$$

$$\Rightarrow \mathcal{R}_c(\eta_*) \approx (1 - F_k)\,\mathcal{R}_c(\eta_k) + \frac{D_k}{3\mathcal{H}_k}\mathcal{R}_c'(\eta_k)$$

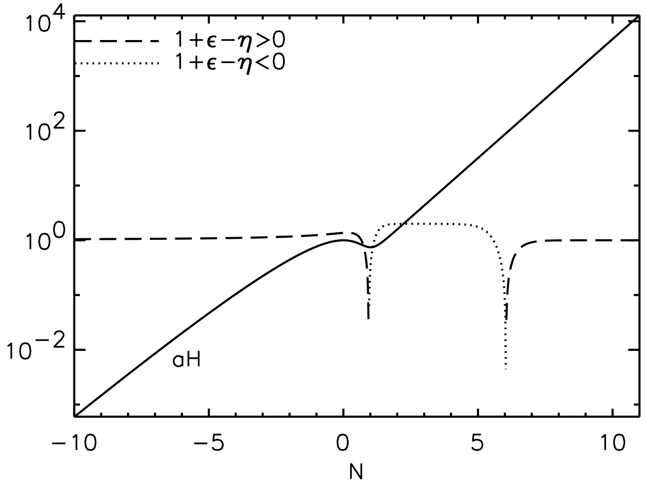
$$|\mathcal{R}_c(\eta_*)| \gg |\mathcal{R}_c(\eta_k)| \quad \text{if} \quad F_k \gg 1 \text{ and/or } D_k \gg 1$$

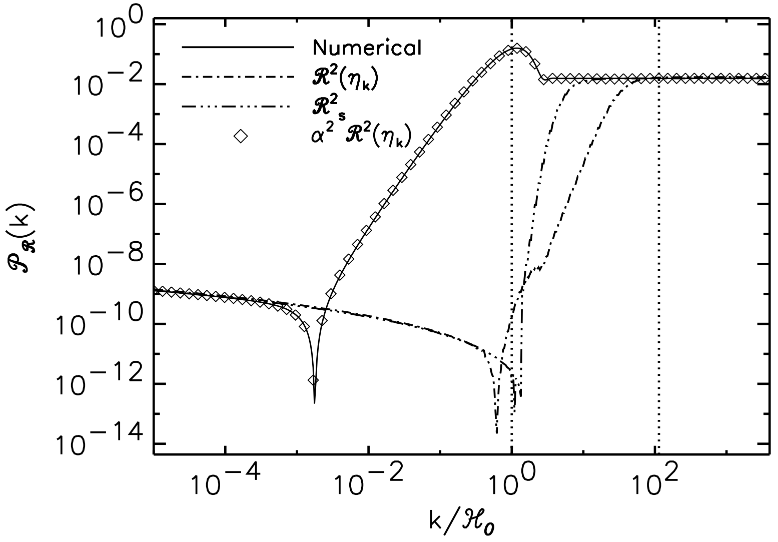
⋆ Leach-Liddle model

Leach & Liddle (2000)

Quartic potential with vacuum energy:

$$V(\phi) = \frac{M^4}{4}\left[1 + B\frac{64\pi^2}{m_{pl}^4}\phi^4\right] \; ; \quad B = 0.55$$





Inflation suspended
at $\phi \sim m_{pl}$

To summarize,

- Spectral formula for non-slow-roll inflation:

$$\mathcal{R}_c(\eta_*) \;= (1 - F_k)\,\mathcal{R}_c(\eta_k) + D_k \frac{k}{3\mathcal{H}_k}\left(\frac{d}{kd\eta}\mathcal{R}_c(\eta_k)\right)$$

$$D_k = \; 3\mathcal{H}_k \int_{\eta_k}^{\eta_*} d\eta' \, \frac{z^2(\eta_k)}{z^2(\eta')} \,,$$

$$F_k = \; k^2 \int_{\eta_k}^{\eta_*} \frac{d\eta'}{z^2(\eta')} \int_{\eta_k}^{\eta'} z^2(\eta'') d\eta'' \,, \qquad z^2 = \left(\frac{a\dot{\phi}}{H}\right)^2$$

- $\mathcal{R}_c(\eta_*)$ in linear combination of $\mathcal{R}'_c(\eta_k)$ and $\mathcal{R}_c(\eta_k)$ with coefficients expressed in terms of background quantities.

⋆ For slow-roll case, we have $D_k \sim 1$, $F_k \sim 0$ for $k/\mathcal{H}_k \lesssim 0.1$
$\Rightarrow \; \mathcal{R}_c(\eta_*) \approx \mathcal{R}_c(\eta_k)$.

⋆ For non-slow-roll case, an enhancement of $\mathcal{R}_c$ can occur:
· A sharp dip appears in the spectrum at $F_k \sim 1$.
· A kink appears at $D_k \gtrsim 1$.

§5. Extension to multi-field inflation

MS & Tanaka (1998)

- n-component scalar field:

$$S_\phi = -\int d^4x \frac{\sqrt{-g}}{2}\left(g^{\mu\nu}\nabla_\mu\vec{\phi}\cdot\nabla_\nu\vec{\phi} + V(\vec{\phi})\right),$$

$$\vec{\phi}\cdot\vec{\phi} = \delta_{pq}\,\phi^p\phi^q \quad (p,q = 1,2,\cdots,n).$$

- Homogeneous background solutions:

(N as a time coordinate; $a = e^{+N}$)

· $G^0{}_0 = T^0{}_0$: (units : $8\pi G = 1$)

$$H^2\left(1 - \frac{1}{6}\vec{\phi}_N^2\right) = \frac{1}{3}V\,; \quad \vec{\phi}_N \equiv \frac{d\vec{\phi}}{dN}$$

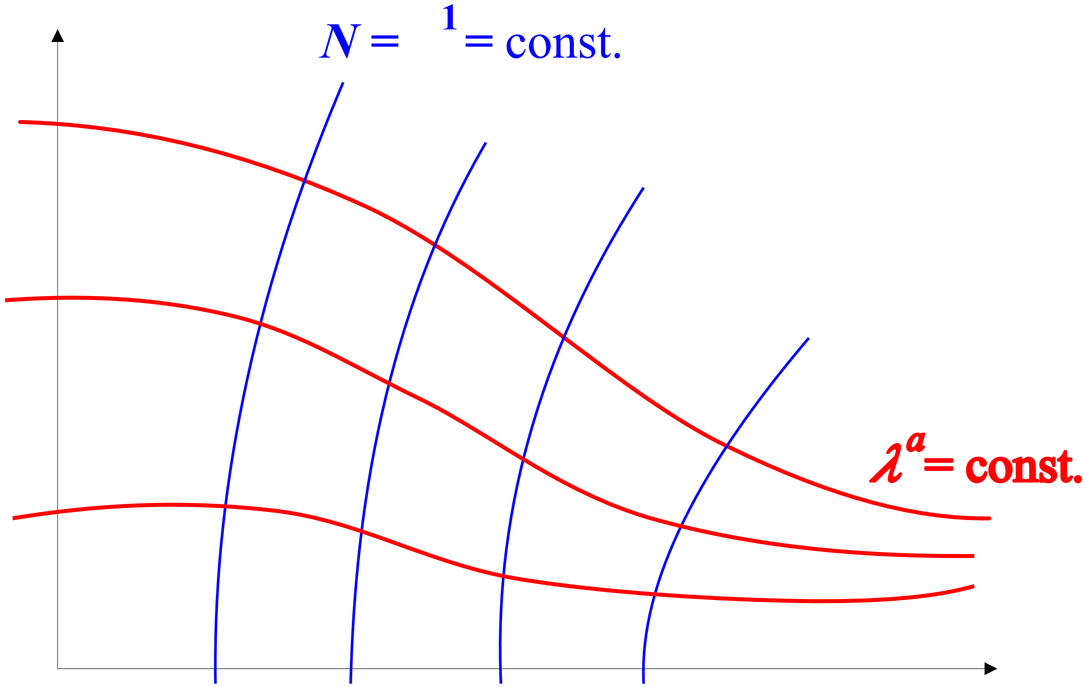
· Scalar field equation :

$$\left[H\frac{d}{dN}H\frac{d}{dN} + 3H^2\frac{d}{dN}\right]\phi^p + V^{|p} = 0\,.$$

General solution is parametrized by $2n$ parameters:

$$\left\{ \vec{\phi} = \vec{\phi}(\lambda^\alpha),\ \vec{\pi} = \vec{\pi}(\lambda^\alpha) \right\} \qquad \left(\vec{\pi} = H e^{3N} \vec{\phi}_N \right)$$

$$\lambda^\alpha = \{N, \lambda^a\} \qquad (\alpha = 1 \sim 2n\,,\ a = 2 \sim 2n)$$



λ^α can be regarded as a new set of phase space coordinates for the homogeneous solutions.

solution is labeled by λ^a $(a = 2, 3, \cdots, 2n)$

- **Metric perturbation:** (expanded in spherical harmonics)

$$ds^2 = a^2(\eta)\Big[-(1+2A\,Y)d\eta^2 - 2B\,Y_j\,d\eta dx^j + \Big((1+2H_L Y)\delta_{ij} + 2H_T Y_{ij}\Big)dx^i dx^j\Big],$$

- **Scalar field perturbation:** $\vec{\phi} \;\to\; \vec{\phi}(\eta) + \delta\vec{\phi}\,Y$

$$(\overset{(3)}{\Delta} + k^2)Y = 0,\;\; Y_j = -k^{-1}\nabla_j Y,\;\; Y_{ij} = k^{-2}\nabla_i\nabla_j Y + \frac{1}{3}\delta_{ij}Y.$$

- **Perturbed e-folding number $\tilde{N}$:**

$$\tilde{N} = N + \int_{\eta_0}^{\eta}\left(\mathcal{R}' - \frac{1}{3}k\sigma_g\right)Y\,d\eta\,, \qquad \left({}' = \frac{d}{d\eta}\right)$$

$$\mathcal{R} = H_L + \frac{H_T}{3} \;:\; \text{curvature perturbation on } \Sigma(\eta)$$

$$k\sigma_g = H_T' - kB \;:\; \text{shear of } \Sigma(\eta)$$

$$\boxed{\delta N = 0 \text{ for } \mathcal{R}' - \tfrac{1}{3}k\sigma_g = 0 \;\; (H_L' = B = 0)}$$

“Constant e-fold gauge”

- **Super-horizon scale perturbation on $\mathcal{R}' - \frac{1}{3}k\sigma_g = 0$ slices**

(constant e-fold ($\delta N = 0$) gauge at $k^2/a^2 \ll H^2$)

A & $\delta\vec{\phi}$

- $$\frac{\delta H}{H} = -A = \frac{H^2 \vec{\phi}_N \cdot \delta\vec{\phi}_N + V_{|p}\delta\phi^p}{2V} \qquad \Leftarrow \qquad A \text{ is subject to } \delta\vec{\phi}\,.$$
- $$\left[H\frac{d}{dN}\left(H\frac{d}{dN}\right) + 3H^2\frac{d}{dN}\right]\delta\phi^p + V^{|p}{}_{|q}\delta\phi^q + 2V^{|p}A - H^2\phi^p_N(\vec{\phi}_N \cdot \delta\vec{\phi}_N) = 0\,.$$

This turns out to be the same as the equation for $\partial\vec{\phi}/\partial\lambda^\alpha$. Hence,

$$\delta\vec{\phi} = c^\alpha \frac{\partial\vec{\phi}}{\partial\lambda^\alpha}, \quad \lambda^\alpha = \{N, \lambda^a\}\,; \quad \frac{\partial\vec{\phi}}{\partial\lambda^1} \equiv \frac{\partial\vec{\phi}}{\partial N} \ : \ \text{time-translation mode}$$

$\delta\vec{\phi}$ in this gauge is completely decsribed by the knowledge of (a congruence of) background solutions.

$\mathcal{R}$ & $k\sigma_g$

From $\delta G^0{}_j = \delta T^0{}_j$ and traceless part of $\delta G^i{}_j = \delta T^i{}_j$,

$$\mathcal{R} = c^\alpha W_{(\alpha)} \int_{N_b}^{N} \frac{dN}{a^3 H}, \qquad k\sigma_g = \frac{3c^\alpha W_{(\alpha)}}{a^2};$$

$$W_{(\alpha)} = \frac{a^3 H^3}{2V}\left(\frac{d\vec{\phi}_N}{dN}\cdot\vec{\chi}_{(\alpha)} - \vec{\phi}_N \cdot \frac{d\vec{\chi}_{(\alpha)}}{dN}\right) = \text{const.},$$

$$\vec{\chi}_{(\alpha)} \equiv \frac{\partial\vec{\phi}}{\partial\lambda^\alpha}$$

where $W_{(1)} = 0$ and N_b is an arbitrary constant.

(One can always choose λ^a such that $W_{(2)} \neq 0$ and $W_{(a)} = 0$ for $a = 3 \sim 2n$.)

⋆ Perturbation in the constant e-fold gauge is parametrized by $2n+1$ parameters $\{N_b\,, c^\alpha\}$:

$$\delta\vec{\phi} = c^\alpha \frac{\partial\vec{\phi}}{\partial\lambda^\alpha}, \quad A = A(\delta\vec{\phi}), \quad \mathcal{R} = c^\alpha W_{(\alpha)} \int_{N_b}^{N} \frac{dN}{a^3 H}, \quad k\sigma_g = \frac{3c^\alpha W_{(\alpha)}}{a^2}.$$

$$\Downarrow$$

one gauge degree of freedom.

- **Gauge mode**

An infinitesimal change of time slicing:

$$N \longrightarrow N - \delta_g N\,,$$
$$\mathcal{R} \longrightarrow \mathcal{R} + \delta_g N\,, \qquad k\sigma_g \longrightarrow k\sigma_g + O(k^2)\,,$$
$$\delta\vec{\phi} \longrightarrow \delta\vec{\phi} + \vec{\phi}_N \delta_g N\,.$$

The condition $\mathcal{R}' - \frac{1}{3}k\sigma_g = 0$ is maintained for $\delta_g N =$ const. in the limit $k^2/a^2 \ll H^2$.

$\Rightarrow$ $(\delta\vec{\phi}\,, \mathcal{R}\,, k\sigma_g) = (c\,\vec{\phi}_N\,, c\,, 0)$ is pure gauge.

Time-Translation Mode

$\Downarrow$

Necessary to construct gauge-invariant quantities such as $\mathcal{R}_c$.

- **Construction of $\mathcal{R}_c$**

Comoving slice condition $\vec{\phi}_N \cdot (\delta\vec{\phi})_c = 0$ defines a surface in the phase space as

$$\vec{\phi}_N(N,0) \cdot \Big(\vec{\phi}(N + \Delta N, \lambda^a) - \vec{\phi}(N,0)\Big) = \vec{\phi}_N(N,0) \cdot \Big(\delta\vec{\phi} + \vec{\phi}_N \Delta N\Big) = 0.$$

$$\Rightarrow \quad \Delta N(\lambda^\alpha) = -\frac{\vec{\phi}_N \cdot \delta\vec{\phi}}{\vec{\phi}_N^2}$$

ΔN depends on x^μ only through its dependence on λ^α.

- Gauge transformation to the comoving slice:

$$\mathcal{R}_c(\lambda^\alpha) = \mathcal{R}(\lambda^\alpha) + \Delta N(\lambda^\alpha)\,; \qquad \lambda^\alpha = \lambda^\alpha(x^\mu)$$
$$= \mathcal{R} - \frac{\vec{\phi}_N \cdot \delta\vec{\phi}}{\vec{\phi}_N^2} = -\frac{\vec{\phi}_N \cdot \vec{\chi}_F}{\vec{\phi}_N^2}\,;$$
$$\vec{\chi}_F := \delta\vec{\phi} - \vec{\phi}_N \mathcal{R}$$
$$= c^1\,\vec{\phi}_N + c^a\left(\frac{\partial\vec{\phi}}{\partial\lambda^a} - W_{(a)}\,\vec{\phi}_N \int_{N_b}^{N} \frac{dN}{a^3 H}\right).$$

$\vec{\chi}_F \sim \delta\vec{\phi}$ on flat slice (contains all the information)

$\cdots$ convenient quantity for evaluating quantum fluctuations

⋆ Change of N_b is absorbed by the redefinition of c^1.

- $c^1 \sim$ adiabatic growing mode amplitude
- $c^a W_{(a)}$ $(= c^2 W_{(2)}$ with $W_{(a)} = 0$ for $a \geq 3)$

  $\sim$ adiabatic decaying mode amplitude
- The rest $(c^a;\, a \geq 3)$ are entropy perturbations

Generally $\mathcal{R}_c$ varies in time even on super-horizon scales

- Slow roll limit:

$$\dot{\vec{\phi}}_N^2 \ll 1, \qquad \left|\frac{D\vec{\phi}_N}{dN}\right| \ll |\vec{\phi}_N| \quad \Rightarrow \quad \phi_N^p = -\frac{V^{|p}}{3H^2} = -(\ln V)^{|p}, \quad H^2 = \frac{1}{3}V\,.$$

2n-d phase space $\Rightarrow$ n-d configuration space

⋆ Slow roll kills all the decaying modes:

$\delta\vec{\phi} = \vec{\chi}_F\,,\ \mathcal{R} = k\sigma_g = 0$ on $\delta N = 0$ slice

$\delta N = 0$ slice becomes equivalent to flat ($\mathcal{R} = 0$) slice in slow-roll limit.

$$\Rightarrow \quad \mathcal{R}_c = \Delta N = -\frac{\vec{\phi}_N \cdot \vec{\chi}_F}{\vec{\phi}_N^2} \quad \text{where} \quad \vec{\chi}_F = C^\alpha \frac{\partial\vec{\phi}}{\partial\lambda^\alpha} \quad (\alpha = 1 \sim n)$$

⋆ $\mathcal{R}_c$ expressed in terms of $(\vec{\chi}_F)_{N=N_k}$ (N_k: horizon crossing time)

$$\mathcal{R}_c(N) = -\left[\frac{\partial N}{\partial\phi^p}\chi_F^p\right]_{N=N_k} - \frac{C^a}{\vec{\phi}_N^2}\,\vec{\phi}_N \cdot \frac{\partial\vec{\phi}}{\partial\lambda^a}\,; \quad C^a = \left[\frac{\partial\lambda^a}{\partial\phi^p}\chi_F^p\right]_{N=N_k} \quad (a = 2 \sim n)$$

$\vec{\chi}_F|_{N=N_k}$: to be evaluated by quantization

⋆ $\mathcal{R}_c = \Delta N$ at the end of inflation ($H^2 = \frac{1}{3}V$ =const. at $N = N_f$)

$$\vec{\phi}_N \cdot \frac{\partial\vec{\phi}}{\partial\lambda^a} = -\frac{\partial \ln V}{\partial\lambda^a} = 0 \quad \Rightarrow \quad \boxed{\mathcal{R}_c(N_e) = -\left[\frac{\partial N}{\partial\phi^p}\chi_F^p\right]_{N_k} = -\left[\frac{\partial N}{\partial\phi^p}\delta\phi^p\right]_{N_k}}$$

§5. Summary

- **Super-horizon scale perturbations are described solely by (a congruence of) homogeneous background solutions.**
  - ⋆ **Correlations among various quantities can be easily calculated.** **(e.g., correlation between entropy and adiabatic perturbations)**
- **Curvature perturbation $\mathcal{R}_c$ may vary in time on super-horizon scales either for multi-field or non-slow-roll inflation.**
  - ⋆ $\mathcal{R}_c \approx \Delta N$ **in the slow-roll case.**
  - ⋆ **Large enhancement on super-horizon scales can occur even for single-field inflation.**

$$\Downarrow$$

  - · **Simple (slow-roll) models predict scale-invariant spectrum, but other, more complicated spectral shapes are possible.**
- **Tensor perturbations may be non-negligible.**

On-going and future observations

- SDSS $\cdots$ $\sim 10^6$ galaxies
- MAP, PLANK $\cdots$ CMB anisotropy map with resolution of $\theta \lesssim 10'$

$\Downarrow$

Inflaton potential may be determined

$\Downarrow$

Understanding of physics of the early universe ($\approx$ extreme high energy physics)