Gravitational self force

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- Difficulties with the description of the self-force in terms of the "tail part" of the field.
- Description of the self-force in terms of the "Singular" and "Regular" fields.
- A "Poor Man's" treatment of the self-force for the gravitational field.

Difficulties with the description of the self force via the "tail" part of the field:

Maxwell's equation

$$\nabla^2 A^a - R^a{}_b A^b = -4\pi J^a. \tag{1}$$

With the decomposition

$$A_a^{\mathsf{ret}} \equiv A_a^{\mathsf{dir}} + A_a^{\mathsf{tail}}, \tag{2}$$

or

$$A_a^{\mathsf{tail}} = A_a^{\mathsf{ret}} - A_a^{\mathsf{dir}}, \tag{3}$$

DeWitt and Brehme show that the electromagnetic self force is

$$F^a = q g^{ac}(\nabla_c A_b^{\mathsf{tail}} - \nabla_b A_c^{\mathsf{tail}})u^b. \tag{4}$$

However:

- If $(R_{ab} - \frac{1}{6}g_{ab}R)u^b \neq 0$, then A_a^{tail} is not differentiable at the particle and averaging around a two-sphere is required.
- For A^a_{tail},
$$\nabla^2 A^a_{\mathsf{tail}} - R^a{}_b A^b_{\mathsf{tail}} \equiv -4\pi J^a_{\mathsf{tail}} \neq 0,$$
but an observer near the charge would observe no physical charge distribution J^a_{tail}.
- A^a_{tail} is a convenient expression which may be used to describe the self-force, but it does not describe an actual electromagnetic field.
- We conclude that the DeWitt-Brehme construction does not describe the self-force in terms of the charge interacting with an electromagnetic field.

- Resolution by use of a new Green's function of curved spacetime, $G^{\mathsf{S}}(x,z)$, for a scalar field and similar Green's functions for electromagnetic and gravitational fields.

$$A^a_{\mathsf{ret}} = A^a_{\mathsf{S}} + A^a_{\mathsf{R}}$$

or

$$A^a_{\mathsf{R}} \equiv A^a_{\mathsf{ret}} - A^a_{\mathsf{S}}$$

- A^a_{ret} and A^a_{S} satisfy the *same* differential equation with the same source.
- Thus, A^a_{R} is a vacuum solution to the field equation and is also guaranteed to be differentiable.
- Once we find A^a_{ret} and A^a_{S}, we no longer require the Green's function and may change these fields to any other gauge.
- In terms of A^a_{R} the self-force is

$$F^a = qg^{ac}(\nabla_c A^{\mathsf{R}}_b - \nabla_b A^{\mathsf{R}}_c)u^b. \tag{5}$$

- What is this new Green's function and what are the features of the corresponding ψ^{S}, A^a_{S} and h^{S}_{ab}?
- In principle, the Hadamard expansion of the Green's function answers these questions.
- We find a "Poor Man's" approach to the self-force for gravity, (with no use of Green's functions or the Hadamard expansion) to be quite helpful.
- Gravity is simpler than the other fields because the dynamical equations follow from the Einstein field equations via the Bianchi identity.
  - No mass renormalization is required.
- For simplicity, we consider gravity with a small black hole of mass μ moving in a background gravitational field with a length scale $\mathcal{R}$ with $\mu \ll \mathcal{R}$.

- Consider the perturbation of a geometry $\overset{\circ}{g}_{ab}$ by a source $T^{(1)}_{ab} = O(\mu)$.

$$E_{ab}(h) = -8\pi T^{(1)}_{ab} + O(\mu^2), \tag{6}$$

where

$$\begin{aligned} 2E_{ab}(h) &= \nabla^2 h_{ab} + \nabla_a \nabla_b h - 2\nabla_{(a}\nabla^c h_{b)c} \\ &\quad +2R_a{}^c{}_b{}^d h_{cd} + g_{ab}(\nabla^c \nabla^d h_{cd} - \nabla^2 h). \end{aligned} \tag{7}$$

After solving this equation for h_{ab}, it follows that

$$G_{ab}(g+h) = 8\pi T^{(1)}_{ab} + O(\mu^2). \tag{8}$$

- An integrability condition for the existence of a solution to $E_{ab}(h) = -8\pi T^{(1)}_{ab}$ is that

$$\nabla^a T^{(1)}_{ab} = O(\mu^2).$$

  For a particle of small mass, the worldline Γ must be close to a geodesic of g°_{ab}.
- Now, try second order perturbation theory.

- Formally, perturbation theory at the second order is no more difficult than at the first order.
- However, the integrability condition for the second order equation is that

$$\nabla^a_{(g^\circ + h^{(1)})} T^{(2)}_{ab} = O(\mu^3).$$

  (See the Appendix for a proof.)

  Thus, after solving the first order equation for $h^{(1)}_{ab}$, $T^{(1)}_{ab}$ must be changed at $O(\mu^2)$, so that $\nabla^a_{g^\circ + h} T^{(2)}_{ab} = O(\mu^3)$. For a particle of small mass, this implies that the worldline Γ must now be adjusted to be close to a geodesic of $g^\circ_{ab} + h^{(1)}_{ab}$. This adjustment is sometimes called the self force.
- But for a particle of small mass, h_{ab} is singular on Γ?

- Rethink the problem after replacing the point mass by a small black hole.
- Consider a small black hole inserted into a smooth, vacuum solution of the Einstein equations g°_{ab} with length and time scale $\mathcal{R}$. (Ultimately, we will put the small black hole into the geometry $g^{\circ}_{ab} + h^{\mathsf{R}}_{ab}$, but disregard this confusion and forget about the self-force for the time being.)

  Normal coordinates near a geodesic provide an expansion of the metric

$$g^{\circ}_{ab} = \eta_{ab} + \frac{1}{2} x^i x^j g^{\circ}_{ab,ij} + O(r^3), \qquad r \to 0, \tag{9}$$

  Cartesian x, y and z :: r, θ, ϕ.

The Thorne, Hartle and Zhang choice of normal coordinates (THZ coordinates) is defined in a neighborhood about a geodesic in a vacuum spacetime.

$$g^{\circ}_{ab} = \eta_{ab} + {}_2H_{ab} + O(r^3/\mathcal{R}^3), \quad r \to 0, \tag{10}$$

$${}_2H_{ab}dx^a dx^b = -\mathcal{E}_{ij}x^i x^j(dt^2 + \delta_{kl}dx^k dx^l) + \frac{4}{3}\epsilon_{kpq}\mathcal{B}^q{}_i x^p x^i dt\, dx^k, \tag{11}$$

$\mathcal{E}$ and $\mathcal{B}$ are spatial, symmetric, tracefree and related to the Riemann tensor and its derivatives evaluated on the geodesic; in particular, $\mathcal{E}_{ij} = R_{titj}$ and $\mathcal{B}_{ij} = \epsilon_i{}^{pq}R_{pqjt}/2$.

- The “2” in ${}_2H_{ab}$ refers to the quadrupole term or $\ell = 2$.
- There is no term in the expansion which is linear in x^i. If there were, then the Christoffel symbols would not vanish on Γ, and Γ would not be a geodesic.

Very low frequency perturbations of a Schwarzschild black hole

Put a small black hole, $\mu \ll \mathcal{R}$, on the geodesic.
The metric is perturbed by ${}_2h_{ab}$, where $E^{\mathsf{Schw}}_{ab}({}_2h) = 0$ with ${}_2h_{ab} \to {}_2H_{ab}$ for $\mu \ll r \ll \mathcal{R}$. In the time independent limit, Zerilli (1970) shows us that

$$
\begin{aligned}
{}_2h_{ab}dx^a dx^b \quad &= \\
& -\mathcal{E}_{ij}x^i x^j \left[(1-2\mu/r)^2 dt^2 + dr^2 + (r^2 - 2\mu^2)(d\theta^2 + \sin^2\theta d\phi^2)\right] \\
& \qquad + \frac{4}{3}\epsilon_{kpq}\mathcal{B}^q{}_i x^p x^i (1-2\mu/r) dt\, dx^k
\end{aligned}
\tag{12}
$$

- The black hole is "at rest" on the geodesic, from its own perspective.
- The black hole is acted upon by no external force to move it off the geodesic.

- If you are concerned about the absence of a dipole term in the above expansions: A dipole ($\ell = 1$) distortion of a black hole implies that the black hole does not sit at the center of the coordinate system. But, a dipole term would also imply that the worldline is NOT a geodesic. A coordinate transformation that removed the dipole perturbation of the black hole, would also change the worldline to a geodesic.

Do asymptotic matching in the region $\mu \ll r \ll \mathcal{R}$:

- Far from the black hole $\mu \ll r$, the geometry is

$$\begin{aligned} g^{\circ}_{ab} dx^a dx^b &= \eta_{ab} dx^a dx^b \\ &\quad - \mathcal{E}_{ij} x^i x^j (dt^2 + \delta_{kl} dx^k dx^l) + \frac{4}{3} \epsilon_{kpq} \mathcal{B}^q{}_i x^p x^i dt\, dx^k + O(\mu/r). \end{aligned} \tag{13}$$

- Close to the world line $r \ll \mathcal{R}$, the geometry is

$$\begin{aligned} (g^{\mathsf{Schw}}_{ab} + {}_2h_{ab}) dx^a dx^b &= (1 - 2\mu/r) dt^2 + (1 - 2\mu/r)^{-1} dr^2 + r^2 d\Omega^2 \\ &\quad - \mathcal{E}_{ij} x^i x^j \left[(1 - 2\mu/r)^2 dt^2 + dr^2 + (r^2 - 2\mu^2)(d\theta^2 + \sin^2\theta d\phi^2) \right] \\ &\quad + \frac{4}{3} \epsilon_{kpq} \mathcal{B}^q{}_i x^p x^i (1 - 2\mu/r) dt\, dx^k + O(r^3/\mathcal{R}^3). \end{aligned} \tag{14}$$

- In the restricted region $\mu/r \ll r^2/\mathcal{R}^2 \ll 1$ the geometry in Eq. (14) reduces to that in Eq. (14) which implies that the geometries asymptotically match.

A Simple expansion for h^{S}_{ab}

- An expansion of h^{S}_{ab} about the worldline of the black hole includes all terms in Eq. (14) which are linear in μ. These terms are singular or nondifferentiable, but exert no "force" on the black hole itself—the black hole is moving along a geodesic of the background geometry.
- Thus, in THZ coordinates

$$
\begin{aligned}
h^{\mathsf{S}}_{ab} dx^a dx^b \quad &= \quad \frac{2\mu}{r}(dt^2 + dr^2) \\
&\quad + \frac{4\mu}{r}\mathcal{E}_{ij}x^i x^j dt^2 - \frac{8\mu}{3r}\epsilon_{kpq}\mathcal{B}^q{}_i x^p x^i dt\, dx^k \\
&\quad + O(\mu r^2/\mathcal{R}^3).
\end{aligned}
$$

- The $\mu x^i x^j \mathcal{E}_{ij}/r$ term and the similar $B^q{}_i$ terms in h^{S}_{ab} result from the tidal distortion of the hole's monopole field by the background geometry, and are not differentiable on the worldline; these must be included in h^{S}_{ab} in order that the resulting h^{R}_{ab} is differentiable.
- For calculating the self force effect on the worldline, it is sufficient to use just these terms to approximate h^{S}_{ab}. Higher order terms in the expansion all have vanishing first derivative on the world line.
- "Simple" expressions for the $O(\mu r^2/\mathcal{R}^3)$ terms in h^{S}_{ab}, in THZ coordinates, have also been found. (SD, 2000)
- Careful reflection shows that actual geometry is $g^{\circ}_{ab} + h^{\mathsf{ret}}_{ab} = g^{\circ} + h^{\mathsf{R}}_{ab} + h^{\mathsf{S}}_{ab}$, that $\mathcal{E}_{ij}$ and $\mathcal{B}_{ij}$ are actually the multipole moments of the perturbed geometry $g^{\circ} + h^{\mathsf{R}}_{ab}$ (which is a solution of the Einstein Equations up to $O(\mu^2)$, and that the small black hole must move along a geodesic of $g^{\circ} + h^{\mathsf{R}}_{ab}$ with corrections of $O(\mu^2)$.

Appendix
General Perturbation Analysis

A particle of small mass, μ, which interacts with a black hole changes the metric, $g_{ab} \rightarrow g_{ab} + h^{(1)}_{ab}$, with $h^{(1)}_{ab} = O(\mu)$.

First we define $E_{ab}(h)$, a linear, second order differential operator on symmetric, two-indexed tensors, by

$$E_{ab}(h) \equiv -\frac{\delta G_{ab}}{\delta g_{cd}} h_{cd}, \tag{15}$$

where G_{ab} is the Einstein tensor of g_{ab}, so that

$$\begin{aligned} 2E_{ab}(h) \quad &\equiv \quad \nabla^2 h_{ab} + \nabla_a \nabla_b h - 2\nabla_{(a}\nabla^c h_{b)c} \\ &\quad\quad +2R_a{}^c{}_b{}^d h_{cd} + g_{ab}(\nabla^c \nabla^d h_{cd} - \nabla^2 h) \end{aligned} \tag{16}$$

where $h \equiv h_{ab} g^{ab}$ and ∇_a and $R_a{}^c{}_b{}^d$ are the derivative operator and Riemann tensor of g_{ab}

In general perturbation analysis, let the g_{ab} of eqn. (16) be an exact solution to the vacuum Einstein equations, g^0_{ab}, and iteratively define

$$g^{(n)}_{ab} = g^{(n-1)}_{ab} + h^{(n)}_{ab} \tag{17}$$

where

$$h^{(n)}_{ab} = O(\mu^n). \tag{18}$$

Assume that we are given $g^{(n-1)}_{ab}$ and $T^{(n)}_{ab} = O(\mu)$, with

$$2G^{(n-1)}_{ab} - 16\pi T^{(n)}_{ab} = O(\mu^n). \tag{19}$$

Now, if we find $h^{(n)}_{ab}$ from

$$E_{ab}(h^{(n)}) = 2G^{(n-1)}_{ab} - 16\pi T^{(n)}_{ab} + O(\mu^{n+1}). \tag{20}$$

then it will follow that

$$2G^{(n)}_{ab} - 16\pi T^{(n)}_{ab} = O(\mu^{n+1}). \tag{21}$$

and we will have iteratively improved our approximate solution to the Einstein equations.

But, it follows easily from eqn. (16) that

$$\nabla^a E_{ab}(h) = 0 \tag{22}$$

for any tensor field h_{ab}; this is a consequence of the Bianchi identity. Thus an integrability condition of eqn. (20) is that

$$\nabla^a (2G_{ab}^{(n-1)} - 16\pi T_{ab}^{(n)}) = O(\mu^{n+1}). \tag{23}$$

In other words, before eqn. (20) can be solved for $h_{ab}^{(n)}$, it is necessary to be certain that the perturbing stress tensor satisfies eqn. (23).

Note,

$$\begin{aligned} \nabla^a (2G_{ab}^{(n-1)} - 16\pi T_{ab}^{(n)}) \quad &= \quad \nabla^a_{(n-1)} (2G_{ab}^{(n-1)} - 16\pi T_{ab}^{(n)}) \\ &\quad + \Gamma^a_{ac} (2G_b^{(n-1)c} - 16\pi T_b^{(n)c}) \\ &\quad - \Gamma^c_{ab} (2G_c^{(n-1)a} - 16\pi T_c^{(n)a}), \end{aligned} \tag{24}$$

where $\nabla^a_{(n-1)}$ is the derivative operator of $g^{(n-1)}_{ab}$, and Γ^a_{bc} is the connection relating the derivative operators ∇^a and $\nabla^a_{(n-1)}$. But,

$$\nabla^a_{(n-1)} G^{(n-1)}_{ab} = 0 \tag{25}$$

from the Bianchi identity, and the terms in eqn. (24)involving Γ^a_{bc} are order μ^{n+1} because of eqn. (19) and the fact that $\Gamma^a_{bc} = O(\mu)$. Thus the right hand side of Eq. (24) is

$$\nabla^a_{(n-1)} T^{(n)}_{ab} + O(\mu^{n+1}). \tag{26}$$

It follows, now, that the integrability condition for eqn. (20) is

$$\nabla^a_{(n-1)} T^{(n)}_{ab} = O(\mu^{n+1}). \tag{27}$$

In other words, before solving for the nth order metric perturbation $h_{ab}^{(n)}$, it is first necessary to adjust the perturbing stress tensor so that it is conserved with the metric $g_{ab}^{(n-1)}$. If this result seems confusing, it might help to recall that $T_{ab}^{(n)} = O(\mu)$. Thus, if the stress-tensor were that of a point particle, then the integrability condition requires that the particle's world line be a geodesic of $g_{ab}^{(n-1)}$, in order to find the correction $h^{(n)}$ which is $O(\mu)$.