Capra Meeting 6 (6/23-6/25/2003)

Radiative Self-force and its Calculation

Index

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Radiative Self-force and its Calculation

1: Introduction

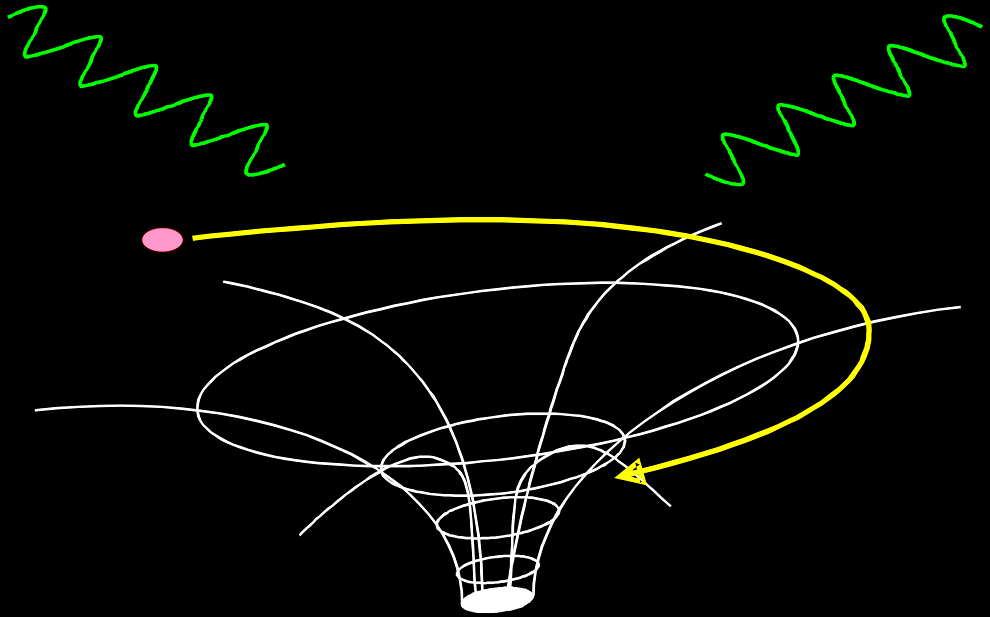
We want to calculate the gravitational wave form from an extreme mass-ratio binary system for LISA project.

$$\frac{M_{\mathrm{small}}}{M_{\mathrm{large}}} \approx 10^{-6}$$

The central black hole is considered to be a Kerr black hole. For its extreme mass-ratio, we expect that a linear perturbation is an effective method of investigation.

Radiative Self-force and its Calculation

- One can calculate the gravitational wave form by a linear perturbation, given an orbital evolution of the binary system.
- We need the orbital evolution of 10^5 (10^8) cycles for one-year observation of gravitational waves.

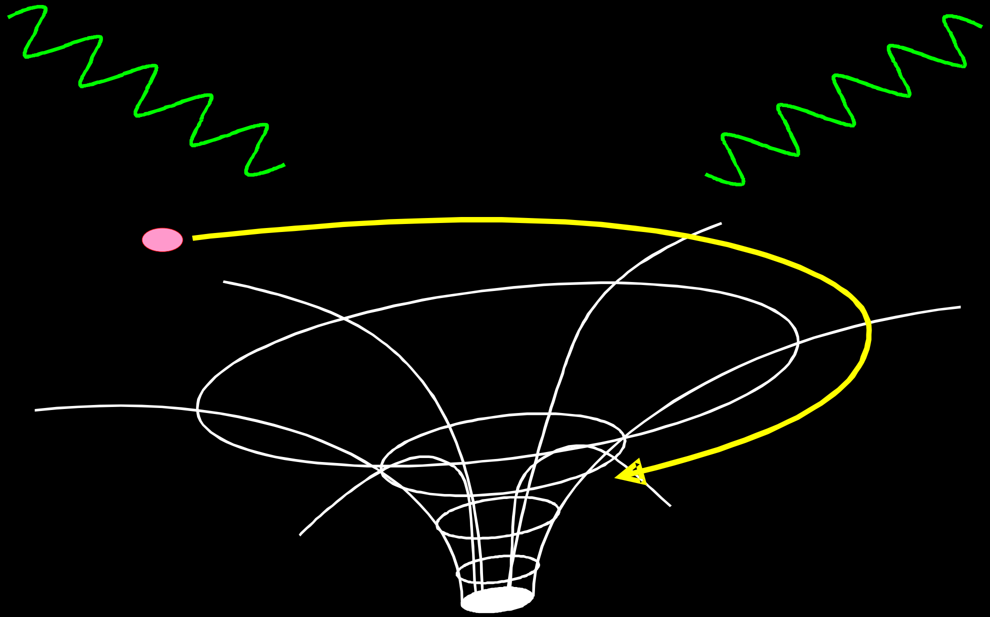


Beyond 10^3 cycles, the orbit deviates from a geodesic by the secular effect of radiation reaction.

Radiative Self-force and its Calculation

We want to know the evolution equation of the orbit, namely, we want to solve "The self-force problem".

We can use the linear perturbation with a geodesic source since the instantaneous deviation from a geodesic is small (~10^-6).



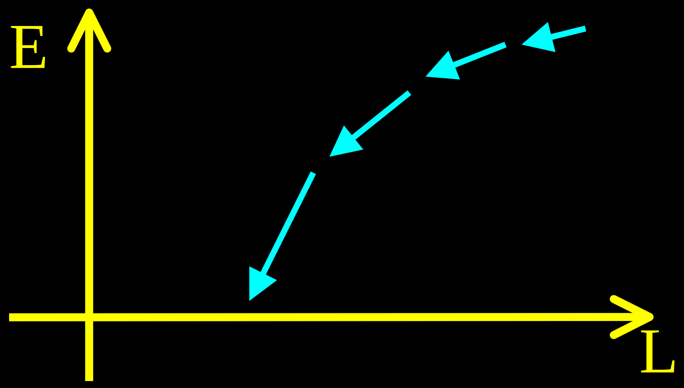
$$\boxed{\frac{D}{d\tau}V^{\mu} = ?}$$

Radiative Self-force and its Calculation

We re-consider “Poor-man’s method”… really poor?

A calculation by the “energy balance” equation;

- We approximate the orbit at a given instant by a geodesic of constants (E,L,K).
- Instead of integrating the orbital equation, we consider the evolution of these constants.



$$\left\langle \frac{dE}{d\tau} \right\rangle = \sum_{LM\omega} \frac{\left| Z_{LM\omega} \right|^2}{4\pi\omega^2}$$

$$\left\langle \frac{dL}{d\tau} \right\rangle = \sum_{LM\omega} \frac{M\left| Z_{LM\omega} \right|^2}{4\pi\omega^3}$$

Radiative Self-force and its Calculation

Geodesics around a Kerr black hole are characterized by 6 constants;

$$\boxed{\frac{D}{d\tau}\frac{d}{d\tau}z^{\alpha}(\tau)=0} \;\Leftarrow\; \boxed{\left\{\frac{d}{d\tau}z^{i}(\tau_0), z^{i}(\tau_0)\right\}}$$

1. We consider an evolution equation of (E,L,K) based on our understanding of the self-force.
2. By integrating the orbital equation perturbatively, we derive the evolution of the rest of constants.
3. We derive the “adiabatic evolution” of the orbit.

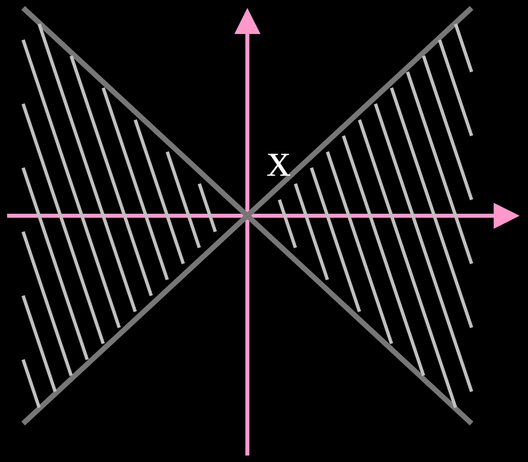
Radiative Self-force and its Calculation

We use a Green function regularization to calculate the self-force.

$$\boxed{G^{ret.}(x,z) = G^{S}(x,z) + G^{R-ret.}(x,z)}$$

$\boxed{G^{S}(x,z)}$

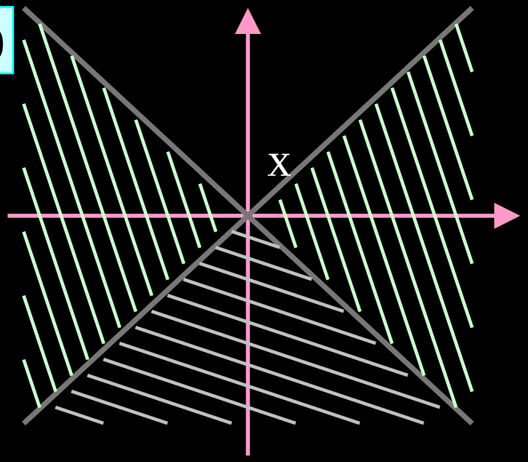
Inhomogeneous Green function



Singular, Symmetric, …..

$\boxed{G^{R-ret.}(x,z)}$

Homogeneous Green function



Regular, Radiative, Residual, ….

The self-force is described by the R-part of the Green function.

$$\boxed{F_{\alpha} = F_{\alpha}\left[\phi^{R-ret.}\right]} \quad \boxed{\phi^{R-ret.} = \int d\tau G^{R-ret.}(x, z(\tau)) J(\tau)}$$

Radiative Self-force and its Calculation

2: Geodesic

$$\left(\frac{dr}{d\lambda}\right)^2 = \left[\left(r^2+a^2\right)E - aL\right]^2 - \Delta\left(r^2+K\right)$$

$$\left(\frac{d\theta}{d\lambda}\right)^2 = -\left(aE\sin\theta - \frac{L}{\sin\theta}\right)^2 - a^2\cos^2\theta + K$$

$$\frac{dt}{d\lambda} = \frac{1}{\Delta}\left(\Sigma^2 E - 2aMrL\right)$$

$$\frac{d\phi}{d\lambda} = \frac{1}{\Delta}\left[2aMrE + \left(\rho^2 - 2Mr\right)\frac{L}{\sin^2\theta}\right]$$

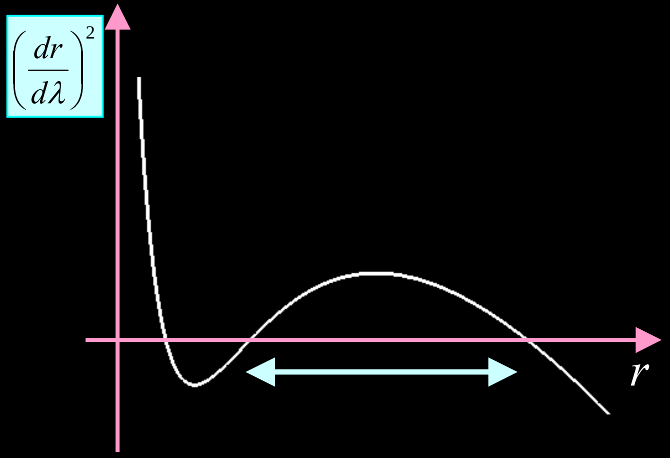
$$\frac{d\tau}{d\lambda} = \rho^2$$

$$\rho^2 = r^2 + a^2\cos^2\theta$$

$$\Delta = r^2 - 2Mr + a^2$$

$$\Sigma^2 = \left(r^2+a^2\right)\rho^2 + 2a^2Mr\sin^2\theta$$

We consider a geodesic rotating without falling into the horizon. *r*- and θ-motion is bounded in a finite domain, and become periodic.



Radiative Self-force and its Calculation

r-motion and θ-motion

$$\left(\frac{dr}{d\lambda}\right)^2 = \left[\left(r^2+a^2\right)E - aL\right]^2 - \Delta\left(r^2+K\right)$$

$$\left(\frac{d\theta}{d\lambda}\right)^2 = -\left(aE\sin\theta - \frac{L}{\sin\theta}\right)^2 - a^2\cos^2\theta + K$$

They are independent periodic motions by bound potentials.

$$r(\lambda) = R(E,L,K;\lambda-\lambda_r) = \sum_n R^{(n)}(E,L,K)e^{in\Omega_r(\lambda-\lambda_r)}$$

$$\theta(\lambda) = \Theta(E,L,K;\lambda-\lambda_\theta) = \sum_n \Theta^{(n)}(E,L,K)e^{in\Omega_\theta(\lambda-\lambda_\theta)}$$

We have two integral constants with freedom to add periods.

$$\left(\lambda_r,\lambda_\theta\right) \rightarrow \left(\lambda_r + n_r 2\pi/\Omega_r, \lambda_\theta + n_\theta 2\pi/\Omega_\theta\right)$$

Radiative Self-force and its Calculation

t-motion and ϕ-motion

$$\frac{dt}{d\lambda} = \frac{1}{\Delta}\left(\Sigma^2 E - 2aMrL\right) = T(r) + T(\theta)$$

$$\frac{d\phi}{d\lambda} = \frac{1}{\Delta}\left[2aMrE + \left(\rho^2 - 2Mr\right)\frac{L}{\sin^2\theta}\right] = \Phi(r) + \Phi(\theta)$$

$$t(\lambda) = T(E, L, K, \overline{\lambda}_r, \overline{\lambda}_\theta; \lambda) + \bar{t} = \dot{T}\lambda + \sum_n \left(T_r^{(n)} e^{in\Omega_r(\lambda - \lambda_r)} + T_\theta^{(n)} e^{in\Omega_\theta(\lambda - \lambda_\theta)}\right) + \bar{t}$$

$$\phi(\lambda) = \Phi(E, L, K, \overline{\lambda}_r, \overline{\lambda}_\theta; \lambda) + \bar{\phi} = \Omega_\phi \lambda + \sum_n \left(\Phi_r^{(n)} e^{in\Omega_r(\lambda - \lambda_r)} + \Phi_\theta^{(n)} e^{in\Omega_\theta(\lambda - \lambda_\theta)}\right) + \bar{\phi}$$

A family of geodesics is characterized by 7(6) constants.

$$E, L, K, \lambda_r, \lambda_\theta, \bar{t}, \bar{\phi}$$

Note: $\lambda_r - \lambda_\theta$: fourth constant of motion

$\Omega_r / \dot{T}, \Omega_\theta / \dot{T}, \Omega_\phi / \dot{T}$: three principal frequencies

3: Symmetry and Self-force induced by a geodesic

We consider the self-force acting on (E,L,K).

$$E = \eta^E_\alpha V^\alpha , \frac{d}{d\lambda} E = \eta^E_\alpha \frac{D}{d\lambda} V^\alpha$$

$$L = \eta^L_\alpha V^\alpha , \frac{d}{d\lambda} L = \eta^L_\alpha \frac{D}{d\lambda} V^\alpha$$

$$K = \frac{1}{2} \eta^K_{\alpha\beta} V^\alpha V^\beta , \frac{d}{d\lambda} K = \eta^K_{\alpha\beta} V^\alpha \frac{D}{d\lambda} V^\beta$$

Radiative Self-force and its Calculation

Transformation law of
4-velocity and self-force

$$v'^{\alpha} = \left(\frac{\partial x'^{\alpha}}{\partial x^{\beta}}\right)\left(\frac{d\tau}{d\tau'}\right)v^{\beta}$$

$$\frac{D}{d\tau'}v'^{\alpha} = \left(\frac{d\tau}{d\tau'}\right)\frac{D}{d\tau}\left[\left(\frac{\partial x'^{\alpha}}{\partial x^{\beta}}\right)\left(\frac{d\tau}{d\tau'}\right)v^{\beta}\right]$$

A) *t*- and ϕ-translation symmetry

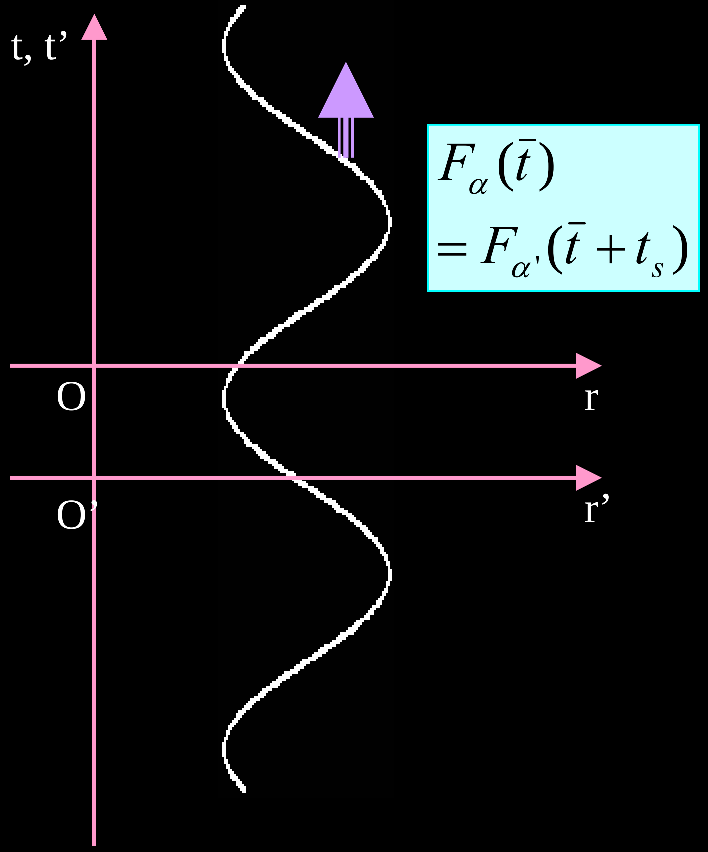
$$t \to t + t_s, r \to r, \theta \to \theta, \phi \to \phi + \phi_s$$

$$F_{\alpha}(E, L, K, \lambda_r, \lambda_{\theta}, \bar{t}, \bar{\phi}; \lambda) = F_{\alpha}(E, L, K, \lambda_r, \lambda_{\theta}, \bar{t} - t_s, \bar{\phi} - \phi_s; \lambda)$$

Self-force is independent on $\bar{t}, \bar{\phi}$

Radiative Self-force and its Calculation

A’) t-(and ϕ-) translation symmetry



$$F_{\alpha}(\bar{t}) \\ = F_{\alpha'}(\bar{t} + t_s)$$

Radiative Self-force and its Calculation

By the freedom $\boxed{(\lambda_r,\lambda_\theta)\rightarrow(\lambda_r+n_r 2\pi/\Omega_r,\lambda_\theta+n_\theta 2\pi/\Omega_\theta)}$

$$\boxed{F_\alpha(E,L,K,\lambda_r,\lambda_\theta;\lambda)=\sum_{m,n}F_\alpha^{(m,n)}(E,L,K;\lambda)\exp[-i(m\Omega_r\lambda_r+n\Omega_\theta\lambda_\theta)]}$$

B) λ-translation symmetry

$$\boxed{\lambda\rightarrow\lambda+\lambda_s}$$

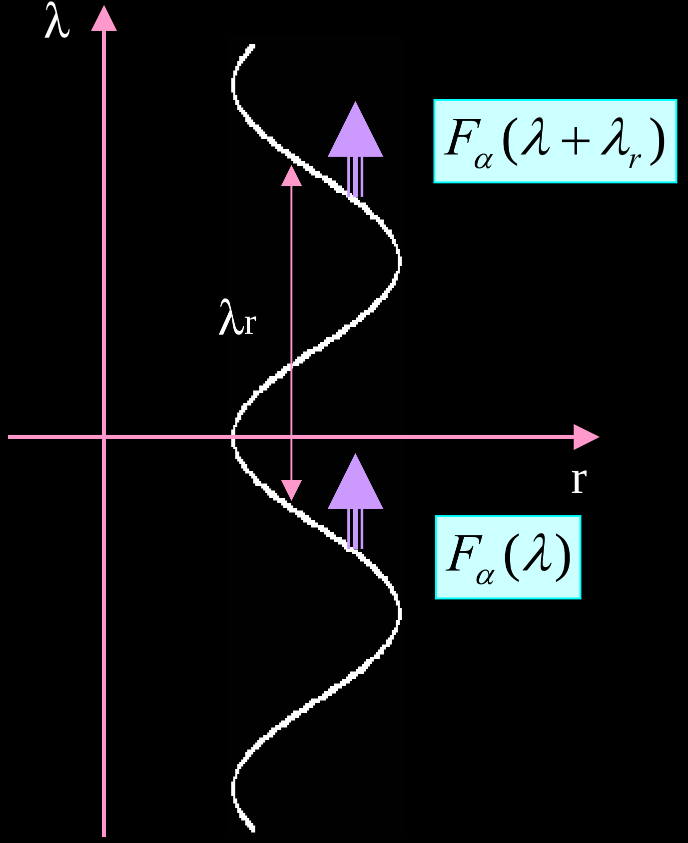
$$\boxed{F^\alpha(E,L,K,\bar{\lambda}_r,\bar{\lambda}_\theta;\lambda)=F^\alpha(E,L,K,\bar{\lambda}_r+\lambda_s,\bar{\lambda}_\theta+\lambda_s;\lambda+\lambda_s)}$$

We choose $\boxed{\lambda_s=-\lambda}$

$$\boxed{F_\alpha(E,L,K,\bar{\lambda}_r,\bar{\lambda}_\theta;\lambda)=\sum_{m,n}F_\alpha^{(m,n)}(E,L,K)\exp[i(m\Omega_r(\lambda-\lambda_r)+n\Omega_\theta(\lambda-\lambda_\theta))]}$$

Radiative Self-force and its Calculation

B') λ-translation symmetry



Since we have two independent periodicity in r and θ, the exact statement is more complicated.

Radiative Self-force and its Calculation

By using t-, ϕ-, λ-translation, one can, in general, have

$$\boxed{\lambda_r = \lambda_\theta = \bar{t} = \bar{\phi} = 0}$$

Note:

This does not mean, we do not need the evolution equation of these constants by the self-force. One can set so once, but, one cannot at the later stage of evolution.

In fact, λr describes the peri-astron advance, and λθ describes the precession of the rotation plane.

Radiative Self-force and its Calculation

C) Geodesic Preserving Symmetry (GPS)

$$t \to -t, r \to r, \theta \to \theta, \phi \to -\phi, (\lambda \to -\lambda)$$

Geodesic :

$$\{E, L, K, \lambda_r, \lambda_\theta, \bar{t}, \bar{\phi}\} \to \{E, L, K, -\lambda_r, -\lambda_\theta, -\bar{t}, -\bar{\phi}\}$$

Self-force vector :

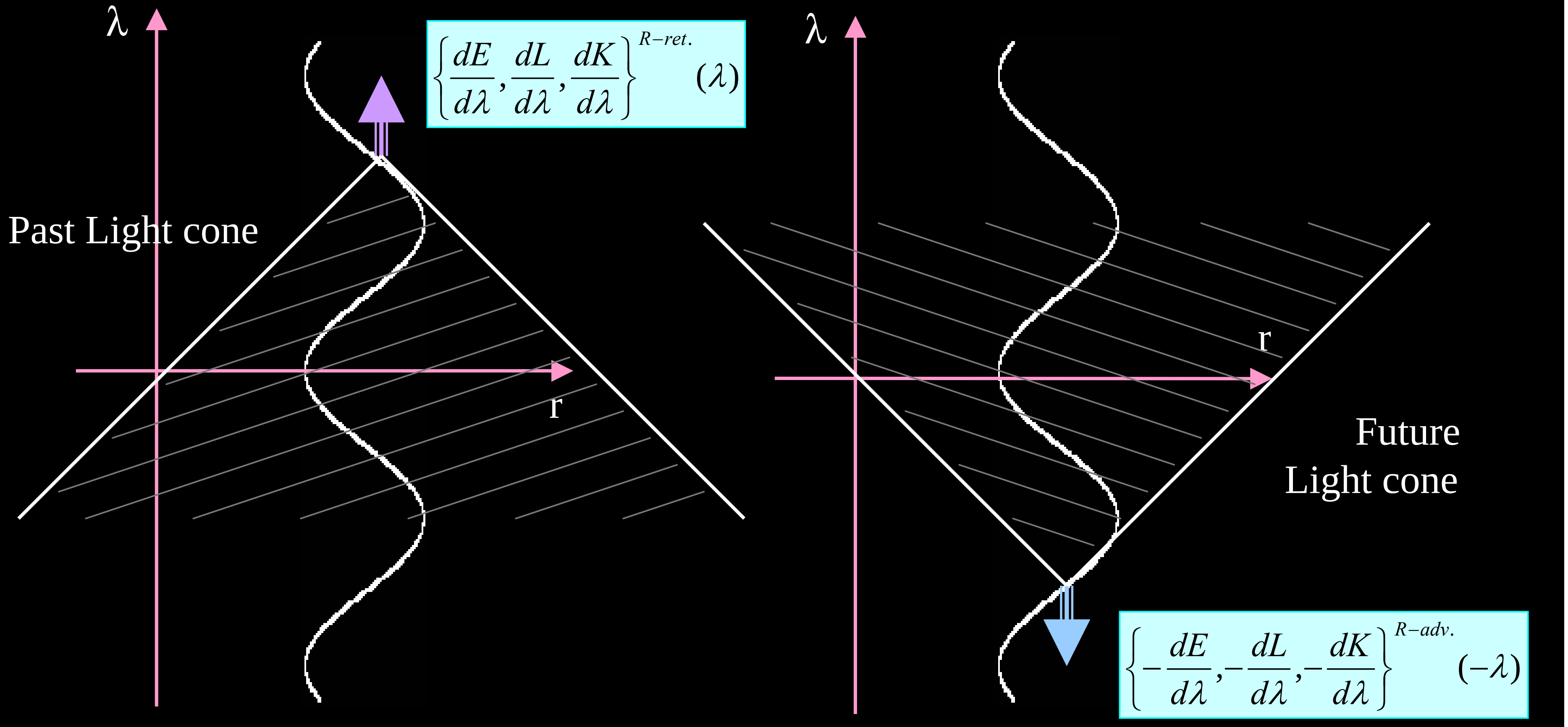
$$\begin{aligned} &\{F_t, F_r, F_\theta, F_\phi\}^{R-ret.}(E, L, K, \lambda_r, \lambda_\theta, \bar{t}, \bar{\phi}; \lambda) \\ &= \{-F_t, F_r, F_\theta, -F_\phi\}^{R-adv.}(E, L, K, -\lambda_r, -\lambda_\theta, -\bar{t}, -\bar{\phi}; -\lambda) \end{aligned}$$

Self-force on the “constants” :

$$\left\{\frac{dE}{d\lambda}, \frac{dL}{d\lambda}, \frac{dK}{d\lambda}\right\}^{R-ret.}(\lambda) = \left\{-\frac{dE}{d\lambda}, -\frac{dL}{d\lambda}, -\frac{dK}{d\lambda}\right\}^{R-adv.}(-\lambda)$$

Radiative Self-force and its Calculation

C’) GPS transformation



$$\left\{\frac{dE}{d\lambda},\frac{dL}{d\lambda},\frac{dK}{d\lambda}\right\}^{R-ret.}(\lambda)$$

$$\left\{-\frac{dE}{d\lambda},-\frac{dL}{d\lambda},-\frac{dK}{d\lambda}\right\}^{R-adv.}(-\lambda)$$

Radiative Self-force and its Calculation

Using Geodesic Preserving Symmetry, two-point averaged self-force can be derived by a radiative Green function

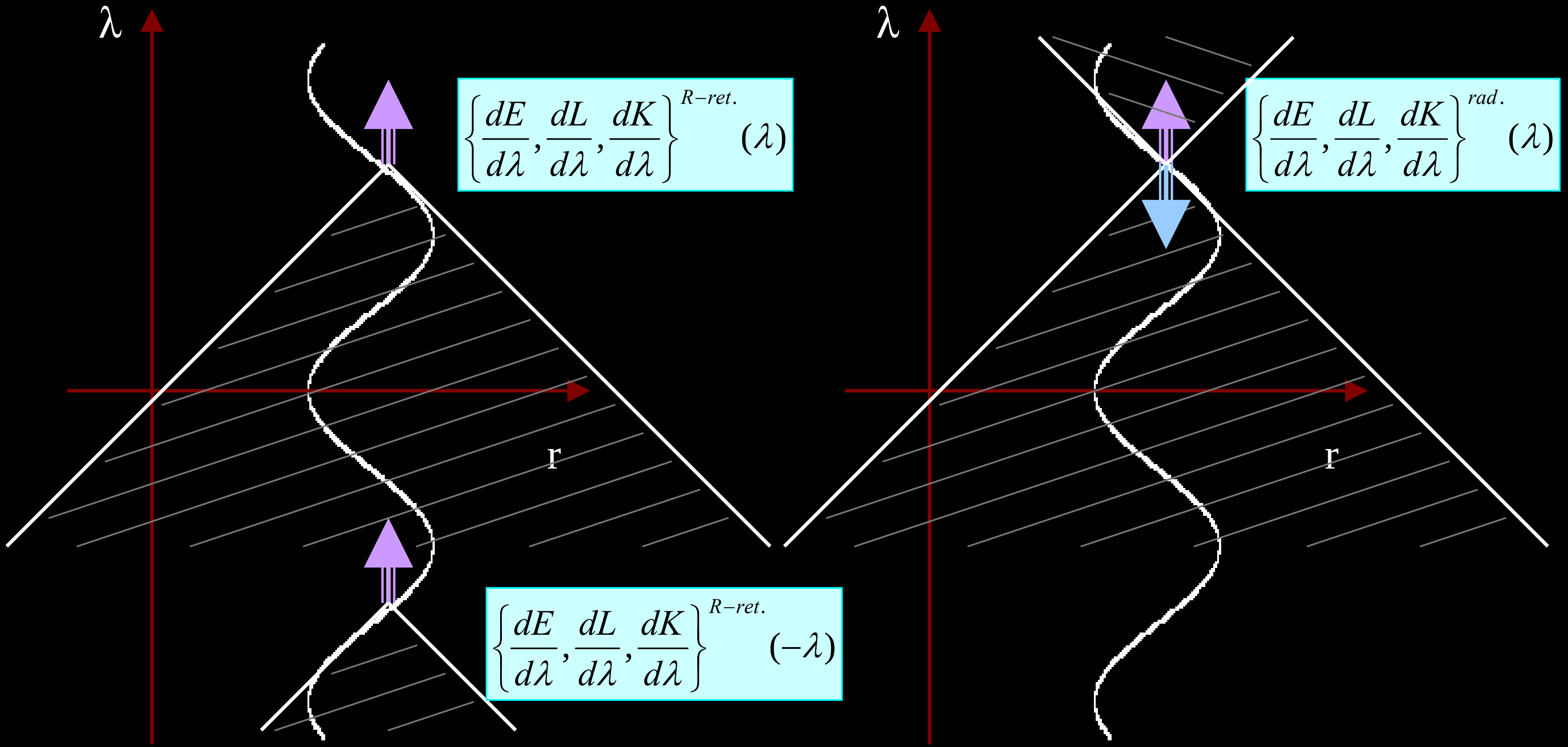
$$\frac{1}{2}\left\{\frac{dE}{d\lambda},\frac{dL}{d\lambda},\frac{dK}{d\lambda}\right\}^{R-ret.}(\lambda)+\frac{1}{2}\left\{\frac{dE}{d\lambda},\frac{dL}{d\lambda},\frac{dK}{d\lambda}\right\}^{R-ret.}(-\lambda)$$
$$=\frac{1}{2}\left\{\frac{dE}{d\lambda},\frac{dL}{d\lambda},\frac{dK}{d\lambda}\right\}^{R-ret.}(\lambda)-\frac{1}{2}\left\{\frac{dE}{d\lambda},\frac{dL}{d\lambda},\frac{dK}{d\lambda}\right\}^{R-adv.}(\lambda)$$
$$=\left\{\frac{dE}{d\lambda},\frac{dL}{d\lambda},\frac{dK}{d\lambda}\right\}^{rad.}(\lambda)$$

Here we use

$$\frac{1}{2}\left(G^{R-ret.}-G^{R-adv.}\right)=\frac{1}{2}\left(\left(G^{ret.}-G^{S}\right)-\left(G^{adv.}-G^{S}\right)\right)=\frac{1}{2}\left(G^{ret.}-G^{adv.}\right)=G^{rad.}$$

Radiative Self-force and its Calculation

2-point averaged self-force



$$\left\{\frac{dE}{d\lambda},\frac{dL}{d\lambda},\frac{dK}{d\lambda}\right\}^{R-ret.}(\lambda)$$

$$\left\{\frac{dE}{d\lambda},\frac{dL}{d\lambda},\frac{dK}{d\lambda}\right\}^{R-ret.}(-\lambda)$$

$$\left\{\frac{dE}{d\lambda},\frac{dL}{d\lambda},\frac{dK}{d\lambda}\right\}^{rad.}(\lambda)$$

Radiative Self-force and its Calculation

We set $\boxed{(\lambda_r, \lambda_\theta) \to (0,0)}$

$$\left\{\frac{dE}{d\lambda}, \frac{dL}{d\lambda}, \frac{dK}{d\lambda}\right\}^{R-ret.} = \sum_{m,n} \left\{\dot{E}, \dot{L}, \dot{K}\right\}^{R-ret.(m,n)} (E, L, K) \exp\left[i\left(m\Omega_r + n\Omega_\theta\right)\lambda\right]$$

$$\left\{\frac{dE}{d\lambda}, \frac{dL}{d\lambda}, \frac{dK}{d\lambda}\right\}^{rad.} = \sum_{m,n} \left\{\dot{E}, \dot{L}, \dot{K}\right\}^{rad.(m,n)} (E, L, K) \exp\left[i\left(m\Omega_r + n\Omega_\theta\right)\lambda\right]$$

Two-point averaged self-force can be read as

$$\frac{1}{2}\left[\left\{\dot{E}, \dot{L}, \dot{K}\right\}^{R-ret.(m,n)} + \left\{\dot{E}, \dot{L}, \dot{K}\right\}^{R-ret.(-m,-n)}\right] = \left\{\dot{E}, \dot{L}, \dot{K}\right\}^{rad.(m,n)}$$

Half of the self-force can be derived by a radiative Green function without a complicated regularization calculation.

Radiative Self-force and its Calculation

One can calculate the time-averaged radiation reaction to E,L,K, by using the radiative Green function.

$$\left\{\dot{E},\dot{L},\dot{K}\right\}^{R-ret.(0,0)} = \left\{\dot{E},\dot{L},\dot{K}\right\}^{rad.(0,0)}$$

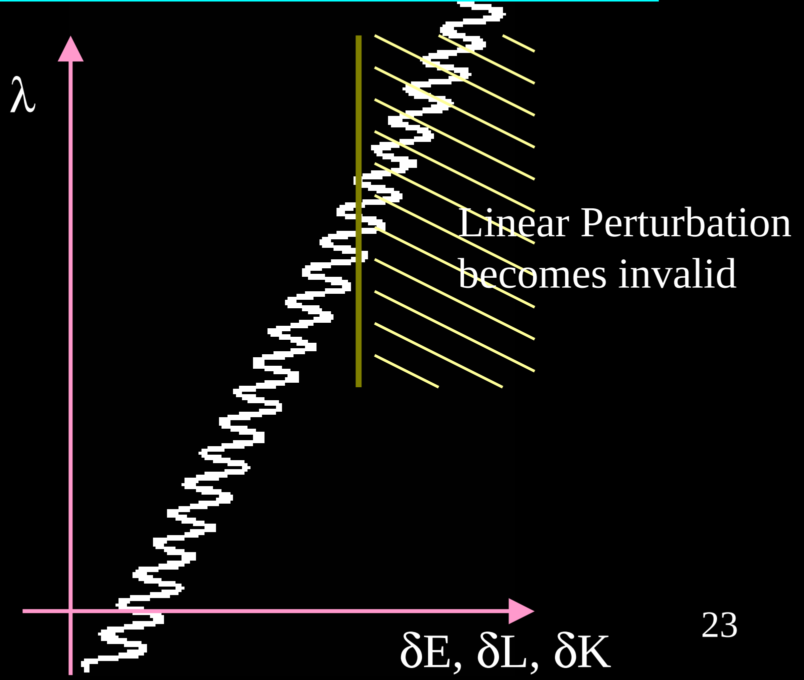
Note:
The same conclusion for energy and angular momentum was derive by Gal’tsov (J.Phy. A: Math. Gen. **15** 3737 (1982)) by showing the equivalence of the balance formula.

4: Perturbative Evolution of an orbit by a self-force

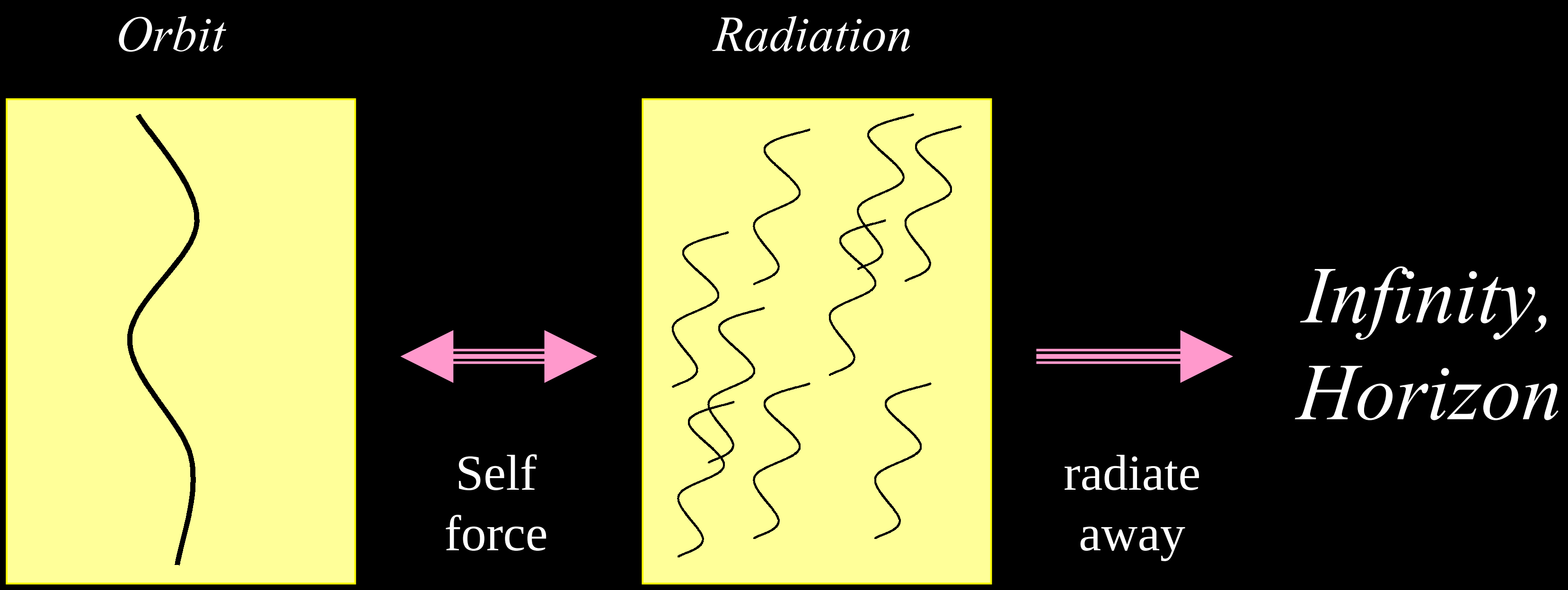
We now obtain a perturbative evolution of E, L, K.

$$\left\{\delta E, \delta L, \delta K\right\} = \left\{\dot{E}, \dot{L}, \dot{K}\right\}^{(0,0)} \lambda + \sum_{m,n} \left\{E, L, K\right\}^{(m,n)} e^{i(m\Omega_r + n\Omega_\theta)\lambda}$$

In the short time scale (of the order of the dynamical time scale), the orbit just exchanges the energy with radiation. In the long time scale, the orbital energy radiates away, and the orbital energy tends to lose.



Radiative Self-force and its Calculation



Orbit

Radiation

Self force

radiate away

Infinity, Horizon

Radiative Self-force and its Calculation

Starting from a geodesic, we perturbatively integrate the orbital equation. We assume the self-force begins to act at λ=0, an we describe an orbital evolution by an evolution of the 'constants'.

constants of the initial geodesic : $\left\{E_0, L_0, K_0, \lambda_{r0}, \lambda_{\theta 0}, \bar{t}_0, \bar{\phi}_0\right\}$

$\Downarrow$ self-force

evolution of the 'orbital constants' :

$$\left\{\begin{matrix} E_0+\delta E, L_0+\delta L, K_0+\delta K, \\ \lambda_{r0}+\delta\lambda_r, \lambda_{\theta 0}+\delta\lambda_\theta, \bar{t}_0+\delta\bar{t}, \bar{\phi}_0+\delta\bar{\phi} \end{matrix}\right\}$$

Radiative Self-force and its Calculation

We formally integrate with varying (E,L,K)

$$\left(\frac{dr}{d\lambda}\right)^2 = \left[\left(r^2+a^2\right)E - aL\right]^2 - \Delta\left(r^2+K\right)$$

$$\left(\frac{d\theta}{d\lambda}\right)^2 = -\left(aE\sin\theta - \frac{L}{\sin\theta}\right)^2 - a^2\cos^2\theta + K$$

$$\frac{dt}{d\lambda} = \frac{1}{\Delta}\left(\Sigma^2 E - 2aMrL\right)$$

$$\frac{d\phi}{d\lambda} = \frac{1}{\Delta}\left[2aMrE + \left(\rho^2 - 2Mr\right)\frac{L}{\sin^2\theta}\right]$$

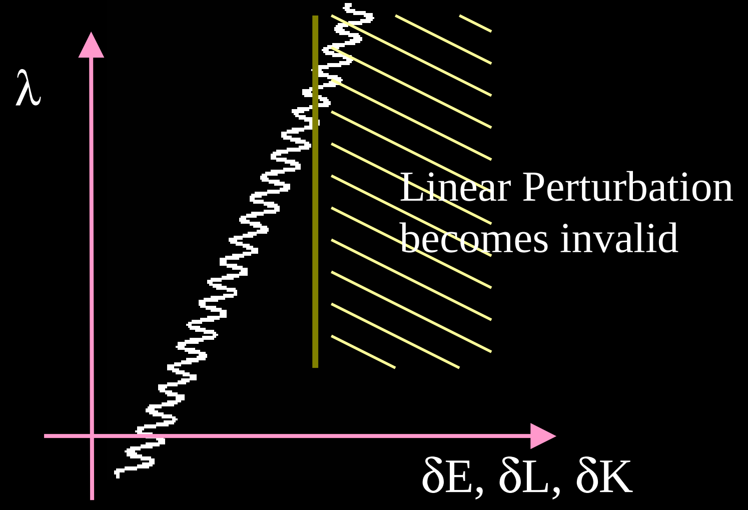
Note:
The perturbed equations have some singular points. By assuming the analyticity of the evolution, one can avoid the singularity.

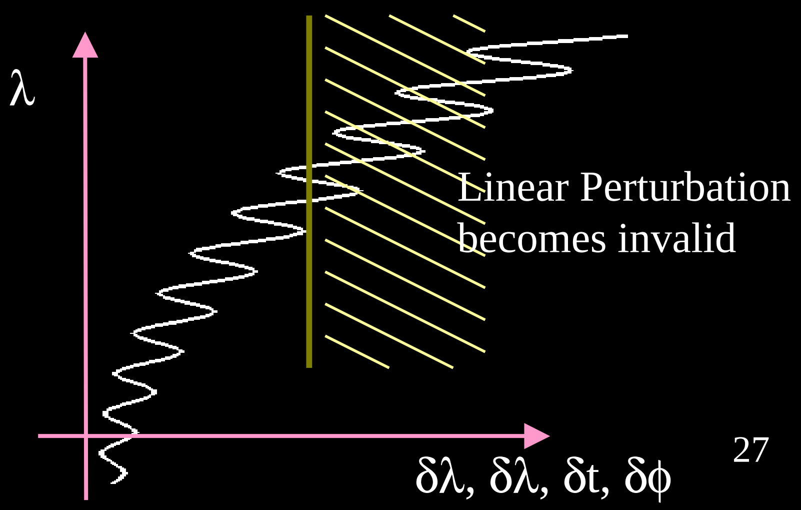
Radiative Self-force and its Calculation

$$\left\{\delta E, \delta L, \delta K\right\} = \left\{\dot{E}, \dot{L}, \dot{K}\right\}^{(0,0)} \lambda + \sum_{m,n} \left\{E, L, K\right\}^{(m,n)} e^{i(m\Omega_r + n\Omega_\theta)\lambda}$$

$$\left\{\delta\lambda_r, \delta\lambda_\theta\right\} = \left\{\ddot{\lambda}_r, \ddot{\lambda}_\theta\right\}^{(0,0)} \frac{\lambda^2}{2} + \sum_{m,n} \left(\left\{\dot{\lambda}_r, \dot{\lambda}_\theta\right\}^{(m,n)} \lambda + \left\{\lambda_r, \lambda_\theta\right\}^{(m,n)}\right) e^{i(m\Omega_r + n\Omega_\theta)\lambda}$$

$$\left\{\delta\bar{t}, \delta\bar{\phi}\right\} = \left\{\ddot{t}, \ddot{\phi}\right\}^{(0,0)} \frac{\lambda^2}{2} + \sum_{m,n} \left(\left\{\dot{t}, \dot{\phi}\right\}^{(m,n)} \lambda + \left\{t, \phi\right\}^{(m,n)}\right) e^{i(m\Omega_r + n\Omega_\theta)\lambda}$$





Radiative Self-force and its Calculation

Linear perturbation is valid at (~10^3 cycles)

$$\boxed{\lambda \approx O(\mu^{\alpha}), \quad 0 \le \alpha < -1/2}$$

Orbital evolution is dominated by the non-oscillating parts as

$$\boxed{\{\delta E, \delta L, \delta K\}^S = \{\dot E, \dot L, \dot K\}^{(0,0)} \lambda}$$

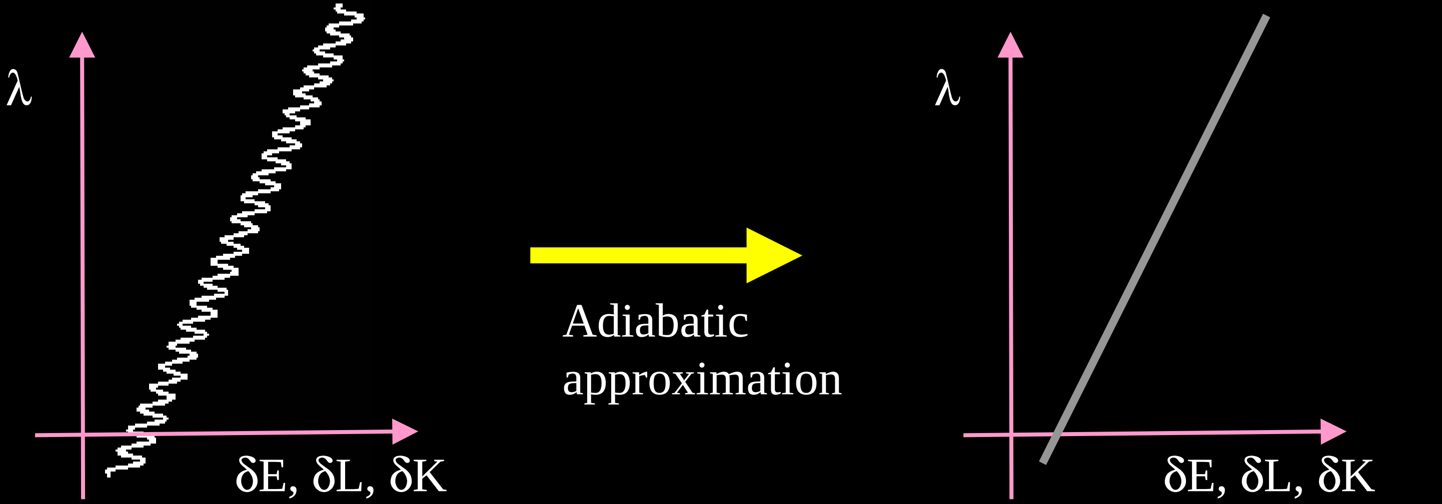
$$\boxed{\{\delta\lambda_r, \delta\lambda_\theta\}^S = \{\ddot\lambda_r, \ddot\lambda_\theta\}^{(0,0)} \frac{\lambda^2}{2}, \quad \{\ddot\lambda_r, \ddot\lambda_\theta\}^{(0,0)} = \left\{\frac{\Omega_{r,i}}{\Omega_r}, \frac{\Omega_{\theta,i}}{\Omega_\theta}\right\} \dot E^{i(0,0)}}$$

$$\boxed{\{\delta\bar t, \delta\bar\phi\}^S = \{\ddot t, \ddot\phi\}^{(0,0)} \frac{\lambda^2}{2}, \quad \{\ddot t, \ddot\phi\}^{(0,0)} = \{\dot T_{,i}, \Omega_{\phi,i}\} \dot E^{i(0,0)}}$$

These dominant parts can be derived by the radiative Green function.

5: "Adiabatic" Evolution

Commonly(?), the adiabatic approximation is understood as a prescription to ignore the oscillation in the short time scale.

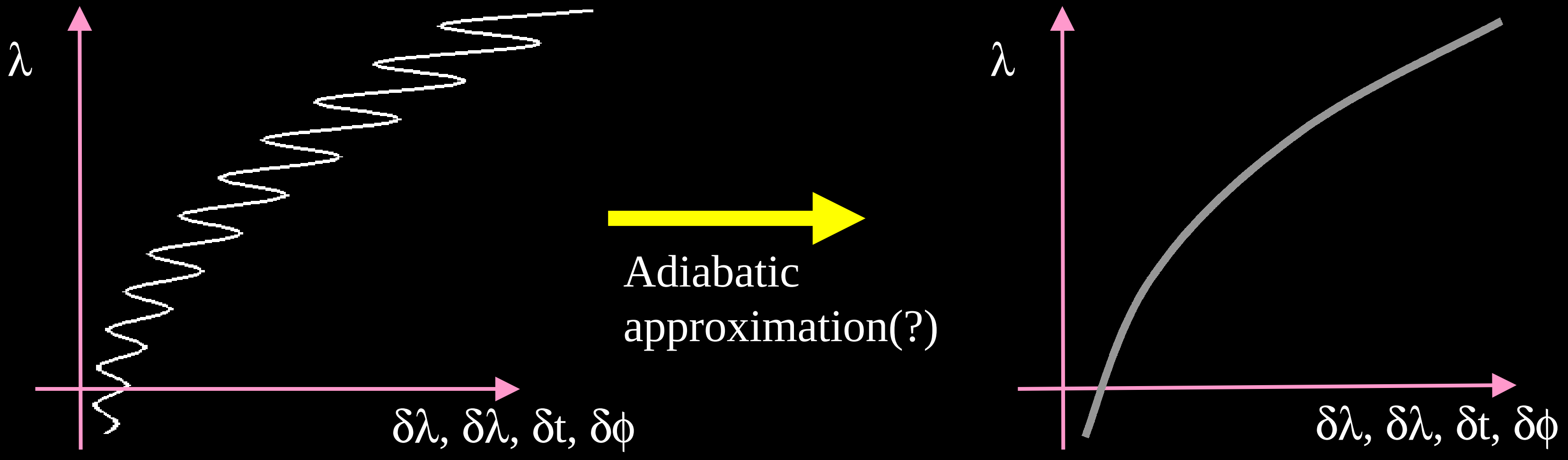


Adiabatic approximation

$$\{\delta E, \delta L, \delta K\} = \{\dot{E}, \dot{L}, \dot{K}\}^{(0,0)}\lambda + \sum_{m,n} \{E, L, K\}^{(m,n)} e^{i(m\Omega_r + n\Omega_\theta)\lambda}$$

$$\{\delta E, \delta L, \delta K\}^S = \{\dot{E}, \dot{L}, \dot{K}\}^{(0,0)}\lambda$$

Radiative Self-force and its Calculation



$$\left\{\delta\lambda_r,\delta\lambda_\theta\right\}=\left\{\ddot{\lambda}_r,\ddot{\lambda}_\theta\right\}^{(0,0)}\frac{\lambda^2}{2}+\sum_{m,n}\left(\left\{\dot{\lambda}_r,\dot{\lambda}_\theta\right\}^{(m,n)}\lambda+\left\{\lambda_r,\lambda_\theta\right\}^{(m,n)}\right)e^{i(m\Omega_r+n\Omega_\theta)\lambda}$$

$$\left\{\delta\bar{t},\delta\bar{\phi}\right\}=\left\{\ddot{t},\ddot{\phi}\right\}^{(0,0)}\frac{\lambda^2}{2}+\sum_{m,n}\left(\left\{\dot{t},\dot{\phi}\right\}^{(m,n)}\lambda+\left\{t,\phi\right\}^{(m,n)}\right)e^{i(m\Omega_r+n\Omega_\theta)\lambda}$$

$$\left\{\delta\lambda_r,\delta\lambda_\theta\right\}^S=\left\{\ddot{\lambda}_r,\ddot{\lambda}_\theta\right\}^{(0,0)}\frac{\lambda^2}{2}$$

$$\left\{\delta\bar{t},\delta\bar{\phi}\right\}^S=\left\{\ddot{t},\ddot{\phi}\right\}^{(0,0)}\frac{\lambda^2}{2}$$

The approximation seems no good since the error grows.

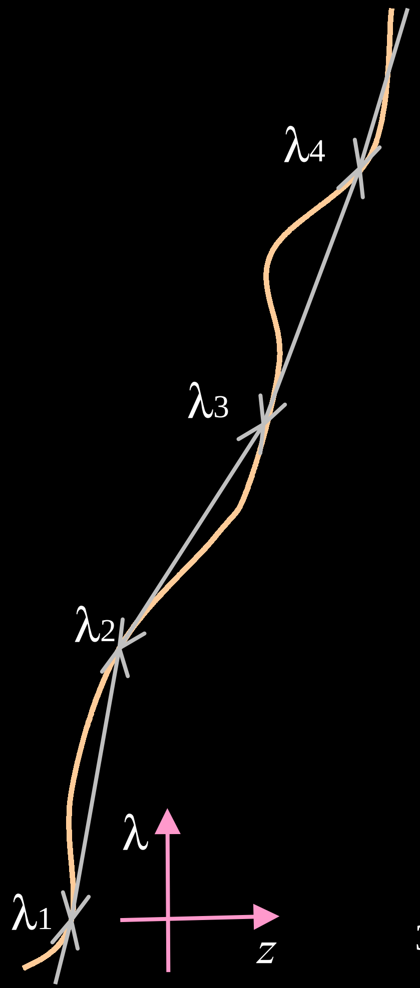
Radiative Self-force and its Calculation

We divide the domain of evolution into a sum of finite domains, at which we can apply the perturbative calculation.

$$\boxed{\lambda_{k+1} - \lambda_k \approx O(\mu^{\alpha}), \quad \alpha \to -1/2}$$

We can calculate the evolution using the perturbative result, and derive the constants at each connecting point.

$$\boxed{\left\{E, L, K, \lambda_r, \lambda_\theta, \bar{t}, \bar{\phi}\right\}(\lambda = \lambda_k)}$$



Radiative Self-force and its Calculation

We define the “adiabatic” evolution is the evolution calculated by the most-dominante terms of the perturbative result.

$$\left\{E,L,K,\lambda_r,\lambda_\theta,\bar{t},\bar{\phi}\right\}^{adi.}(\lambda=\lambda_k)$$

We consider the evolution from λ_0 to λ_N.

$$N\approx O(\mu^{\beta})\quad \lambda_N\approx O(\mu^{\alpha+\beta})$$

$$\left\{E,L,K\right\}^{adi.}_{\lambda_N}\approx O(\mu^{1+\alpha+\beta}),\quad \left\{\lambda_r,\lambda_\theta,\bar{t},\bar{\phi}\right\}^{adi.}_{\lambda_N}\approx O(\mu^{1+2\alpha+\beta})$$

The contribution by the next dominant terms in the perturbative result becomes

$$\left\{E,L,K\right\}^{err.}_{\lambda_N}\approx O(\mu^{1+\beta/2}),\quad \left\{\lambda_r,\lambda_\theta,\bar{t},\bar{\phi}\right\}^{err.}_{\lambda_N}\approx O(\mu^{1+\alpha+\beta/2})$$

*Here we assume that, because of the oscillation, it acts like a gaussian noise.

Radiative Self-force and its Calculation

Predictability of the “adiabatic” evolution;

1) rotation velocity

$$\boxed{\left[\frac{\phi}{t}\right]^{err.} \approx O(\mu^{-\alpha-\beta/2})} \qquad \boxed{\beta < -2\alpha \to 1}$$

We can always make a correct prediction by the “adiabatic” evolution.

2) rotation angle

$$\boxed{[\phi]^{err.} \approx O(\mu^{1+\alpha+\beta/2})} \qquad \boxed{\beta > -2\alpha - 2 \to -1}$$

We can apply the perturbation less than μ^(-1) times. We cannot make a prediction beyond λ>μ^(-3/2) (10^9 cycles).

6: Some Issues in Gravity

A) Green function?

1st step : prepare all homogeneous solutions

$$h_{\mu\nu}(lm\omega P) = \hat{I}_P \hat{\tau}^{+}_{\mu\nu} \Phi(lm\omega)$$

One can use the Chrzanovsky-Misner-Wald algorithm. Because the algorithm makes a complex solutions in general, we separate the real part and the imaginary part by using the parity operation. The calculation by Chrzanovsky justifies we have a necessary set of homogeneous metric perturnations.

Radiative Self-force and its Calculation

2nd step : expand the Green function

$$G^{ret.}_{\alpha\beta,\mu\nu}(x,z)=\sum_{lm\omega P}\left(G^{+}_{\mu\nu}(lm\omega P,z)h^{up}_{\alpha\beta}(lm\omega P,x)\theta(x^r-z^r)+\ldots\right)$$

$$G^{+}_{\mu\nu}(lm\omega P,z)=\sum_{l'm'\omega'P'}w^{+}(lm\omega P;l'm'\omega'P')h^{in}_{\mu\nu}(l'm'\omega'P',z)$$

By the ingenious Green’s reciprocal theorem, we can expand the Green function with the homogeneous solutions. However, the equation is not necessarily a Strum-Liouville form because of the gauge, and we have a radially local term.

$$G^{ret.}_{\alpha\beta,\mu\nu}(x,z)=\sum_{\substack{lm\omega P\\ l'm'\omega'P'}}\left(w^{+}h^{up}_{\alpha\beta}(x)h^{in}_{\mu\nu}(z)\theta(x^r-z^r)+\ldots\right)+\hat{X}^{ret.}_{\alpha\beta,\mu\nu}(x,z)\delta(x^r-z^r)$$

Radiative Self-force and its Calculation

3rd step : determine the weighting matrices

One can determine the weighting matrices at spatial infinity, where we know how to construct the complete Green function. Chrzanowsky-Misner derive a general formula by focusing the transverse-traceless mode of solutions.

$$w^{+}(lm\omega P; l'm'\omega'P') = 1/ < h^{up}_{\alpha\beta}(lm\omega P), h^{in}_{\mu\nu}(l'm'\omega'P') >_{I^{\pm}}$$

Note:
We have an unknown radial local term, and it is responsible for the gauge problem. (Capra3)

Radiative Self-force and its Calculation

4th step : radiative Green function?

$$G^{rad.}_{\alpha\beta,\mu\nu}(x,z) = \sum_{lm\omega P}\left(h^{in*}_{\alpha\beta}(x)h^{in}_{\mu\nu}(z) + \ldots\right) + \frac{1}{2}\left(\hat{X}^{ret.}_{\alpha\beta,\mu\nu} - \hat{X}^{adv.}_{\alpha\beta,\mu\nu}\right)\delta(x^{r} - z^{r})$$

The radiative (half-retarded-minus-half-advanced) Green function can be derived with the unknown local terms. Because the radiative Green function is a homogeneous solution, the unknown local terms should vanish.

$$\hat{X}^{ret.}_{\alpha\beta,\mu\nu}(x,z) - \hat{X}^{adv..}_{\alpha\beta,\mu\nu}(x,z) = 0$$

Radiative Self-force and its Calculation

B) Gauge Dependence?

Our approximate adiabatic formula depends only on the time averaged self-force to the constants.

$$\boxed{\delta E(\tau) = \int \eta_\alpha(z(\tau)) F^\alpha(\tau)\, d\tau, \quad \delta K(\tau) = \int \eta_{\alpha\beta}(z(\tau)) V^\alpha(\tau) F^\beta(\tau)\, d\tau}$$

Gauge dependent? We consider a gauge transformation.

$$\boxed{\delta_\xi F^\alpha(\tau) = -\left(\xi^\alpha{}_{;\beta\gamma} + R^\alpha{}_{\beta\delta\gamma}\xi^\delta\right) V^\beta V^\gamma}$$

Radiative Self-force and its Calculation

$$\frac{1}{T}\int_0^T d\tau\left[\frac{d}{d\tau}E(\tau)\right]_\xi = \frac{1}{T}\left[-\eta_\alpha\frac{D}{d\tau}\xi^\alpha+\xi^\alpha\frac{D}{d\tau}\eta_\alpha\right]_0^T \to 0$$

$$\frac{1}{T}\int_0^T d\tau\left[\frac{d}{d\tau}Q(\tau)\right]_\xi = \frac{1}{T}\left[-\eta_{\alpha\beta}V^\alpha\frac{D}{d\tau}\xi^\beta+\xi^\alpha V^\beta\frac{D}{d\tau}\eta_{\alpha\beta}\right]_0^T \to 0$$

The orbital evolution by the formula is gauge independent!

The part we ignore by the “adiabatic approximation” is negligibly small (up to 10^8cycles), and is not observable. The formula considers the secular part of the evolution, and is observable from the waves. Thus, it is naturally gauge invariant.

7: Conclusion

We prove:

- The evolution of λ_r, λ_θ, t_0, ϕ_0 is as important as that of E, L, K.
- The orbit does not evolve adiabatically in an exact sense.
- One can define an adiabatic evolution of the orbit as an approximation.
- The `adiabatic’ evolution equations of all `constants’ are derived in a consistent (gauge-invariant) manner.
- The `adiabatic’ evolution equations can be derived by using an appropriate radiative Green function.

Radiative Self-force and its Calculation

We have a prediction of the orbit up to 10^6 cycles.

The procedure in making a template bank;

1) derive a trajectory of (E,L,K)-space

$$\boxed{\frac{d}{d\lambda}\{E, L, K\} = \{\dot{E}, \dot{L}, \dot{K}\}^{(0,0)}}$$

2) derive a trajectory of (λ_r,λ_θ,t_0,ϕ_0)-space

$$\boxed{\frac{d^2}{d\lambda^2}\{\lambda_r, \lambda_\theta\} = \{\ddot{\lambda}_r, \ddot{\lambda}_\theta\}^{(0,0)}} \quad \boxed{\frac{d^2}{d\lambda^2}\{\bar{t}, \bar{\phi}\} = \{\ddot{t}, \ddot{\phi}\}^{(0,0)}}$$

3) derive the wave-form

Radiative Self-force and its Calculation

We need as much as 10^12 ~ 10^24 templates.

Semi-analytic approach?

The wave functions are efficiently derived in the frequency domain.