Parametric decay instability of circularly polarized Alfvén wave in magnetically dominated plasma

W. Ishizaki and K. Ioka, Phys. Rev. E 110, 015205 (arXiv: 2404.15689)

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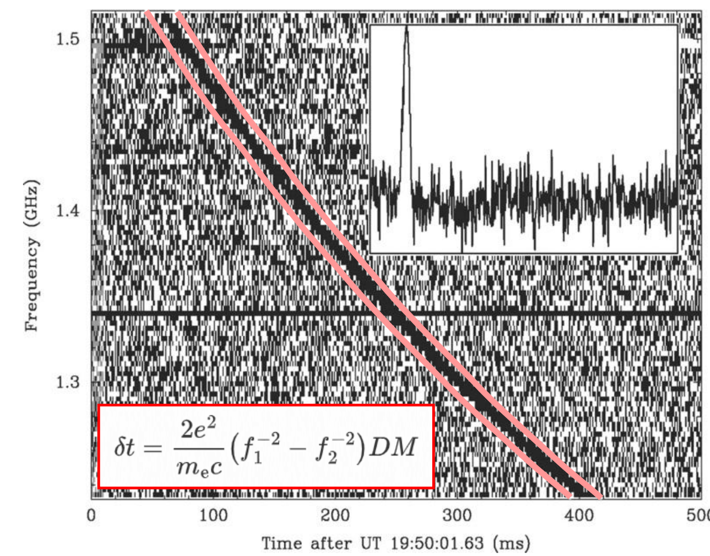
Fast Radio Burst

$$DM = \int_0^D n_e ds \sim \langle n_e \rangle D$$



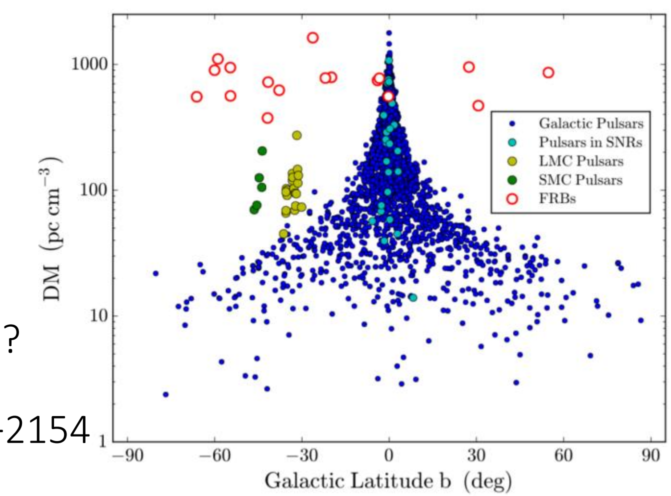
(Lorimer et al., 2007)

- Transient phenomenon in the radio bands
  - Luminosity~$10^{35\text{-}43}$ erg/s , Frequency~O(GHz)
  - Duration : δt~msec
  - Excess from the Dispersion Measure (DM) of Galaxy

    DM~O(1000) pc cm^{-3} → Cosmological
  - Extremely high brightness temperature → coherent emission

$$T_b \simeq \frac{\mathcal{S}_\nu D^2}{2\pi k_B (\nu \delta t)^2} = 1.2 \times 10^{36}\ \mathrm{K} \left(\frac{D}{10^{28}\ \mathrm{cm}}\right)^2 \frac{\mathcal{S}_\nu}{\mathrm{Jy}} \left(\frac{\nu}{\mathrm{GHz}}\right)^{-2} \left(\frac{\delta t}{\mathrm{ms}}\right)^{-2}$$

- Source/Mechanism → Still Unknown
  - Short duration → compact star origin?
  - Repeating sources (Repeater) → cataclysmic origin difficult?
  - Association with X-ray flare of galactic magnetar SGR1935+2154



⇒Promising candidate: **Magnetosphere of magnetars**

(Cordes et al., 2016)

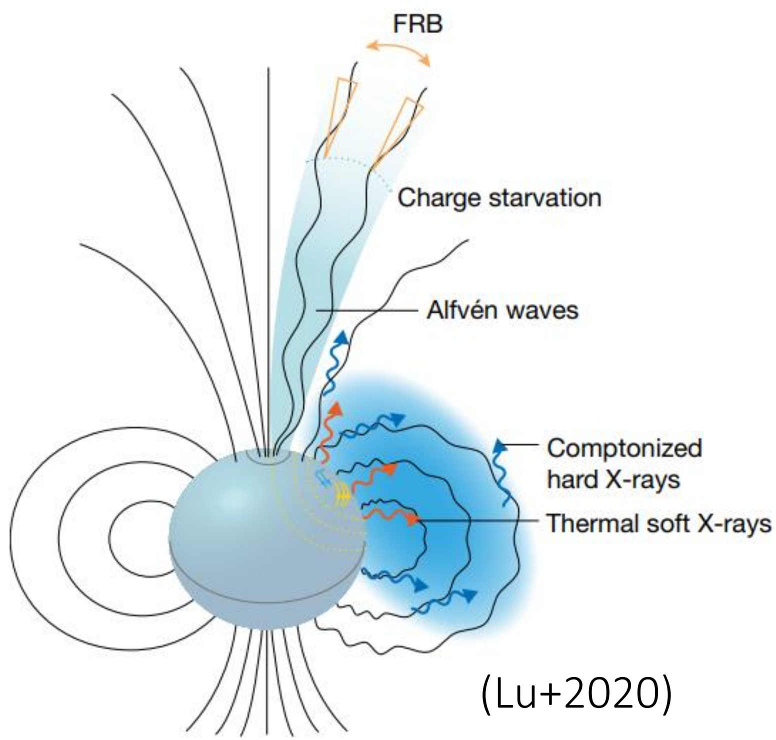
(magnetar: highly magnetized neutron star ($B_{surface}$~10^{15} G))

Emission site

- Even if magnetars are the origin of FRBs, the **emission site remains unknown**
  - Two scenarios have been discussed extensively

Magnetospheric model

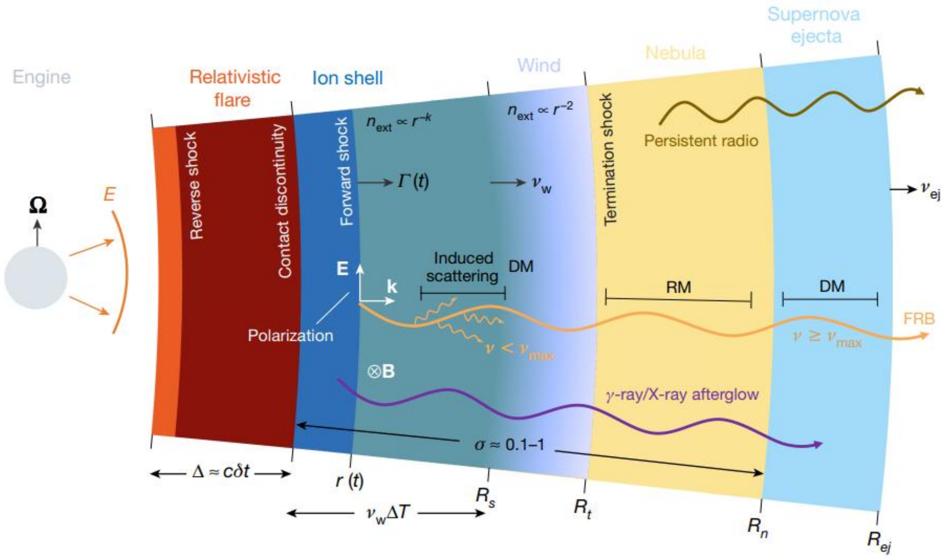
e.g., Katz 2014, Lu & Kumar 2018,
Yang & Zhang 2018, Kumar & Bosnjak 2020



(Lu+2020)

External shock model

e.g., Lyubarsky 2014, Murase+2016,
Margalit+2020, Beloborodov 2020



(B. Zhang 2020)

Emission site

- Even if magnetars are origin of FRBs, an **emission site remains unknown**
    - Two scenarios are discussed extensively

Some observational difficulties are pointed out:

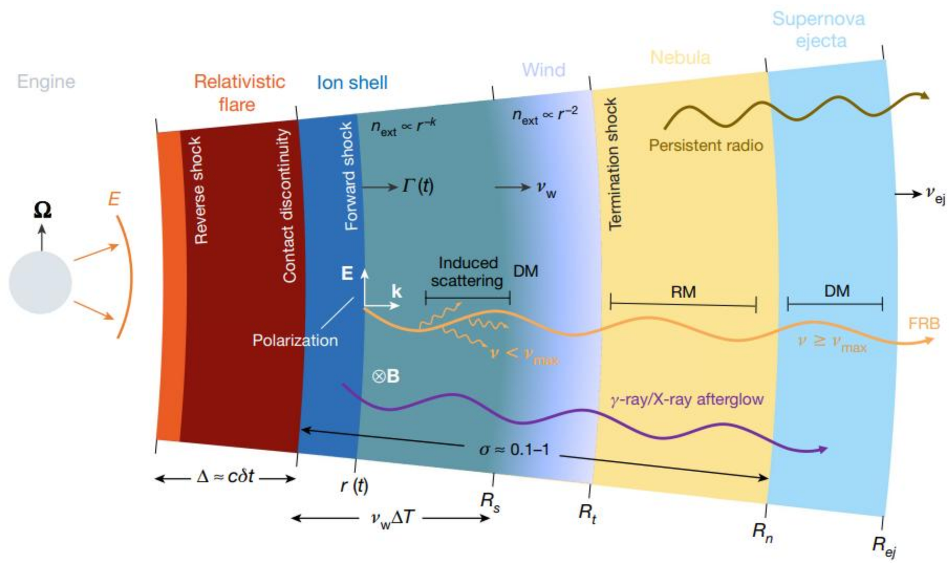
✓ **Difficult to explain the observed circular polarization**
(e.g., Gajjar+18, Michilli+18, Oslowski+19)

✓ **Conflicts with short time-scale variability of some bursts**
(e.g., Beniamini & Kumar 20, Lu+22)

✓ **Inconsistent with scintillation results** (e.g., Nimmo 2024)

External shock model

e.g., Lyubarsky 2014, Murase+2016, Margalit+2020, Beloborodov 2020



(B. Zhang 2020)

Emission site

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Magnetospheric model

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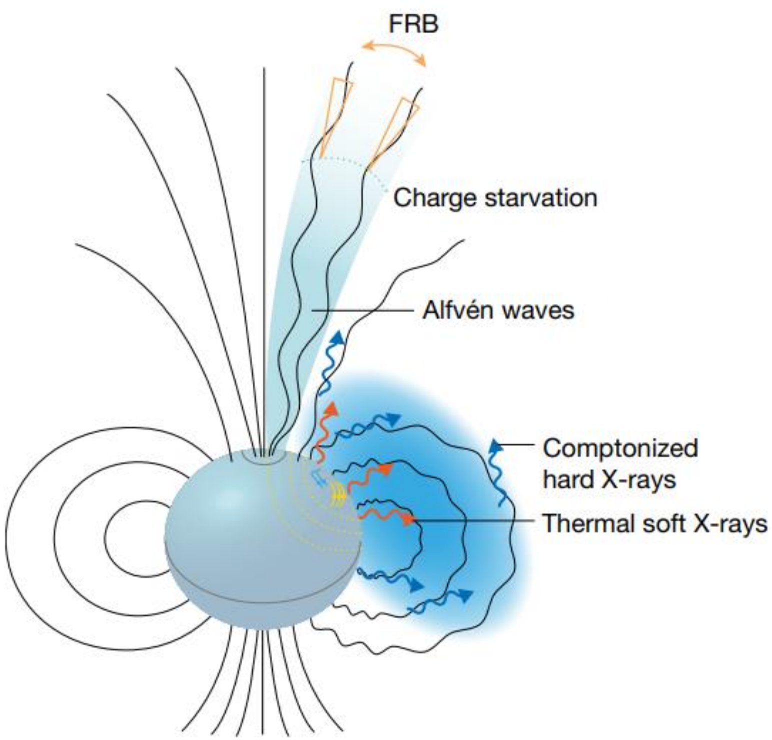


Some theoretical difficulties

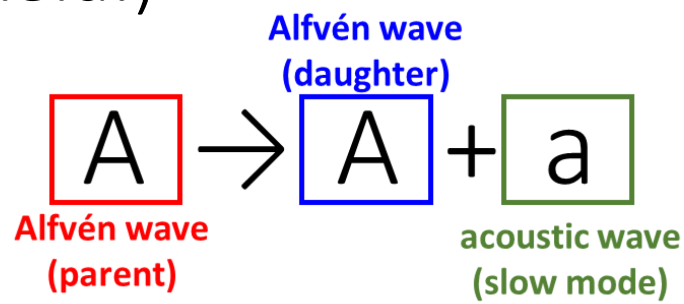
✓ **Poor understanding of the coherent emission mechanism**

✓ **High-intensity photons may not escape from the magnetosphere**

(Idea)
Energy can be transported by Alfvén waves, which can propagate relatively freely in the plasma, then converted into photons where the plasma becomes "optically thin"

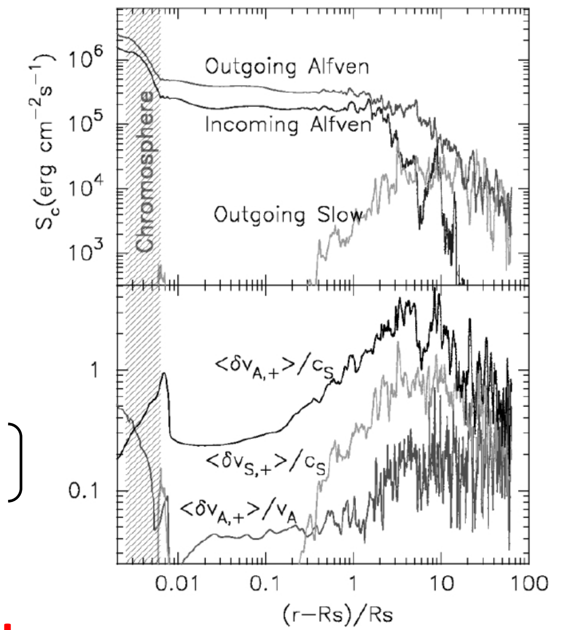
(e.g., Kumar & Bosnjak 20, Yuan+20)

Stability of Alfvén wave (Non-Rela.)



- Parametric Decay Instability of Alfvén wave
  - An instability due to a resonant 3-wave interaction
  - Major process is **Alfven waves decay into slow and Alfven waves** for β<1 (non-rela. regime)

- Example : Coronal heating
  - Alfven waves generated near the chromosphere
  - Converted to slow wave and backward Alfven wave
  - Counterpropagating Alfven wave → particle acceleration
  - Growth rate: $\gamma/\omega_0 = \frac{1}{2}\eta\beta^{-1/4}$ $\left(\eta = \delta B/B_0\,,\ \beta = c_s^2/v_A^2 \right)$



(Suzuki & Inutsuka (2005))

Consider this process in a "relativistically magnetized" plasma

What is “Relativistically magnetized” plasma?

- “Relativistically magnetized” plasma
  - The energy density of the electromagnetic field exceeds the rest mass energy density
  - Magnetization parameter σ : $$\sigma = \frac{\text{(energy density of B-field)}}{\text{(enthalpy density of gas)}} = \frac{B_0^2/4\pi}{\rho c^2 + \epsilon_{\text{int}} + p_{\text{gas}}}$$

e.g., NS magnetosphere

$\sigma \sim 10^{3\text{-}5} \gg 1$

- Extreme limit: Force-free approximation
  - Neglecting the contribution of matter field
  - The normal modes are the Alfvén wave and the fast wave; **there is no slow wave**.
  - In the force-free regime, an Alfvén wave is stable against 3-wave interactions

    (cannot be decayed by satisfying the resonance condition)

Without force-free approximation, we investigate the stability of Alfvén waves in a plasma with relativistic magnetization

MHD wave

- Typical parameter of the plasma in neutron star magnetosphere
  - The magnetosphere consists of a pair plasma formed around a rotating, magnetized neutron star.

    Typical number density : $n_{\rm GJ} = \frac{\Omega B_{\rm NS}}{2\pi c e} \sim 6.9 \times 10^{13}\ {\rm cm}^{-3} \left(\frac{P}{1\ {\rm sec}}\right)^{-1} \left(\frac{B_{\rm NS}}{10^{15}\ {\rm G}}\right)$

    (Goldreich & Julian 1969)
  - Plasma frequency

$$\omega_p = \sqrt{\frac{4\pi \mathcal{M} n_{\rm GJ} e^2}{m_e}} \sim 4.7 \times 10^{11}\ {\rm Hz}\ \mathcal{M}^{1/2} \left(\frac{P}{1\ {\rm sec}}\right)^{-1/2} \left(\frac{B_{\rm NS}}{10^{15}\ {\rm G}}\right)^{1/2}$$

  - Cyclotron frequency

$$\omega_c = \frac{e B_{\rm NS}}{m_e c} \sim 1.8 \times 10^{22}\ {\rm Hz} \left(\frac{B_{\rm NS}}{10^{15}\ {\rm G}}\right)$$

- Frequency of interest: ω_0 (Alfvén wave frequency)
  - Spin period of neutron star? → ω~1-10^3 Hz
  - Frequency of observed EM waves from FRB? → ω~10^9 Hz

In any case: $\omega_0 \ll \omega_p \ll \omega_c \Rightarrow$ We describe the waves using relativistic MHD

Governing Equations

- Basic equation : Special relativistic ideal MHD equations

$$\frac{\partial}{\partial t}\left[(\epsilon+p)\gamma^2 - p + \frac{1}{8\pi}\left(E^2+B^2\right)\right] + \frac{\partial}{\partial z}\left[(\epsilon+p)\gamma^2 v_z + \frac{c}{4\pi}(\boldsymbol{E}\times\boldsymbol{B})_z\right] = 0$$

$$\frac{\partial}{\partial t}\left[(\epsilon+p)\gamma^2 v_x + \frac{c}{4\pi}(\boldsymbol{E}\times\boldsymbol{B})_x\right] + \frac{\partial}{\partial z}\left[(\epsilon+p)\gamma^2 v_x v_z - \frac{c^2}{4\pi}(E_xE_z + B_xB_z)\right] = 0$$

$$\frac{\partial}{\partial t}\left[(\epsilon+p)\gamma^2 v_y + \frac{c}{4\pi}(\boldsymbol{E}\times\boldsymbol{B})_y\right] + \frac{\partial}{\partial z}\left[(\epsilon+p)\gamma^2 v_y v_z - \frac{c^2}{4\pi}(E_yE_z + B_yB_z)\right] = 0$$

$$\frac{\partial}{\partial t}\left[(\epsilon+p)\gamma^2 v_z + \frac{c}{4\pi}(\boldsymbol{E}\times\boldsymbol{B})_z\right] + \frac{\partial}{\partial z}\left[(\epsilon+p)\gamma^2 v_z^2 - \frac{c^2}{4\pi}(E_z^2 + B_z^2)\right] + c^2\frac{\partial}{\partial z}\left[p + \frac{E^2+B^2}{8\pi}\right] = 0$$

$$\frac{\partial B_x}{\partial t} = -\frac{\partial}{\partial z}\left(v_zB_x - v_xB_z\right)$$

$$\frac{\partial B_y}{\partial t} = \frac{\partial}{\partial z}\left(v_yB_z - v_zB_y\right)$$

- Key Assumptions
  - **We consider only parallel propagation**

    ⇒ the slow mode becomes a pure acoustic wave

    ⇒ the fast mode is degenerate with the Alfvén wave
  - For simplicity, we take the parent wave to be a circularly polarized Alfvén wave

Setting

$$\eta \equiv \delta B / B_0$$

- Perturbations are set as:

	Background	Parent wave	Daughter waves	
$\boldsymbol{B} =$	$\boldsymbol{B}_0$	$+\,\boldsymbol{\delta B}$	$+\,\boldsymbol{b}_\perp$	
$\boldsymbol{v} =$		$\boldsymbol{\delta v}$	$+\,\boldsymbol{v}_\perp$	$+\,\boldsymbol{v}_\parallel$
$\epsilon =$	ϵ_0			$+\,\epsilon_\parallel$
		Alfvén wave	Alfvén wave	Sound wave
	$O(1)$	$O(\eta)$	$O(\varepsilon)$	

Uniform plasma with B_0

Alfven wave propagating “clean” plasma

Setting

$$\eta \equiv \delta B / B_0$$

- Perturbations are set as:

	Background	Parent wave	Daughter waves	
				Sound wave
$\boldsymbol{B} =$	$\boldsymbol{B_0}$	$+ \boldsymbol{\delta B}$	$+ \boldsymbol{b}_\perp$	
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$\epsilon =$	ϵ_0			$+ \epsilon_\parallel$
		Alfvén wave	Alfvén wave	
	$O(1)$	$O(\eta)$	$O(\varepsilon)$	

Uniform plasma with B_0

Alfven wave propagating “clean” plasma

An exact solution exists even for finite-amplitude waves!

Setting

$\eta \equiv \delta B / B_0$

- Perturbations are set as:

	Background	Parent wave	Daughter waves	
		Alfvén wave	Alfvén wave	Sound wave
$\boldsymbol{B} =$	$\boldsymbol{B_0}$	$+\ \boldsymbol{\delta B}$	$+\ \boldsymbol{b}_\perp$	
$\boldsymbol{v} =$		$\boldsymbol{\delta v}$	$+\ \boldsymbol{v}_\perp$	$+\ \boldsymbol{v}_\parallel$
$\epsilon =$	ϵ_0			$+\ \epsilon_\parallel$
	$O(1)$	$O(\eta)$	$O(\varepsilon)$	

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Alfven wave propagating “clean” plasma

$+$ perturbations

Setting

$\eta \equiv \delta B / B_0$

- Perturbations are set as:

	Background	Parent wave	Daughter waves	
$B =$	B_0	$+\,\delta B$	$+\,b_\perp$	
$v =$		δv	$+\,v_\perp$	$+\,v_\parallel$
$\epsilon =$	ϵ_0			$+\,\epsilon_\parallel$
		Alfvén wave	Alfvén wave	Sound wave
	$O(1)$	$O(\eta)$	$O(\varepsilon)$	

Uniform plasma with B_0

Alfven wave propagating “clean” plasma

+ perturbations

Do the perturbations grow?

Setting

- We linearize the MHD equations around the background wave:

MHD equations

$$\frac{\partial}{\partial t}\left[(\epsilon+p)\gamma^2 - p + \frac{1}{8\pi}\left(E^2+B^2\right)\right] + \frac{\partial}{\partial z}\left[(\epsilon+p)\gamma^2 v_z + \frac{c}{4\pi}(\boldsymbol{E}\times\boldsymbol{B})_z\right] = 0$$

$$\frac{\partial}{\partial t}\left[(\epsilon+p)\gamma^2 v_x + \frac{c}{4\pi}(\boldsymbol{E}\times\boldsymbol{B})_x\right] + \frac{\partial}{\partial z}\left[(\epsilon+p)\gamma^2 v_x v_z - \frac{c^2}{4\pi}(E_x E_z + B_x B_z)\right] = 0$$

$$\frac{\partial}{\partial t}\left[(\epsilon+p)\gamma^2 v_y + \frac{c}{4\pi}(\boldsymbol{E}\times\boldsymbol{B})_y\right] + \frac{\partial}{\partial z}\left[(\epsilon+p)\gamma^2 v_y v_z - \frac{c^2}{4\pi}(E_y E_z + B_y B_z)\right] = 0$$

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$$\frac{\partial B_x}{\partial t} = -\frac{\partial}{\partial z}\left(v_z B_x - v_x B_z\right)$$

$$\frac{\partial B_y}{\partial t} = \frac{\partial}{\partial z}\left(v_y B_z - v_z B_y\right)$$

+

Perturbations

	Background	Parent wave	Daughter waves	
				Sound wave
$\boldsymbol{B} =$	$\boldsymbol{B_0}$	$+\,\boldsymbol{\delta B}$	$+\,\boldsymbol{b}_\perp$	
$\boldsymbol{v} =$		$\boldsymbol{\delta v}$	$+\,\boldsymbol{v}_\perp$	$+\,\boldsymbol{v}_\parallel$
$\epsilon =$	ϵ_0			$+\,\epsilon_\parallel$
		Alfvén wave	Alfvén wave	
	$O(1)$	$O(\eta)$	$O(\varepsilon)$	

Perturbed equations

	Background	Parent wave	Daughter waves	
				Sound wave
$\boldsymbol{B} =$	$\boldsymbol{B_0}$	$+\ \boldsymbol{\delta B}$	$+\ \boldsymbol{b}_\perp$	
$\boldsymbol{v} =$		$\boldsymbol{\delta v}$	$+\ \boldsymbol{v}_\perp$	$+\ \boldsymbol{v}_\parallel$
$\epsilon =$	ϵ_0			$+\ \epsilon_\parallel$
		Alfvén wave	Alfvén wave	
	$O(1)$	$O(\eta)$	$O(\varepsilon)$	

- Acoustic wave
  - Energy conservation

$$\frac{1+\beta_s^2\delta\beta^2}{1-\delta\beta^2}\frac{1}{c}\frac{\partial\epsilon_\parallel}{\partial t} + \left(\delta\gamma^2\left(\epsilon_0+p_0\right)\right)\frac{\partial\beta_\parallel}{\partial z} = -\frac{1}{c}\frac{\partial}{\partial t}\left[\left(2\delta\gamma^4\left(\epsilon_0+p_0\right)+\frac{B_0^2}{4\pi}\right)\left(\delta\boldsymbol{\beta}\cdot\boldsymbol{\beta}_\perp\right) - \frac{\delta\beta\delta B B_0\beta_\parallel}{4\pi}\right] + \frac{B_0}{4\pi}\left[\boldsymbol{\beta}_\perp\cdot\frac{\partial}{\partial z}\left(\delta\boldsymbol{B}\right) + \delta\boldsymbol{\beta}\cdot\frac{\partial\boldsymbol{b}_\perp}{\partial z}\right]$$

  - Momentum equation (z-component)

$$\left[\delta\gamma^2\left(\epsilon_0+p_0\right)+\frac{\delta B^2}{4\pi}\right]\frac{1}{c}\frac{\partial\beta_\parallel}{\partial t} + \beta_s^2\frac{\partial\epsilon_\parallel}{\partial z} = -\frac{\partial}{\partial z}\left(\frac{\delta\boldsymbol{B}\cdot\boldsymbol{b}_\perp}{4\pi}\right) + \frac{B_0}{4\pi}\left(\delta\boldsymbol{B}\cdot\frac{1}{c}\frac{\partial\boldsymbol{\beta}_\perp}{\partial t} + \boldsymbol{b}_\perp\cdot\frac{1}{c}\frac{\partial}{\partial t}\left(\delta\boldsymbol{\beta}\right)\right)$$

- Alfven wave
  - Momentum equations (in xy-plain)

$$\left[\delta\gamma^2\left(\epsilon_0+p_0\right)+\frac{B_0^2+\delta B^2}{4\pi}\right]\frac{1}{c}\frac{\partial\boldsymbol{\beta}_\perp}{\partial t} - \frac{B_0}{4\pi}\frac{\partial\boldsymbol{b}_\perp}{\partial z} = -\delta\gamma^2\left(\epsilon_0+p_0\right)\beta_\parallel\frac{\partial}{\partial z}\left(\delta\boldsymbol{\beta}\right)$$
$$-\delta\gamma^2\left(1+\beta_s^2\right)\epsilon_\parallel\frac{1}{c}\frac{\partial}{\partial t}\left(\delta\boldsymbol{\beta}\right) - \delta\boldsymbol{\beta}\beta_s^2\frac{1}{c}\frac{\partial\epsilon_\parallel}{\partial t} + \frac{B_0}{4\pi}\frac{1}{c}\frac{\partial}{\partial t}\left(\beta_\parallel\delta\boldsymbol{B}\right)$$
$$-2\delta\gamma^4\left(\epsilon_0+p_0\right)\left(\delta\boldsymbol{\beta}\cdot\boldsymbol{\beta}_\perp\right)\frac{1}{c}\frac{\partial}{\partial t}\left(\delta\boldsymbol{\beta}\right) + \frac{1}{4\pi}\frac{\partial}{\partial z}\left[\hat{z}\cdot\left\{\left(\delta\boldsymbol{\beta}\times\boldsymbol{b}_\perp\right)+\left(\boldsymbol{\beta}_\perp\times\delta\boldsymbol{B}\right)\right\}\right]\left(\delta\boldsymbol{\beta}\times\boldsymbol{B_0}\right)$$
$$-\frac{\delta\boldsymbol{B}\cdot\boldsymbol{b}_\perp}{4\pi}\frac{1}{c}\frac{\partial}{\partial t}\left(\delta\boldsymbol{\beta}\right) - \left(\delta\boldsymbol{B}\cdot\frac{1}{c}\frac{\partial\boldsymbol{b}_\perp}{\partial t}\right)\delta\boldsymbol{\beta} + \frac{\delta B\delta\beta}{4\pi}\frac{1}{c}\frac{\partial\boldsymbol{b}_\perp}{\partial t} + \frac{\delta\boldsymbol{B}}{4\pi}\left(\frac{1}{c}\frac{\partial\boldsymbol{\beta}_\perp}{\partial t}\cdot\delta\boldsymbol{B}\right) + \frac{\delta\boldsymbol{B}\cdot\boldsymbol{\beta}_\perp}{4\pi}\frac{1}{c}\frac{\partial}{\partial t}\left(\delta\boldsymbol{B}\right)$$

  - Induction euations (in xy-plain)

$$\frac{1}{c}\frac{\partial\boldsymbol{b}_\perp}{\partial t} - B_0\frac{\partial\boldsymbol{\beta}_\perp}{\partial z} = -\frac{\partial}{\partial z}\left(\beta_\parallel\delta\boldsymbol{B}\right)$$

Dispersion relation

$$S_0 = k^2\left(\omega^3 + k\omega^2 - 3\omega + k\right)$$
$$S_3 = S_0 - \omega^2(\omega - k) - \eta^2 k\omega\left[\omega - k + \theta^2(\omega + k)(k\omega - 1)\right]$$
$$+\theta^2\left[\omega^5 + 3k\omega^4 + (2k^2 - 3)\omega^3 - k(4k^2 + 7)\omega^2 - k^2(4k^2 - 11)\omega + k^3\right]$$
$$S_4 = \theta^2 k\omega(\omega - k)\left\{(\omega + k)^2 - 4\right\}$$

$\theta \equiv V_S/V_A$

- Dispersion relation with k and ω as independent variables sound wave for $\eta \lesssim 1$

  (unit system ω_0=1, k_0=1)

$$(\omega - k)^2\left(\omega^2 - \theta^2 k^2\right)\left\{(\omega + k)^2 - 4\right\} = \frac{1}{(1+\sigma)^4}\eta^2(\omega - k)\left(S_0 + S_1\sigma + S_2\sigma^2 + S_3\sigma^3 + S_4\sigma^4\right)$$

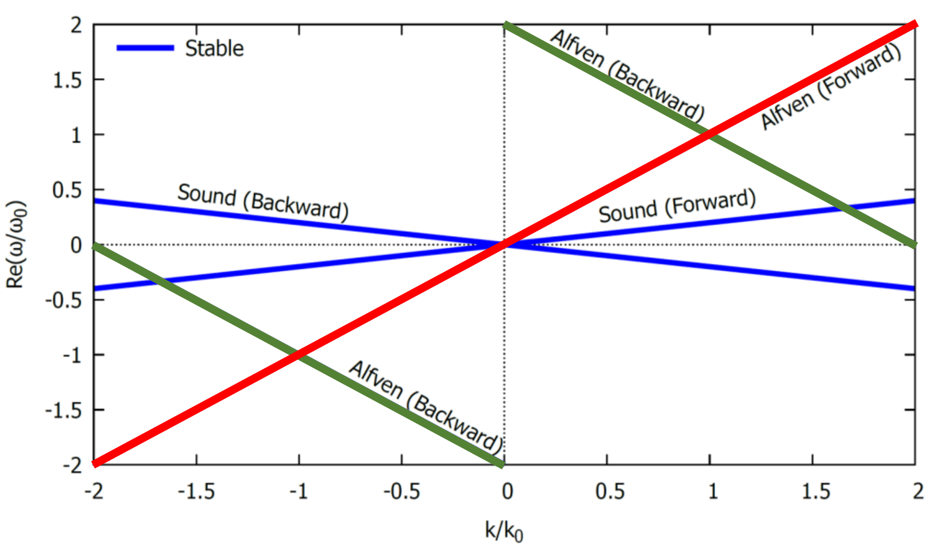
  - η=0 : recover the dispersion relation in a uniform plasma

$\eta \equiv \delta B/B_0$

$k_\pm = k \pm k_0,\quad \omega_\pm = \omega \pm \omega_0$

Parent wave **Parent wave**

$\eta = 0$ (no parent wave)



Forward Alfven wave (satisfying resonance)

Backward Alfven wave (satisfying resonance)

Acoustic wave

Dispersion relation

$$S_0 = k^2\left(\omega^3 + k\omega^2 - 3\omega + k\right)$$
$$S_3 = S_0 - \omega^2(\omega - k) - \eta^2 k\omega\left[\omega - k + \theta^2(\omega + k)(k\omega - 1)\right]$$
$$\quad + \theta^2\left[\omega^5 + 3k\omega^4 + \left(2k^2 - 3\right)\omega^3 - k\left(4k^2 + 7\right)\omega^2 - k^2\left(4k^2 - 11\right)\omega + k^3\right]$$
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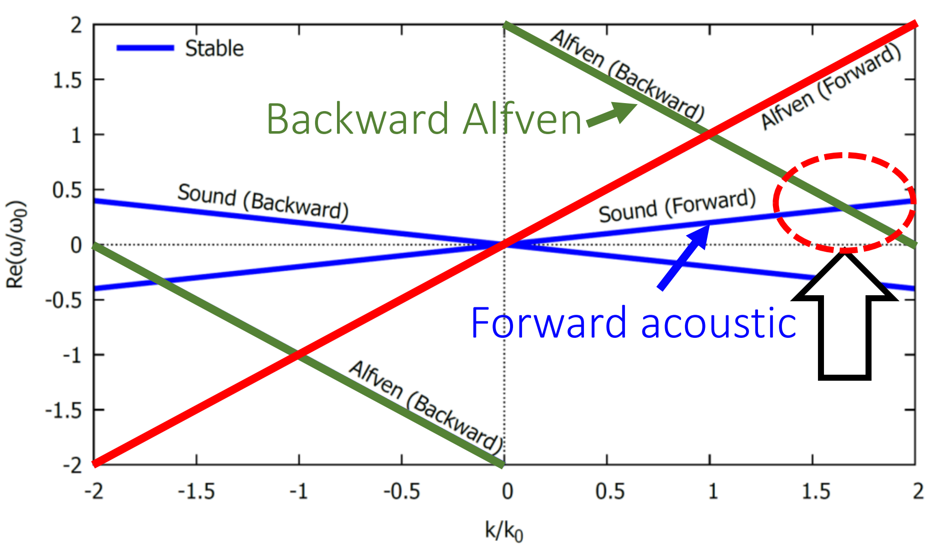
$\theta \equiv V_S/V_A$

- Dispersion relation with k and ω as independent variables sound wave

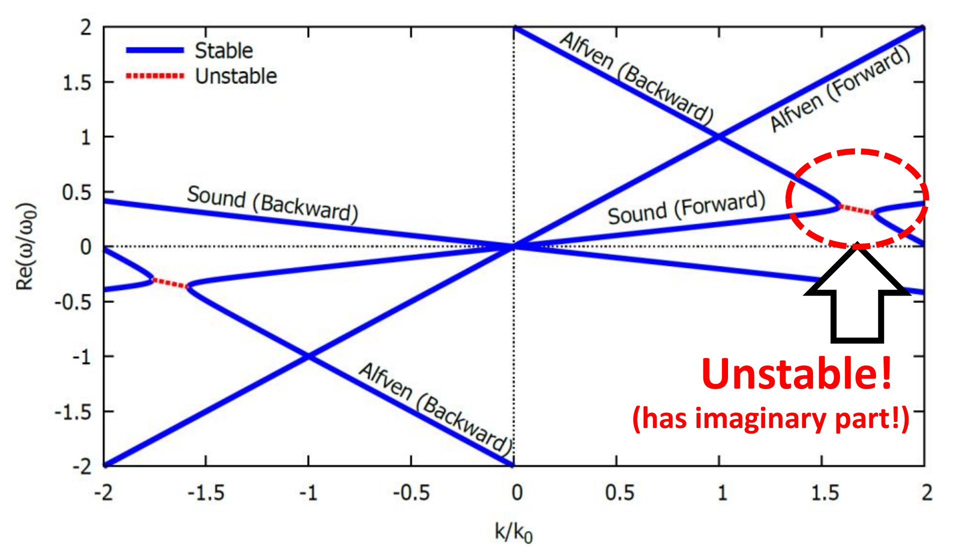
$$(\omega - k)^2\left(\omega^2 - \theta^2 k^2\right)\left\{(\omega + k)^2 - 4\right\} = \frac{1}{(1+\sigma)^4}\eta^2(\omega - k)\left(S_0 + S_1\sigma + S_2\sigma^2 + S_3\sigma^3 + S_4\sigma^4\right)$$

  - η=0 : recover the dispersion relation in a uniform plasma
  - η>0 : an unstable solution appears when the **backward Alfven wave** and the **forward acoustic wave** satisfy the resonance condition

$\eta = 0$ (no parent wave)



$\eta = 0.1$ (parent wave exists)



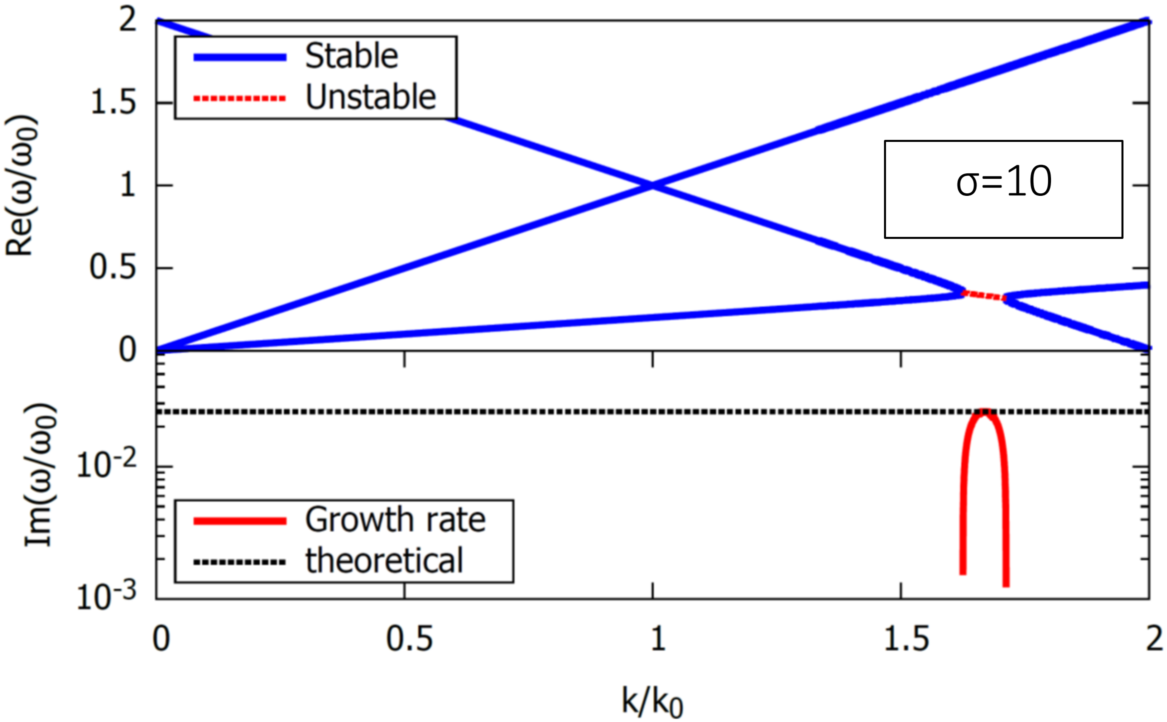
Growth rate of the instability

Alfvén sound Alfvén sound

$$k_0 = k_- + k, \quad \omega_0 = \omega_- + \omega$$

Parent wave **Parent wave**

- Growth rate (per 1 wave period)
  - We expand the dispersion relation around the point that satisfies the resonance condition
  - Growth rate: $\Gamma = \frac{1}{2}\eta\theta^{-1/2}\frac{\sqrt{1-\theta}}{1+\theta}\frac{1+\sigma\left(1+\theta^2\right)}{\left(1+\sigma\right)^{3/2}} \sim \frac{1}{2}\eta\theta^{-1/2}\left(1+\sigma\right)^{-1/2}$



Key parameters

Wave amplitude

$$\eta \equiv \delta B / B_0$$

Sound velocity

$$\theta \equiv V_S / V_A$$

Magnetization

$$\sigma \equiv \frac{B_0^2}{4\pi(\rho c^2 + \epsilon_{\rm int} + p)}$$

Growth rate of the instability

e.g., NS magnetosphere

- Growth rate (per 1 wave period)

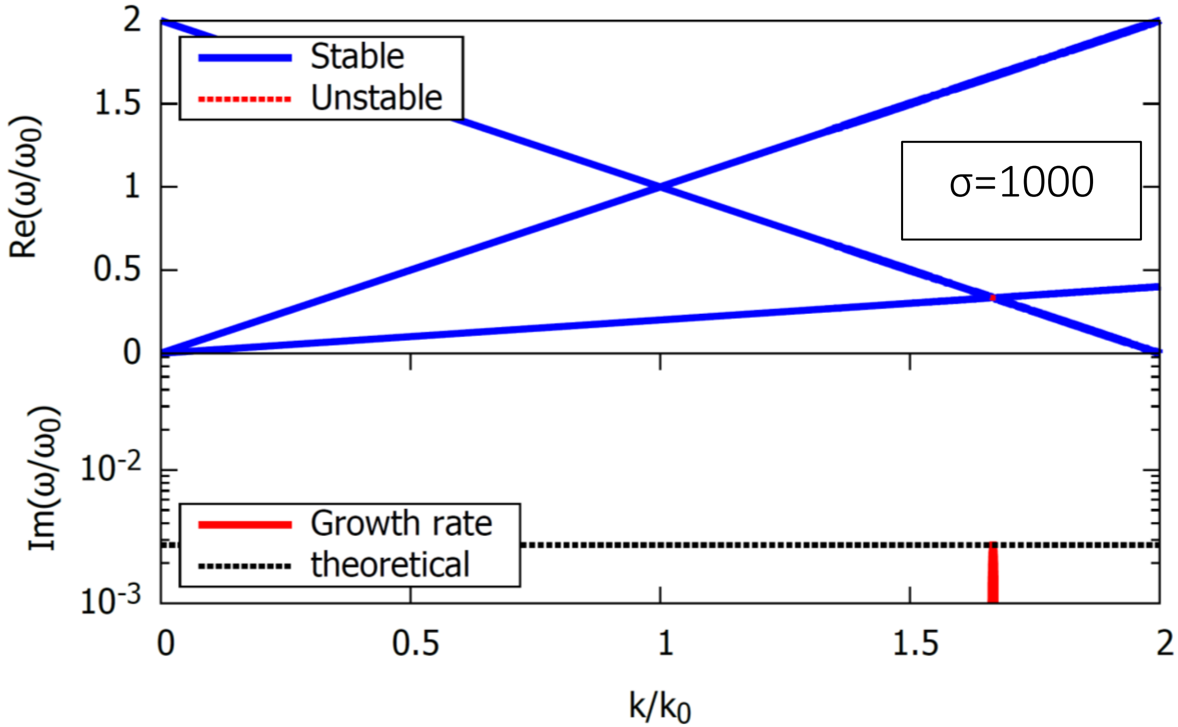
$$\Gamma = \frac{1}{2}\eta\theta^{-1/2}\frac{\sqrt{1-\theta}}{1+\theta}\frac{1+\sigma\left(1+\theta^2\right)}{(1+\sigma)^{3/2}} \sim \frac{1}{2}\eta\theta^{-1/2}(1+\sigma)^{-1/2}$$

$\frac{1}{2}\eta\theta^{-1/2}$: Same as non-rela case

$(1+\sigma)^{-1/2}$: **Rela. correction** **~10^{-2} (for σ~$10^{3\text{-}5}$)**

Even though a larger σ suppresses the instability,

Alfven waves are still unstable with respect to the decay instability.



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Wave amplitude

$$\eta \equiv \delta B/B_0$$

Sound velocity

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Magnetization

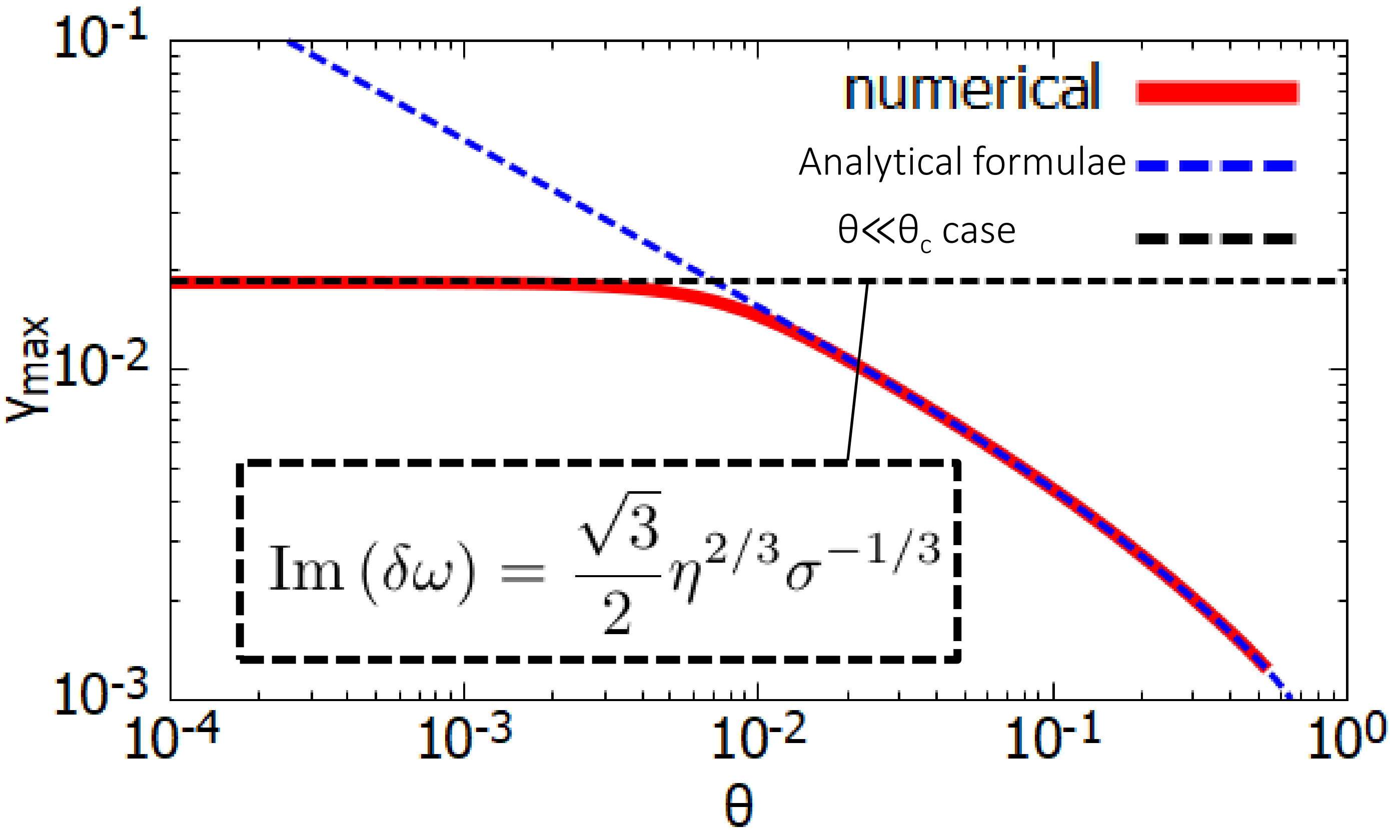
$$\sigma \equiv \frac{B_0^2}{4\pi(\rho c^2 + \epsilon_{\rm int} + p)}$$

θ dependence

$\theta \equiv V_S/V_A$

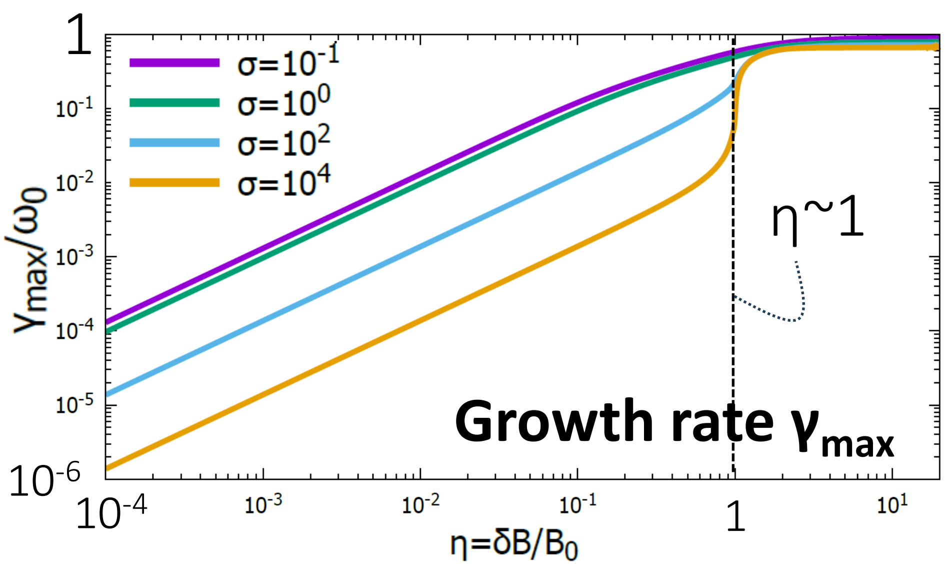
$$\sigma \equiv \frac{B_0^2}{4\pi(\rho c^2 + \epsilon_{\mathrm{int}} + p)}$$

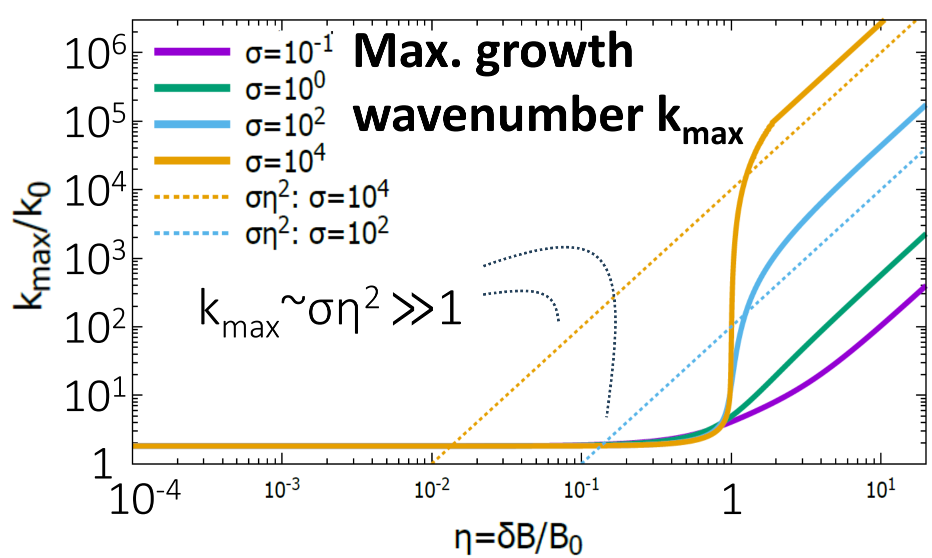
- Analytical formulae (for general sigma)
  - Formulae : $\Gamma = \frac{1}{2}\eta\theta^{-1/2}\frac{\sqrt{1-\theta}}{1+\theta}\frac{1+\sigma\left(1+\theta^2\right)}{(1+\sigma)^{3/2}} \sim \frac{1}{2}\eta\theta^{-1/2}(1+\sigma)^{-1/2}$
  - For θ<<1, a different expansion is needed, and the growth rate no longer depends on θ.



η dependence (Preliminary)

- Dependence on the parent wave amplitude $\eta=\delta B/B_0$
  - Thanks to exact finite-amplitude wave solutions, our model remains valid even when the parent wave's amplitude is larger than unity
  - Where the amplitude η becomes unity, the growth rate $\gamma \sim \omega_0$, roughly independent of σ or θ
  - The maximum growth wavenumber $k_{max} \sim \sigma\eta^2$,which becomes extremely short wavelength
  - Although ideal MHD predicts very rapid decay, the actual evolution may be regulated by kinetic or dissipative processes (e.g., the Landau resonance)





Summary

arXiv: 2404.15689

- What do we want to know?
  - To understand the mechanism of FRBs, we study the propagation of Alfvén waves in relativistically magnetized plasma without the force-free approximation.
- What did we do?
  - We derived the equations of sound and Alfven-like perturbations on a background with finite amplitude Alfven waves and obtained the dispersion relation for perturbations.
- What did we find?
  - As in the non-relativistic case, even for relativistically magnetized plasma, there is a decay instability in which Alfven waves excite forward-propagating sound waves and backward-propagating Alfven waves.
  - Obtained an analytical expression for the growth rate of the instability, finding that growth rate of the decay instability becomes smaller for large σ.
  - For sufficiently strong parent waves, the decay occurs on a timescale of about one wave period, so strong Alfvén waves cannot propagate over long distances.

Backup

Emission site

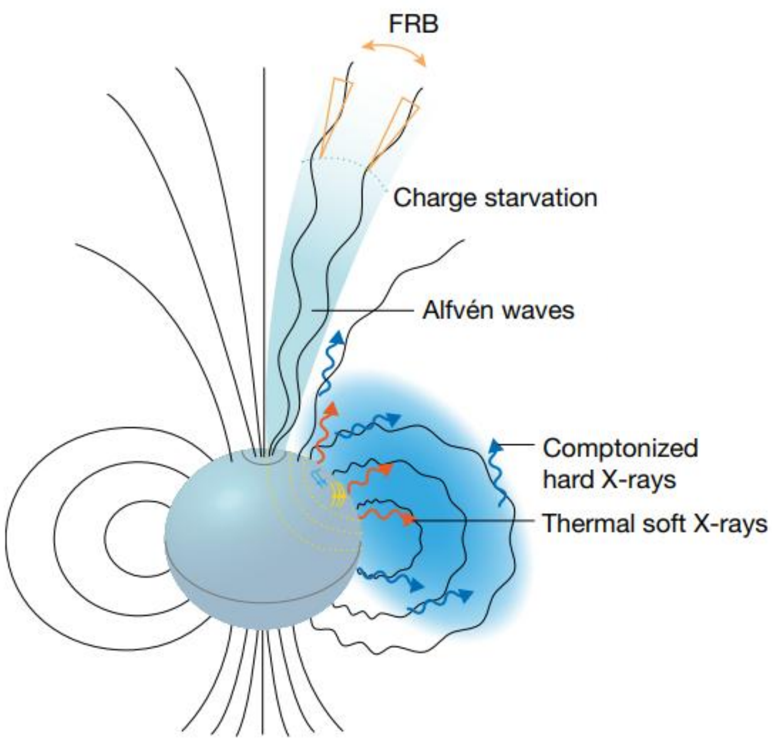
- Even if magnetars are origin of FRBs, an **emission site are still unknown**
  - Two scenarios are discussed extensively

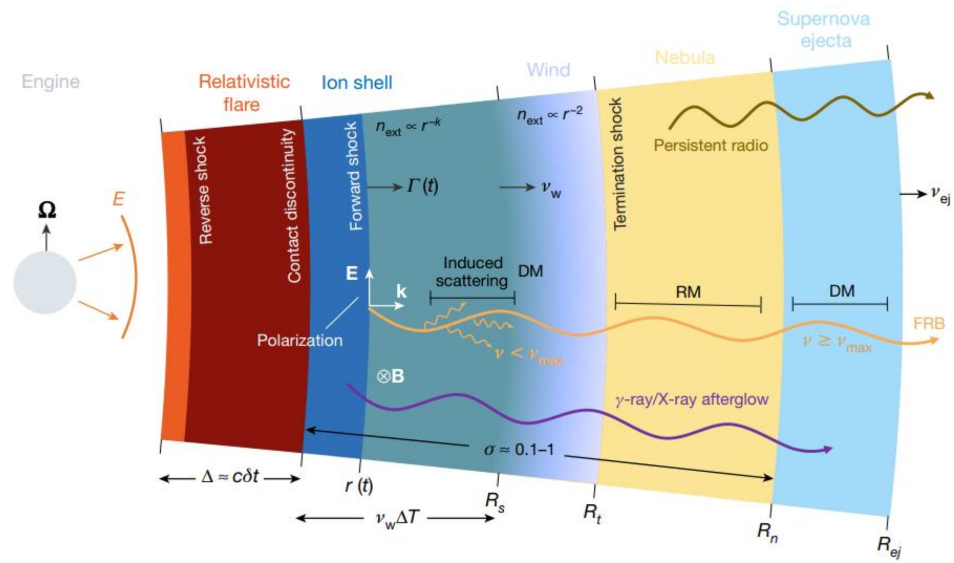
Magnetospheric model

e.g., Katz 2014, Lu & Kumar 2018,
Yang & Zhang 2018, Kumar & Bosnjak 2020

External shock model

e.g., Lyubarsky 2014, Murase+2016,
Margalit+2020, Beloborodov 2020





(B. Zhang 2020)

Emission site

- Even if magnetars are origin of FRBs, an **emission site are still unknown**
    - Two scenarios are discussed extensively

Some difficulties are pointed out

✓ **Difficult to explain the observed circularly polarization**

(e.g., Gajjar+18, Michilli+18, Oslowski+19)

✓ **Conflicts with short time variability of some bursts**

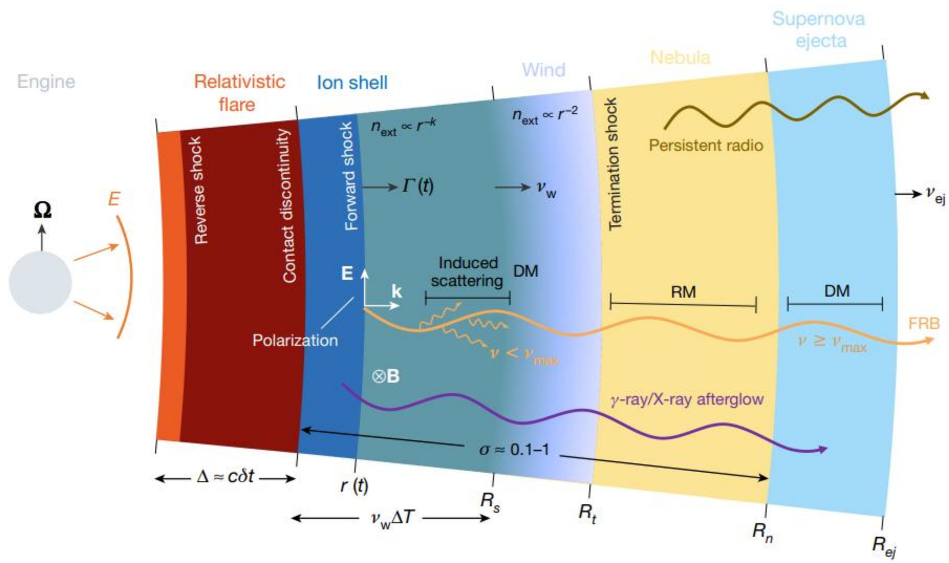
(e.g., Beniamini & Kumar 20, Lu+22)

✓ **Inconsistent with scintillation result**

(e.g., Nimmo 2024)

External shock model

e.g., Lyubarsky 2014, Murase+2016, Margalit+2020, Beloborodov 2020



(B. Zhang 2020)

Emission site

- Even if magnetars are origin of FRBs, an **emission site are still unknown**
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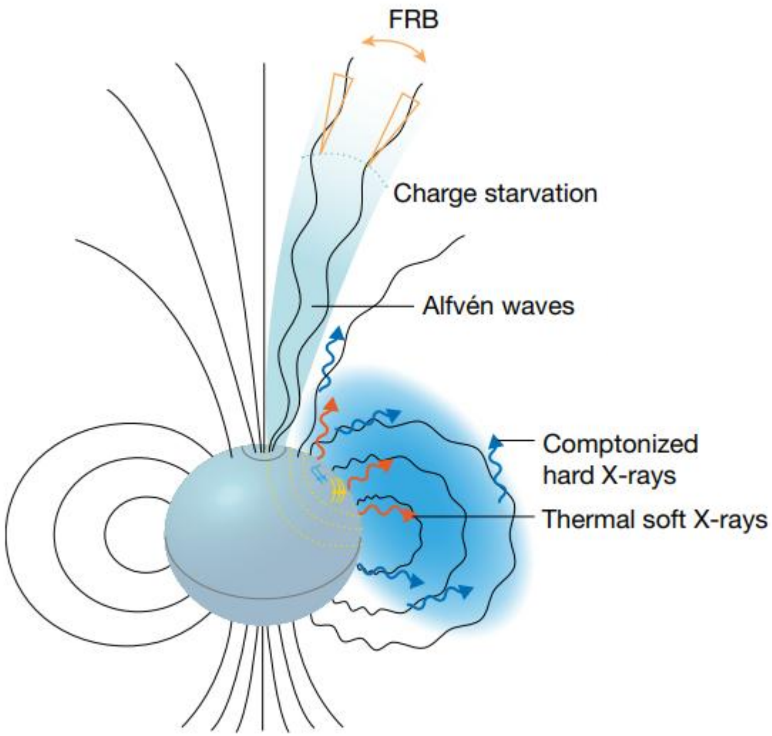


Some theoretical difficulties

- ✓ **Poor understanding to the coherent emission mechanism**
- ✓ **High intensity photons cannot escape from magnetosphere**

(Idea)
Transport energy via Alfven waves, which can propagate relatively freely in plasma

Converted into photons where optically thin

(e.g., Kumar & Bosnjak 20, Yuan+20)

Perturbed equations

	Background	Parent wave	Daughter waves	
		Alfvén wave	Alfvén wave	Sound wave
$\boldsymbol{B} =$	$\boldsymbol{B_0}$	$+\,\boldsymbol{\delta B}$	$+\,\boldsymbol{b}_\perp$	
$\boldsymbol{v} =$		$\boldsymbol{\delta v}$	$+\,\boldsymbol{v}_\perp$	$+\,\boldsymbol{v}_\parallel$
$\epsilon =$	ϵ_0			$+\,\epsilon_\parallel$
	$O(1)$	$O(\eta)$	$O(\varepsilon)$	

- Acoustic wave
  - Energy conservation

$$\frac{1+\beta_s^2\delta\beta^2}{1-\delta\beta^2}\frac{1}{c}\frac{\partial\epsilon_\parallel}{\partial t} + \left(\delta\gamma^2(\epsilon_0+p_0)\right)\frac{\partial\beta_\parallel}{\partial z} = -\frac{1}{c}\frac{\partial}{\partial t}\left[\left(2\delta\gamma^4(\epsilon_0+p_0)+\frac{B_0^2}{4\pi}\right)(\delta\boldsymbol{\beta}\cdot\boldsymbol{\beta}_\perp) - \frac{\delta\beta\delta B B_0\beta_\parallel}{4\pi}\right] + \frac{B_0}{4\pi}\left[\boldsymbol{\beta}_\perp\cdot\frac{\partial}{\partial z}(\delta\boldsymbol{B})+\delta\boldsymbol{\beta}\cdot\frac{\partial \boldsymbol{b}_\perp}{\partial z}\right]$$

  - Momentum equation (z-component)

$$\left[\delta\gamma^2(\epsilon_0+p_0)+\frac{\delta B^2}{4\pi}\right]\frac{1}{c}\frac{\partial\beta_\parallel}{\partial t} + \beta_s^2\frac{\partial\epsilon_\parallel}{\partial z} = -\frac{\partial}{\partial z}\left(\frac{\delta\boldsymbol{B}\cdot\boldsymbol{b}_\perp}{4\pi}\right) + \frac{B_0}{4\pi}\left(\delta\boldsymbol{B}\cdot\frac{1}{c}\frac{\partial\boldsymbol{\beta}_\perp}{\partial t} + \boldsymbol{b}_\perp\cdot\frac{1}{c}\frac{\partial}{\partial t}(\delta\boldsymbol{\beta})\right)$$

- Alfven wave
  - Momentum equations (in xy-plain)

$$\left[\delta\gamma^2(\epsilon_0+p_0)+\frac{B_0^2+\delta B^2}{4\pi}\right]\frac{1}{c}\frac{\partial\boldsymbol{\beta}_\perp}{\partial t} - \frac{B_0}{4\pi}\frac{\partial\boldsymbol{b}_\perp}{\partial z} = -\delta\gamma^2(\epsilon_0+p_0)\beta_\parallel\frac{\partial}{\partial z}(\delta\boldsymbol{\beta})$$
$$-\delta\gamma^2\left(1+\beta_s^2\right)\epsilon_\parallel\frac{1}{c}\frac{\partial}{\partial t}(\delta\boldsymbol{\beta}) - \delta\boldsymbol{\beta}\beta_s^2\frac{1}{c}\frac{\partial\epsilon_\parallel}{\partial t} + \frac{B_0}{4\pi}\frac{1}{c}\frac{\partial}{\partial t}(\beta_\parallel\delta\boldsymbol{B})$$
$$-2\delta\gamma^4(\epsilon_0+p_0)(\delta\boldsymbol{\beta}\cdot\boldsymbol{\beta}_\perp)\frac{1}{c}\frac{\partial}{\partial t}(\delta\boldsymbol{\beta}) + \frac{1}{4\pi}\frac{\partial}{\partial z}\left[\hat{z}\cdot\{(\delta\boldsymbol{\beta}\times\boldsymbol{b}_\perp)+(\boldsymbol{\beta}_\perp\times\delta\boldsymbol{B})\}\right](\delta\boldsymbol{\beta}\times\boldsymbol{B_0})$$
$$-\frac{\delta\boldsymbol{B}\cdot\boldsymbol{b}_\perp}{4\pi}\frac{1}{c}\frac{\partial}{\partial t}(\delta\boldsymbol{\beta}) - \left(\delta\boldsymbol{B}\cdot\frac{1}{c}\frac{\partial\boldsymbol{b}_\perp}{\partial t}\right)\delta\boldsymbol{\beta} + \frac{\delta B\delta\beta}{4\pi}\frac{1}{c}\frac{\partial\boldsymbol{b}_\perp}{\partial t} + \frac{\delta\boldsymbol{B}}{4\pi}\left(\frac{1}{c}\frac{\partial\boldsymbol{\beta}_\perp}{\partial t}\cdot\delta\boldsymbol{B}\right) + \frac{\delta\boldsymbol{B}\cdot\boldsymbol{\beta}_\perp}{4\pi}\frac{1}{c}\frac{\partial}{\partial t}(\delta\boldsymbol{B})$$

  - Induction euations (in xy-plain)

$$\frac{1}{c}\frac{\partial\boldsymbol{b}_\perp}{\partial t} - B_0\frac{\partial\boldsymbol{\beta}_\perp}{\partial z} = -\frac{\partial}{\partial z}(\beta_\parallel\delta\boldsymbol{B})$$

too complicated... → First we ignore $O(\varepsilon\eta^2)$

Perturbed equations

	Background	Parent wave	Daughter waves	
				Sound wave
$\boldsymbol{B} =$	$\boldsymbol{B}_0$	$+\,\boldsymbol{\delta B}$	$+\,\boldsymbol{b}_\perp$	
$\boldsymbol{v} =$		$\boldsymbol{\delta v}$	$+\,\boldsymbol{v}_\perp$	$+\,\boldsymbol{v}_\parallel$
$\epsilon =$	ϵ_0			$+\,\epsilon_\parallel$
		Alfvén wave	Alfvén wave	
	$O(1)$	$O(\eta)$	$O(\varepsilon)$	

- Acoustic wave
  - Energy conservation

$$\frac{1}{c}\frac{\partial \epsilon_\parallel}{\partial t} + (\epsilon_0 + p_0)\frac{\partial \beta_\parallel}{\partial z} = -\frac{1}{c}\frac{\partial}{\partial t}\left[\left(2(\epsilon_0+p_0) + \frac{B_0^2}{4\pi}\right)(\delta\boldsymbol{\beta}\cdot\boldsymbol{\beta}_\perp)\right] + \frac{B_0}{4\pi}\left[\boldsymbol{\beta}_\perp\cdot\frac{\partial}{\partial z}(\delta\boldsymbol{B}) + \delta\boldsymbol{\beta}\cdot\frac{\partial \boldsymbol{b}_\perp}{\partial z}\right]$$

  - Momentum equation (z-component)

$$(\epsilon_0+p_0)\frac{1}{c}\frac{\partial \beta_\parallel}{\partial t} + \beta_s^2\frac{\partial \epsilon_\parallel}{\partial z} = -\frac{\partial}{\partial z}\left(\frac{\delta\boldsymbol{B}\cdot\boldsymbol{b}_\perp}{4\pi}\right) + \frac{B_0}{4\pi}\left(\delta\boldsymbol{B}\cdot\frac{1}{c}\frac{\partial \boldsymbol{\beta}_\perp}{\partial t} + \boldsymbol{b}_\perp\cdot\frac{1}{c}\frac{\partial}{\partial t}(\delta\boldsymbol{\beta})\right)$$

- Alfven wave
  - Momentum equations (in xy-plain)

Neglecting $O(\varepsilon\eta^2)$, $O(\varepsilon^2)$

$$\left(\epsilon_0 + p_0 + \frac{B_0^2}{4\pi}\right)\frac{1}{c}\frac{\partial \boldsymbol{\beta}_\perp}{\partial t} - \frac{B_0}{4\pi}\frac{\partial \boldsymbol{b}_\perp}{\partial z} =$$

$$-(\epsilon_0+p_0)\beta_\parallel\frac{\partial}{\partial z}(\delta\boldsymbol{\beta}) - (1+\beta_s^2)\frac{B_0\epsilon_\parallel}{4\pi\mathcal{E}}\frac{\partial}{\partial z}(\delta\boldsymbol{B}) - \delta\boldsymbol{\beta}\beta_s^2\frac{1}{c}\frac{\partial \epsilon_\parallel}{\partial t} + \frac{B_0}{4\pi}\frac{1}{c}\frac{\partial}{\partial t}(\beta_\parallel\delta\boldsymbol{B})$$

  - Induction euations (in xy-plain)

$$\frac{1}{c}\frac{\partial \boldsymbol{b}_\perp}{\partial t} - B_0\frac{\partial \boldsymbol{\beta}_\perp}{\partial z} = -\frac{\partial}{\partial z}(\beta_\parallel\delta\boldsymbol{B})$$

$$\left(\mathcal{E} \equiv \epsilon_0 + p_0 + \frac{B_0^2}{4\pi}\right)$$

Assumption: the amplitude of the parent wave is not so large

Perturbed equations

- Acoustic wave

	Background	Parent wave	Daughter waves	
$\boldsymbol{B} =$	$\boldsymbol{B}_0$	$+\,\delta\boldsymbol{B}$	$+\,\boldsymbol{b}_\perp$	
$\boldsymbol{v} =$		$\delta\boldsymbol{v}$	$+\,\boldsymbol{v}_\perp$	$+\,\boldsymbol{v}_\parallel$
$\epsilon =$	ϵ_0			$+\,\epsilon_\parallel$
		Alfvén wave	Alfvén wave	Sound wave

Equations are linear with respect to the perturbed quantities

⇩

Finding a solution is equivalent to finding a dispersion relation

⇩

Let us check if there is an unstable mode in the dispersion relation!

Sigma dependence

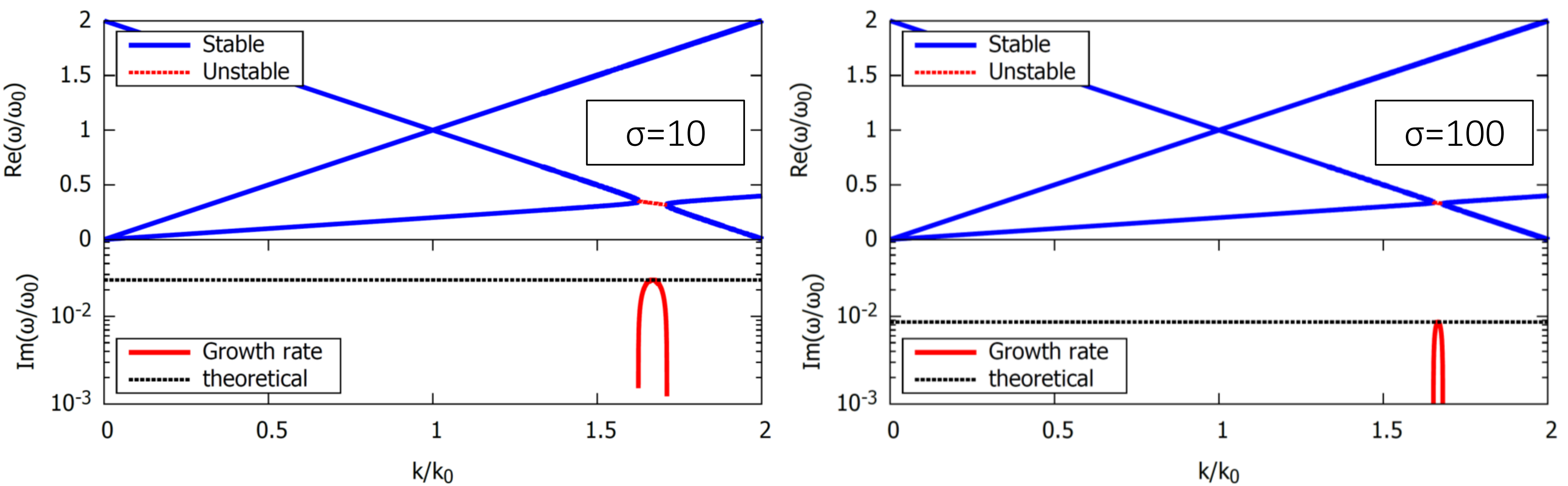
$$\theta \equiv V_S / V_A$$

$$\sigma \equiv \frac{B_0^2}{4\pi(\rho c^2 + \epsilon_{\mathrm{int}} + p)}$$

- Growth rate of the decay instability

$$\Gamma = \frac{1}{2}\eta\theta^{-1/2}\frac{\sqrt{1-\theta}}{1+\theta}\frac{1+\sigma\left(1+\theta^2\right)}{\left(1+\sigma\right)^{3/2}} \sim \frac{1}{2}\eta\theta^{-1/2}\left(1+\sigma\right)^{-1/2}$$

The larger σ, the narrower the range of wavenumbers that become unstable, and the instability disappears at σ → ∞.



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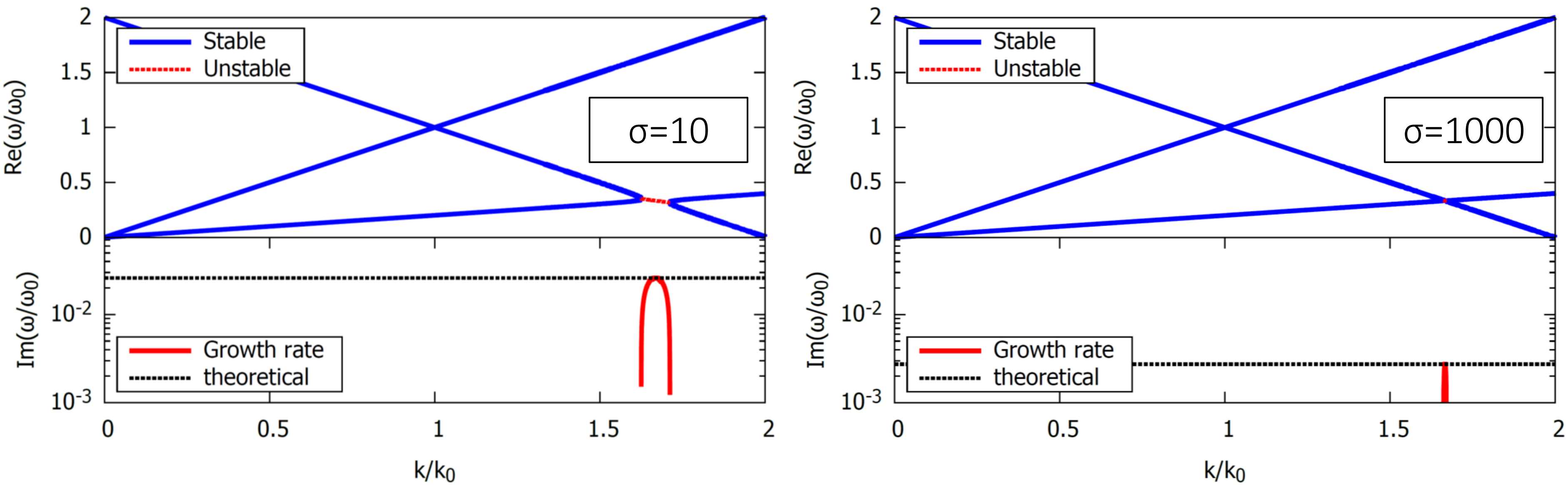
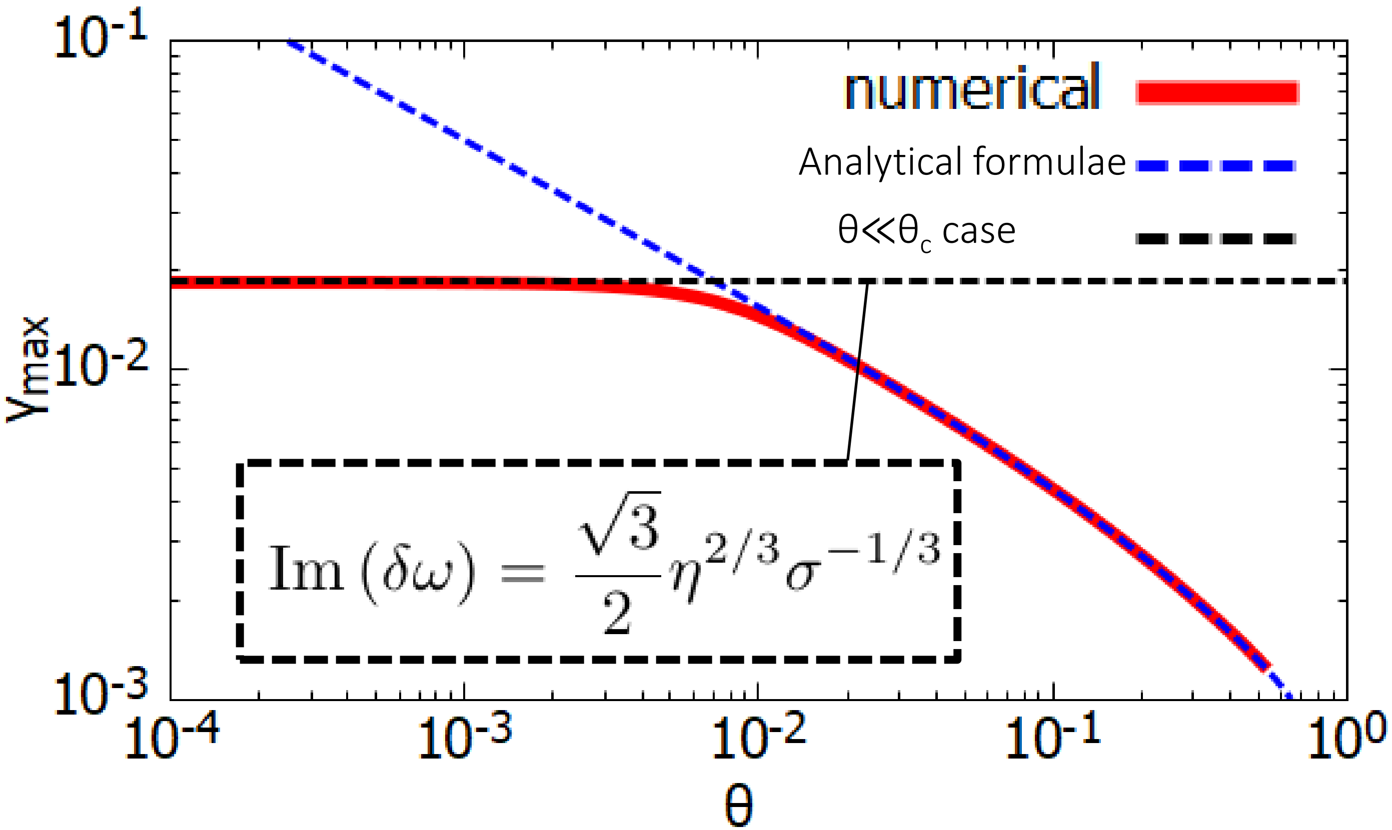


θ dependence

$\theta \equiv V_S / V_A$

$$\sigma \equiv \frac{B_0^2}{4\pi(\rho c^2 + \epsilon_{\mathrm{int}} + p)}$$

- Analytical formulae (for general sigma)
  - Formulae : $\Gamma = \frac{1}{2}\eta\theta^{-1/2}\frac{\sqrt{1-\theta}}{1+\theta}\frac{1+\sigma\left(1+\theta^2\right)}{(1+\sigma)^{3/2}} \sim \frac{1}{2}\eta\theta^{-1/2}(1+\sigma)^{-1/2}$
  - For θ<<1, a different expansion is needed, growth rate will no longer depend on θ



Discussion : Why instability vanishes?

- Why does the instability vanish for σ → ∞?
  - In the non-rela case, why the wave becomes unstable?

    Alfvén + Alfvén : Fluctuation of the magnetic pressure → generates sound wave

    Alfvén + Sound : Fluctuation of the inertia

    →Magnetic tension per mass is fluctuated→generates Alfven wave
  - In the relativistic case, the force perpendicular to the background magnetic field is dominated by that originating from the displacement current.
  - Alfven waves become like free photons and are no longer affected by the plasma.

$$\mathcal{E}\frac{1}{c}\frac{\partial \boldsymbol{\beta}_\perp}{\partial t} - \frac{B_0}{4\pi}\frac{\partial \boldsymbol{b}_\perp}{\partial z} = \boxed{-\left(\epsilon_0 + p_0\right)\beta_\parallel \frac{\partial}{\partial z}\left(\delta\boldsymbol{\beta}\right)} \boxed{-\left(1+\beta_s^2\right)\frac{B_0\epsilon_\parallel}{4\pi\mathcal{E}}\frac{\partial}{\partial z}\left(\delta\boldsymbol{B}\right)} \boxed{-\delta\boldsymbol{\beta}\beta_s^2\frac{1}{c}\frac{\partial \epsilon_\parallel}{\partial t}} \boxed{+\frac{B_0}{4\pi}\frac{1}{c}\frac{\partial}{\partial t}\left(\beta_\parallel \delta\boldsymbol{B}\right)}$$

Convection | Inertia | Inertia (Internal energy) | Displacement current