Unphysical topological charge in nonabelian gauge theory

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Based on

N. Yamanaka, arXiv:2212.10994 [hep-th]

N. Yamanaka, arXiv:2212.11820 [hep-ph]

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History of chiral anomaly/topological charge

- ABJ anomaly (Adler, Bell, Jackiw, 1969)
- Goldstone dipole mechanism resolves $U(1)_A$ problem (Kogut, Susskind, 1974)
- Instanton (Belavin, Polyakov, Schwartz, Tyupkin, 1975)
- Theta-vacuum (Jackiw, Rebbi, Callan, Dashen, Gross, 1976)
- Instanton resolves $U(1)_A$ problem ('t Hooft, 1976)
- Peccei-Quinn mechanism (Peccei, Quinn, 1977)
- Large N_c analysis, and constraint on η' mass (instanton does not resolve $U(1)_A$ problem) (Witten, Veneziano, 1979)
- Chiral perturbation analysis of neutron EDM from θ (Crewther, Veneziano, Witten, 1979)
- Anomaly in path integral (Fujikawa, 1979)

Topological charge

$$\tilde{F}_a^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F^a_{\rho\sigma}$$

$$K_\mu \equiv \frac{\alpha_s}{8\pi}\epsilon_{\mu\nu\rho\sigma}\left[A_a^\nu F_a^{\rho\sigma} - \frac{g_s}{3} f_{abc} A_a^\nu A_b^\rho A_c^\sigma\right]$$

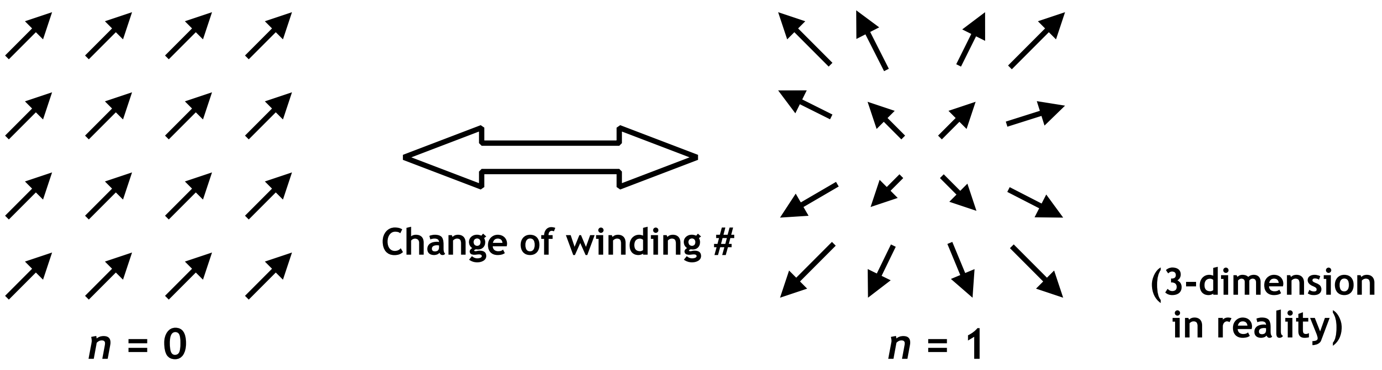
$$\int d^4x\, F_{\mu\nu,a}\tilde{F}_a^{\mu\nu} = \int d^4x\, \partial_\mu K^\mu$$

$$= \frac{i g_s \alpha_s}{24\pi}\int d^3\vec{x}\, f_{abc}\epsilon_{ijk} A_{ia}(\vec{x}) A_{jb}(\vec{x}) A_{kc}(\vec{x})\Big|_{t=-\infty}^{t=+\infty}$$

$$= \Delta n$$ **Integer!**

Integral of total derivative, but nonzero!

⇒ Δ*n* = change of winding number of gauge configurations



Theta-term and Strong CP problem

QCD vacuum is a coherent superposition of vacua with different winding number (topological charge)

$$|\theta\rangle = \sum_n e^{-in\theta}|n\rangle$$

A correlator in the path integral formulation is written as

$$\begin{aligned}\langle\theta|O_{\rm phys}|\theta\rangle &= \sum_{n,m} e^{-i(n-m)\theta}\langle m|O_{\rm phys}|n\rangle \\ &= \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar\psi\, O_{\rm phys} e^{i\int d^4x\,\mathcal{L}_{\rm QCD}+i\theta\frac{\alpha_s}{8\pi}\int d^4x\, F_{\mu\nu a}\tilde F^{\mu\nu}_a}\end{aligned}$$

(example of one flavor QCD)

If θ is not zero, $\mathcal{L}_\theta = \theta\frac{\alpha_s}{8\pi}F^a_{\mu\nu}\tilde F^{\mu\nu}_a$ **(θ-term) appears effectively in Lagrangian**

There is in principle no symmetry argument to forbid θ

However, it is known that θ is very small from EDM experiments (|θ| < 10^{-10})

C. Abel et al., Phys. Rev. Lett. **124**, 081803 (2020);
B. Graner et al., Phys. Rev. Lett. **116**, 161601 (2016)

Why is θ so small compared to the QCD coupling??

➡ **Strong CP problem**

List of arguments needed for demonstration

- Gradient of gauge function (longitudinal component of gauge field) is unphysical within perturbation theory and BRST symmetry

- Goal : show that the topological charge always involves the unphysical longitudinal component of gauge field

- BRST symmetry only holds for small field amplitude, within the fundamental modular region (= no Gribov copies)

- Topological charge (chiral anomaly) is perturbative one-loop finite (Adler-Bardeen theorem = ’t Hooft anomaly matching)

- Perturbation theory uses infinitesimal field amplitude (Glauber’s coherent field)

- Extension to fermions with Atiyah-Singer theorem: Chiral Dirac zero-modes are unphysical, QCD Lagrangian is $U(1)_A$ sym.

BRST algebra

Local gauge transform

Quarks (fundamental representation):

$$\psi(x) \to U(x)\psi(x) \qquad\qquad U(x) = e^{it_a\chi_a(x)}$$

Gauge field (adjoint representation):

$$A_a^\mu(x)t_a \to U(x)[A_a^\mu(x)t_a + i\partial^\mu/g_s]U^\dagger(x)$$

$$= A_a^\mu(x)t_a + \partial^\mu\chi_a(x)t_a + O(g_s)$$

$O(g_s)$ terms are irrelevant in perturbation theory

After gauge fixing, the gauge function χ_a becomes Faddeev-Popov ghost

⇒ In perturbation theory, longitudinal component of gluons and ghost are unphysical due to BRST symmetry

BRST algebra

BRST transform:

$$\begin{cases} \delta_B \psi = i\lambda g_s c_a (t_a \psi) \\ \delta_B A^\mu_a = \lambda(\partial^\mu c_a + g_s f_{abc} A^\mu_b c_c) \\ \delta_B c_a = -\frac{1}{2}\lambda g_s f_{abc} c_b c_c \\ \delta_B \bar{c}_a = i\lambda B_a \\ \delta_B B_a = 0 \end{cases}$$

ψ : **quark (fundamental repr.)**
A : **gauge field**
c : **Faddeev-Popov ghost**
$\bar{c}$: **Anti-ghost**
B : **Nakanishi-Lautrup field**

Important property : Nilpotency

$$\delta_B \delta_B x = 0$$

⇒ BRST algebra is closed

Using this property, we may construct BRST invariants, especially Lagrangian

$$\mathcal{L}_{\mathrm{BRS}} = \delta_B f(\psi, A, c, \bar{c}, B)$$

C. Becchi, A. Rouet and R. Stora, Ann. Phys. **98**, 287 (1976);
V. I. Tyutin, arXiv:0812.0580 [hep-th].

BRST quartet mechanism

Physical states must be BRST invariant

$$Q_B|\text{phys}\rangle = 0 \qquad\qquad Q_B : \text{ generator of BRST transformation}$$

(analogue of the Gupta-Bleuler subsidiary condition)

Representations of BRST non-singlet contain only 4 fields (pair of BRST doublets)

$$|\alpha, N\rangle \xrightarrow{Q_B\,:} |\alpha, N+1\rangle \xrightarrow{Q_B\,:} 0 \quad \text{(BRST doublet)}$$

$$\text{(Anti-BRST doublet)} \quad |\alpha, -(N+1)\rangle \xrightarrow{Q_B\,:} |\alpha, -N\rangle \xrightarrow{Q_B\,:} 0$$

N **: ghost number**

α : other quantum #

BRST "quartets" do not contribute to physical observables

Quartet mechanism !

T. Kugo and I. Ojima, Phys. Lett. B **73**, 459(1978);
Prog. Theor. Phys. Suppl. **66**, 1 (1979).

Note:

BRST charge must be well-defined, then quartet mechanism is certified

**In perturbation theory, BRST charge is well-defined,
but BRST symmetry is broken in Gribov region**

Longitudinally polarized gluon

BRST transform of gauge field has derivative of ghost and a composite term

$$\delta_B A_a^\mu = \lambda D^\mu c_a = \lambda(\underline{\partial^\mu c_a} + \underline{g_s f_{abc} A_b^\mu c_c})$$

Derivative term :

Derivative = longitudinal
Cancel longitudinally polarized gluon in perturbation theory

Composite term :

Does not exist in perturbation theory (no bound-state)
May exist with nonperturbative effect (if gauge-ghost bound-state exists)

In perturbation theory, longitudinal polarization of gluon is unphysical, while transverse ones are BRST singlet, physical

(Recall that BRST non-invariant quantity is not observable)

Processes involving longitudinally polarized gluon is unphysical in perturbation theory!

Inspection of topological charge

Topological charge ALWAYS involves longitudinal component

Topological charge :

$$\frac{i g_s \alpha_s}{24\pi}\int d^3\vec{x}\; f_{abc}\epsilon_{ijk}A_{ia}(\vec{x})A_{jb}(\vec{x})A_{kc}(\vec{x})\Big|_{t=-\infty}^{t=+\infty}$$

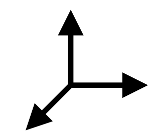
Infinity, time is frozen

Gauge fields do not depend on time, and time component does not contribute

⇒ Restricted to 3-dimensional space

Triple product of gauge field operators, at the same coordinate $\vec{x}$

⇒ Triple product (contraction with ε_{ijk}) covers all 3-dimensional directions



⇒ Also covers the gradient direction of gauge function

⇒ Gradient direction = "longitudinal" component ⇒ Unphysical !

Topological charge involves unphysical longitudinal gauge field

➡ Processes involving topological charge is unphysical !!

Strong CP problem resolved?

Topological charge cannot be detected, unphysical

⇒ θ-term cannot be detected, unphysical

⇒ Strong CP problem resolved?

Our answer is YES , but

so far we have assumed perturbation and BRST symmetry

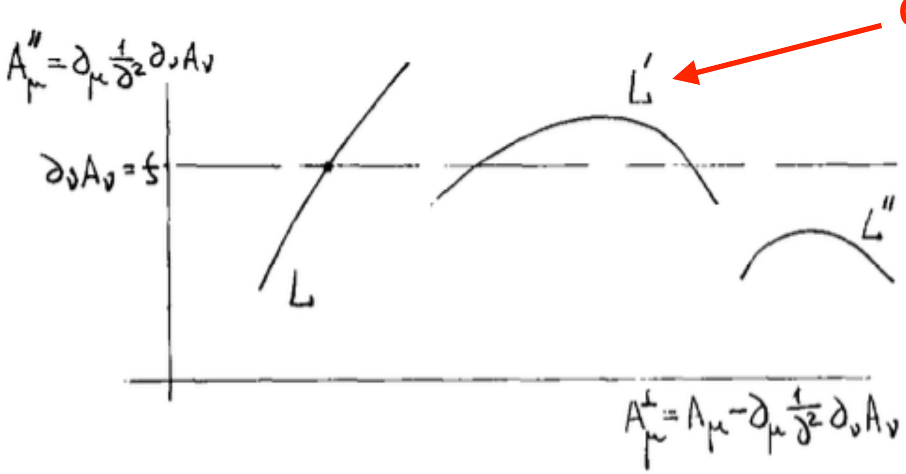
However, restriction to Gribov region might upset BRST symmetry

We now have to show that Gribov ambiguity is not relevant
for the topological charge

Gribov ambiguity and BRST symmetry breaking

Gauge fixing and Gribov ambiguity

In nonabelian gauge theory, Lorentz covariant gauge fixing cannot completely fix the gauge ⇒ Gribov copies



Case of nonabelian gauge theory :
gauge orbit crosses the gauge condition more than once

V. N. Gribov, Nucl. Phys. B **139**, 1 (1978).

To remove Gribov copies, we have to restrict the path integral to a region with smaller field amplitude, ideally so as to fulfill the gauge fixing condition only once.

Gribov copies and Gribov region

If there are Gribov copies, there should be other gauge transformed configurations $A'_\mu = UA_\mu U^\dagger - \frac{i}{g_s}(\partial_\mu U)U^\dagger$ satisfying the gauge condition

Take 4-divergence → Fulfill Landau gauge condition

$$(\partial_\mu U)A_\mu U^\dagger + UA_\mu(\partial_\mu U)^\dagger - \frac{i}{g_s}\Big[(\partial^2 U)U^\dagger + (\partial_\mu U)(\partial_\mu U^\dagger)\Big] = 0$$

Use gauge condition $\partial_\mu A_\mu = 0,\ \partial_\mu A'_\mu = 0$ and take infinitesimal shift

⇒ $-\partial_\mu D_\mu \alpha = 0$ ⇒ Faddeev-Popov operator has zero eigenvalue

Faddeev-Popov operator may have zero eigenvalue for sufficiently large A_μ

Nonabelian gauge theory+Lorentz covariant gauge fixing always affected by Gribov copies (no such problem with gauges with decoupled ghost, QED)

- First Gribov region :
  Restrict to space with positive definite Faddeev-Popov eigenvalue
- Fundamental modular region :
  Space without Gribov copies (≠1st Gribov region), still not established

N. Vandersickel and D. Zwanziger, Phys. Rept. **520**, 175 (2012).

Gribov-Zwanziger action and BRST breaking

Restriction of the path integral to the Gribov region
⇒ Gribov-Zwanziger (GZ) action

$$S_{\rm GZ} = S_{\rm YM} + S_{\rm fix} + \int d^4x \Big[\bar{\varphi}^{ac}_\mu \partial_\nu D^{am}_\nu \varphi^{mc}_\mu - \bar{\omega}^{ac}_\mu \partial_\nu D^{am}_\nu \omega^{mc}_\mu - g(\partial_\nu \bar{\omega}^{ac}_\mu) f^{abm} (D_\nu c)^b \varphi^{mc}_\mu \Big]$$
$$-\gamma^2 \int d^4x \Big[g f^{abc} A^a_\mu \varphi^{bc}_\mu + g f^{abc} A^a_\mu \bar{\varphi}^{bc}_\mu + 4(N_c^2-1)\gamma^2 \Big]$$

BRST transform rules: $\delta_B A^a_\mu = -(D_\mu c)^a, \; \delta_B c^a = \frac{g_s}{2} f^{abc} c^b c^c, \;\; \delta_B \bar{c}^a = b^a, \; \delta_B b^a = 0,$ **(standard BRST)**

$\delta_B \varphi^a_i = \omega^a_i, \;\; \delta_B \omega^a_i = 0, \;\; \delta_B \bar{\omega}^a_i = \bar{\varphi}^a_i, \;\; \delta_B \bar{\varphi}^a_i = 0.$ **(additional part)**

⇒ GZ action softly breaks BRST symmetry

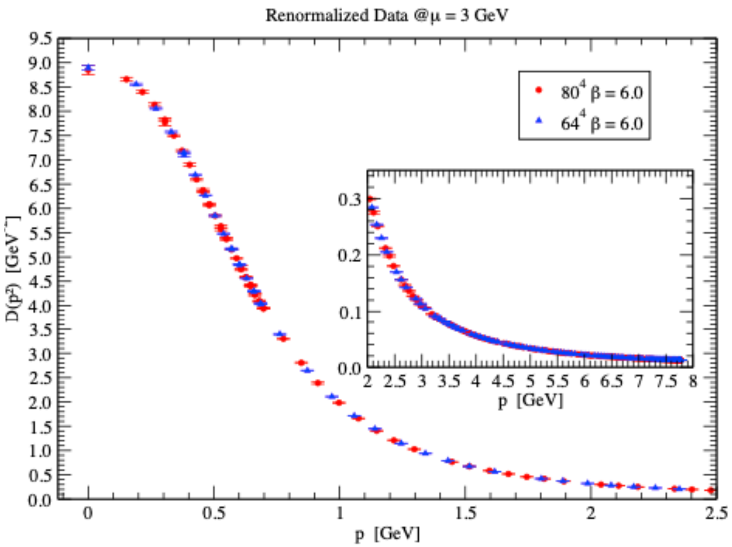
D. Zwanziger, Nucl. Phys. B **321**, 591 (1989).

Refined formulation with Gribov-Zwanziger action yields massive gluons, with modified BRST symmetry (conceived to fit lattice results, see next slide)

D. Dudal et al., Phys. Rev. D **78**, 065047 (2008).

⇒ Modification of original BRST symmetry may upset our discussion?

Gluon propagator in lattice QCD with Landau gauge



Lattice calculation of gluon propagator in Landau gauge yields massive gluon

D. Dudal et al., Ann. Phys. B **397**, 351 (2018).

Lattice Landau gauge does not totally certify the restriction to the fundamental modular region, but the most important feature is OK

⇒ Consistent with refined Gribov-Zwanziger action,

lattice results suggest that (modified) GZ framework is working

Does our argument on the unphysical topological charge still hold??

Glauber's coherent field and perturbative finiteness of anomaly

Field amplitude in quantum field theory

In classical field theory (e.g. Maxwell's electrodynamics), fields are given by

$$\phi(x) = A\sin(px+\delta)$$

⇒ Field amplitude (A) is arbitrary

In quantum field theory ??

⇒ Field amplitude should be the effective number of particles with the same quantum number

⇒ Field amplitude ($N_{\rm eff}$) is extracted by acting the field operator

$$\hat{\phi}(p)|N_{\rm eff}\phi\rangle = N_{\rm eff}\phi(p)|N_{\rm eff}\phi\rangle \qquad \hat{\phi}(p) = \int dx\,[\phi(x)a_p e^{ipx} + \phi^\dagger(x)a_p^\dagger e^{-ipx}]$$

We note that states change in particle number basis

$$\hat{\phi}(p)|N_{\rm eff}\phi\rangle = \hat{\phi}(p)(a_p^\dagger)^{N_{\rm eff}}|0\rangle = N\phi(p)(a_p^\dagger)^{N_{\rm eff}-1}|0\rangle = N\phi(p)|(N_{\rm eff}-1)\phi\rangle$$

(Particle number is due to (anti)commutation relation $[a_x, a_{x'}^\dagger] = \delta(x-x')$)

Note : if $N_{\rm eff}$ is large, $N_{\rm eff}$ -1 ≈ $N_{\rm eff}$ 、 approximate eigenstate,

but not exact eigenstate

⇒ The light we see in real life, laser, etc, are almost classical

Coherent states

In particle number (perturbative) basis, there is no eigenstates of field operator

What does a field operator eigenstate with finite amplitude look like?

⇒ Coherent state! (Coherence = classical wave)

Coherent states are infinite superposition of multi-particle states

$$|N_{\text{eff}}\phi(\alpha)\rangle = e^{N_{\text{eff}}a_\alpha^\dagger - N_{\text{eff}}^* a_\alpha}|0\rangle = \sum_n^\infty \frac{1}{n!}(N_{\text{eff}}a_\alpha^\dagger - N_{\text{eff}}^* a_\alpha)^n|0\rangle$$

(α : given set of quantum numbers)

R. J. Glauber, Phys. Rev. **131**, 2766 (1963).

In perturbation theory, the field amplitude is infinitesimal

$$|\epsilon\rangle = e^{-\frac{1}{2}|\epsilon|^2}\sum_n \frac{(\epsilon a^\dagger)^n}{n!}|0\rangle = (1+\epsilon a^\dagger)|0\rangle$$

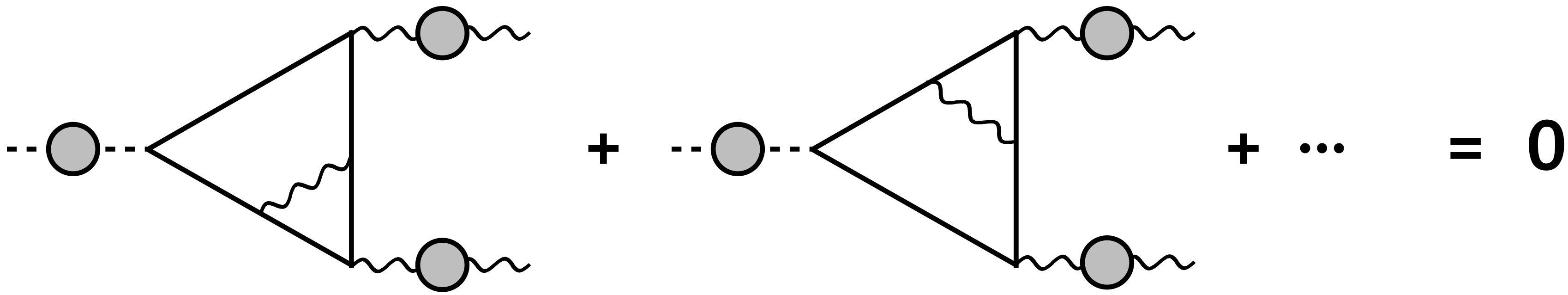
("1" has no physical effect)

In perturbation theory, we can count the number of particles

Adler-Bardeen theorem

Radiative corrections do not contribute to the anomaly up to renormalization of external fields

⇒ Works for nonabelian gauge theories, even with nonrenormalizable interaction



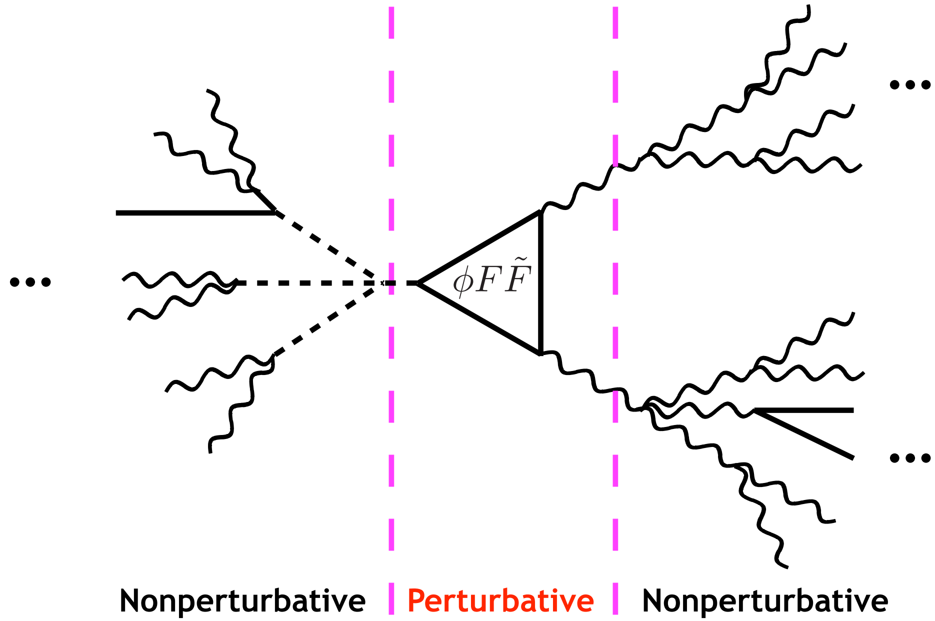
S. L. Adler and W. A. Bardeen, Phys. Rev. **182**, 1517 (1969);
D. Anselmi, Phys. Rev. D **91**, 105016 (2015).

Chiral anomaly and topological charge do not change at any scale, even if external fields are "dressed" by nonperturbative effects ('t Hooft anomaly matching)

⇒ Chiral anomaly and topological charge are perturbative finite!

Adler-Bardeen theorem will be used many times in our discussion!

Perturbative finiteness of anomaly



ϕ : field with $U(1)_A$ charge

At the instant of the operation of the chiral anomaly/topological charge density, external field operators are strictly perturbative, no corrections or mixings (Adler-Bardeen)

"Amplitudes" of external fields may be dynamically corrected, i.e. may be renormalized and interact, but the chiral anomaly contribution is always made of perturbative field operators in the path integral

According to Glauber, perturbative field = infinitesimal field amplitude
⇒ Infinitesimal fields always lie inside fundamental modular region

⇒ Topological charge not affected by Gribov copies! BRST analysis is OK!

Ward-Takahashi identity of topological charge (Another demonstration)

Ward-Takahashi identity of BRST symmetry

Assume the following BRST quartet $\begin{cases} [Q_B, A(x)] = iC(x) \\ \{Q_B, \bar{C}(y)\} = B(y) \end{cases}$

The following Ward-Takahashi (WTI) identity then holds:

$$\langle 0|\{Q_B, T[A(x), \bar{C}(y)]\}|0\rangle$$

$$= \boxed{\langle 0|T[A(x)B(y) - iC(x)\bar{C}(y)]|0\rangle = 0}$$

We previously saw that the topological charge is gauge variant

➡ **Let us apply the WTI to topological charge !**

(We assume BRST charge is well-defined, justified by the perturbative finiteness of chiral anomaly)

T. Kugo and I. Ojima, Phys. Lett. B **73**, 459 (1978);
T. Kugo and I. Ojima, Prog. Theor. Phys. Suppl. **66**,1 (1979).

WTI for topological charge

Consider the following (topological) BRST quartet

$$\begin{cases} K_\mu = \dfrac{\alpha_s}{4\pi}\epsilon_{\mu\nu\rho\sigma}\left[A_a^\nu \partial^\rho A_a^\sigma + \dfrac{1}{3} g_s f_{abc} A_a^\nu A_b^\rho A_c^\sigma\right] \\ \mathcal{C}_\mu = \dfrac{\alpha_s}{4\pi}\epsilon_{\mu\nu\rho\sigma}(\partial^\nu c_a)(\partial^\rho A_a^\sigma) \\ \bar{\mathcal{C}}_\mu = \dfrac{\alpha_s}{8\pi} g_s f_{abc}\epsilon_{\mu\nu\rho\sigma}(\partial^{-2}\partial^\nu \bar{c}_a) A_b^\rho A_c^\sigma \\ \mathcal{B}_\mu = \dfrac{\alpha_s}{8\pi} g_s f_{abc}\epsilon_{\mu\nu\rho\sigma}\left[(\partial^{-2}\partial^\nu B_a) A_a^\rho A_b^\sigma + (\partial^{-2}\partial^\nu \bar{c}_a) F_b^{\rho\sigma} c_c\right] \end{cases}$$

Topological current

T. Kugo, Nucl. Phys. B **155**, 368 (1979).

We have freedom to choose it

$$F_a^{\mu\nu} \equiv \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g_s f_{abc} A_b^\mu A_c^\nu$$

After taking total derivatives, we obtain the following "topological" WTI

$$\langle 0|T[\partial_\mu K^\mu(x)\partial_\nu \mathcal{B}^\nu(y)]|0\rangle = 0$$

(ghost term cancels)

Cluster decompose the two operators $(|x-y| \to \infty)$

$$\sum_{|\Omega\rangle \neq |0\rangle} \underbrace{\langle 0|\partial_\mu K^\mu(x)|\Omega\rangle}_{\text{Topological charge}} \underline{\langle \Omega|\partial_\nu \mathcal{B}^\nu(y)|0\rangle} = 0$$

What's this?

$|\Omega\rangle$ **:vacua belonging to different topological sectors**

Use of equation of motion

We now transform $\partial_\nu \mathcal{B}^\nu$ according to the equation of motion (EOM)

Assuming the most general gauge and BRST invariant Lagrangian,

$$\mathcal{L} = \left[-\frac{1}{4}(\partial^\mu A_a^\nu - \partial^\nu A_a^\mu)^2 - (\partial_\mu B_a) A_a^\mu + \frac{\alpha_r}{2} B_a B_a - i(\partial^\mu \bar{c}_a)(\partial_\mu c_a) + g_s A_a^\mu \Gamma_{\mu a} \right] \phi$$

ϕ : arbitrary BRST invariant scalar operator

$\Gamma_{\mu a}$: arbitrary color vector vertex

Apply variational principle to gauge field (EOM) :

$$\partial^2 A_a^\mu - \partial^\mu (\partial_\nu A_a^\nu) = \partial^\mu B_a - g_s \Gamma_a^\mu$$

By substituting to our $\partial_\nu \mathcal{B}^\nu$, we obtain

$$\partial^\mu \mathcal{B}_\mu = \frac{\alpha_s}{8\pi} g_s f_{abc} \epsilon_{\mu\nu\rho\sigma} \partial^\mu \Big[\Big\{ \underline{A_a^\nu A_b^\rho A_c^\sigma} - \underline{(\partial^{-2}\partial^\nu \partial_\alpha A_a^\alpha) A_b^\rho A_c^\sigma + g_s (\partial^{-2}\Gamma_a^\nu) A_b^\rho A_c^\sigma} \Big\}$$

$$\underline{+2(\partial^{-2}\partial^\nu \bar{c}_a)(\partial^\rho c_b) A_c^\sigma + g_s f_{bde} (\partial^{-2}\partial^\nu \bar{c}_a) A_d^\rho A_e^\sigma c_c} \Big]$$

➡ **First term yields the topological charge !**

Other terms cannot generate topological charge (Adler-Bardeen theorem)
They are also total derivative ⇒ Irrelevant

Final result from topological WTI

Topological WTI yields

$$\sum_{|\Omega\rangle\neq|0\rangle} \langle 0|\partial_\mu K^\mu(x)|\Omega\rangle\langle\Omega|\partial_\nu \mathcal{B}^\nu(y)|0\rangle$$

$$\propto \boxed{\sum_{|\Omega\rangle\neq|0\rangle} \langle 0|F\tilde{F}(x)|\Omega\rangle\langle\Omega|F\tilde{F}(y)|0\rangle = 0}$$

➡ **Topological charge of vacuum is unphysical !**

Operator product expansion (OPE)

Is it possible to probe the topological sector of physical states?

⇒ What is the value of $\langle \mathrm{phys}' | F\tilde{F} | \mathrm{phys} \rangle \equiv \langle 0 | F\tilde{F}(x)\phi(x') | 0 \rangle$?

<u>Operator product expansion:</u>

Operators separated by finite distances may be rewritten as a sum of local operators O_i

$$F_{\mu\nu,a}\tilde{F}_a^{\mu\nu}(x)\phi(y) = \sum_{O_i \neq F\tilde{F}} C_i O_i\left(\frac{x+y}{2}\right)$$

ϕ : arbitrary operator

Important point:
Adler-Bardeen theorem forbids generation of single topological charge density operator

⇒ We lose the information of topological sector for finitely distanced and correlated operators

We must therefore separate with infinite distance (or isolate them to not make them interact)

Finitely separated operators do not change topology (correlated)

Generalized topological WTI

To avoid OPE, we must isolate operators by infinite distances

The topological WTI for arbitrary Green's function:

$$\sum_{|\Omega\rangle \neq |0\rangle} \langle 0|\phi(x')|0\rangle\langle 0|F\tilde{F}(x)|\Omega\rangle\langle\Omega|F\tilde{F}(y)|0\rangle\langle 0|\phi(y')|0\rangle = 0$$

ϕ **: arbitrary scalar BRST singlet function**

The topological WTI with higher power of FF̃ for arbitrary Green's function:

$$\sum_{\Omega_1,\cdots,\Omega_{2n-1}} \langle 0|\psi(x')|0\rangle\langle 0|F_{\mu\nu,a}\tilde{F}_a^{\mu\nu}(x_1)|\Omega_1\rangle\langle\Omega_1|F_{\mu\nu,a}\tilde{F}_a^{\mu\nu}(x_2)|\Omega_2\rangle \times\cdots$$
$$\times\langle\Omega_{2n-2}|F_{\rho\sigma,b}\tilde{F}_b^{\rho\sigma}(y_2)|\Omega_{2n-1}\rangle\langle\Omega_{2n-1}|F_{\rho\sigma,b}\tilde{F}_b^{\rho\sigma}(y_1)|0\rangle\langle 0|\psi(y')|0\rangle = 0$$

➡ **Topological sectors and θ are unphysical !**

(vanishes at the level of observables even if amplitude is finite)

Consistency with the cluster decomposition principle

Cluster decomposition principle and need for θ-vacua

Cluster decomposition principle (CDP):

$$\langle \mathrm{vac}|A(x)B(y)|\mathrm{vac}\rangle \underset{|x-y|\to\infty}{\to} \langle \mathrm{vac}|A(x)|\mathrm{vac}\rangle\langle \mathrm{vac}|B(y)|\mathrm{vac}\rangle$$

Case of topological charge:

$$\langle 0|F\tilde{F}(x)F\tilde{F}(y)|0\rangle \underset{|x-y|\to\infty}{\to} \sum_{|\Omega\rangle\neq|0\rangle} \langle 0|F\tilde{F}(x)|\Omega\rangle\langle\Omega|F\tilde{F}(y)|0\rangle$$

$|\Omega\rangle$ **: vacua with different topology**

Previous argument :
Change of topology in the intermediate state leads to violation of CDP

Previous resolution :
Introducing θ vacuum $|\theta\rangle = \sum_n e^{-in\theta}|\mathrm{vac}(n)\rangle$ **resolves the problem**

$$\langle \theta|F\tilde{F}(x)F\tilde{F}(y)|\theta\rangle \underset{|x-y|\to\infty}{\to} \langle \theta|F\tilde{F}(x)|\theta\rangle\langle\theta|F\tilde{F}(y)|\theta\rangle = 0$$

θ vacua unchanged after topological transition ⇒ Fulfill CDP !!

R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37, 172 (1976);
C. G. Callan, R. F. Dashen and D. J. Gross, Phys. Lett. B **63** (1976) 334.

Consistency of the unobservabilty of topology and CDP

Our result : topological charge is unphysical

$$\sum_{|\Omega\rangle \neq |0\rangle} \langle 0|F\tilde{F}(x)\underline{|\Omega\rangle\langle\Omega|}F\tilde{F}(y)|0\rangle = 0$$

summation over
indefinite metric

It is not a problem even if the CDP is violated
since CDP violation due to the change of topology is not observable!

Indefinite metric space may not fulfill CDP, but it is not physical
⇒ This is consistent with Strocchi's CDP theorem.

F. Strocchi, Phys. Lett. B **62** (1976) 60; Phys. Rev. D **17**, 2010 (1978).

Singlet axial charge :

It is important to note that only the topology changing contribution of singlet axial charge is unphysical, but there is a remaining part which is physical.

The anomalous WTI must be split into physical and unphysical parts (see later).

Instanton

Classical Yang-Mills action

Rewrite the Euclidean YM action

Topological charge : constant

$$S_{\mathrm{YM}} = \frac{1}{4}\int d^4x\, G^a_{\mu\nu}G^a_{\mu\nu} = \frac{1}{4}\int d^4x\left[\pm G^a_{\mu\nu}\tilde{G}^a_{\mu\nu} + \underbrace{\frac{1}{2}\left(G^a_{\mu\nu} \mp \tilde{G}^a_{\mu\nu}\right)^2}_{\text{Always positive!}}\right]$$

⇒ YM action minimal with $G^a_{\mu\nu} = \pm\tilde{G}^a_{\mu\nu}$ ((anti)self-dual field)

⇒ Classical Euclidean Yang-Mills theory = (anti)self-dual field

Solution of EOM of classical Euclidean theory = tunneling (instanton)

⇒ Tunneling occurs between different topological sectors

$$P_{\mathrm{tunneling}} \sim e^{-\frac{8\pi^2}{g_s^2}}$$

⇒ We expect the most frequent configurations to be topological

Instanton (classical solution) may be calculated

Yang-Mills instanton

(Anti)self-duality

$$G^a_{\mu\nu} = \pm \tilde{G}^a_{\mu\nu}$$

Ansatz: $A^a_\mu = 2f(x^2)\eta_{a\mu\nu}\dfrac{x^\nu}{x^2}$

$$\eta_{a\mu\nu} = \begin{cases} \epsilon_{a\mu\nu} & (\mu,\nu = 1,2,3) \\ \delta_{a\mu} & (\nu = 4) \\ -\delta_{a\nu} & (\mu = 4) \end{cases}$$

(SU(2) case)

From (anti)self-duality, we obtain the differential equation

$$x^2 f' - f(1-f) = 0$$

Instanton solution: $A^a_\mu(x) = 2\eta_{a\mu\nu}\dfrac{x^\nu}{x^2+\rho^2}$

(ρ : free parameter, instanton size)

Hedgehog-like at spatial infinity = topological

We may also impose the topology at x=0 by spatial inversion (x → 1/x)

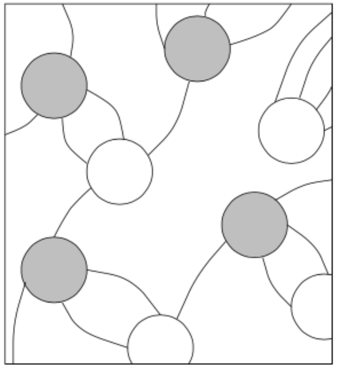
(YM lagrangian is conformal)

$$A'^a_\mu(x) = 2\bar{\eta}_{a\mu\nu}\frac{x^\nu}{x^2}\frac{\rho^2}{x^2+\rho^2}$$

$$\bar{\eta}_{a\mu\nu} = \begin{cases} \epsilon_{a\mu\nu} & (\mu,\nu = 1,2,3) \\ -\delta_{a\mu} & (\nu = 4) \\ \delta_{a\nu} & (\mu = 4) \end{cases}$$

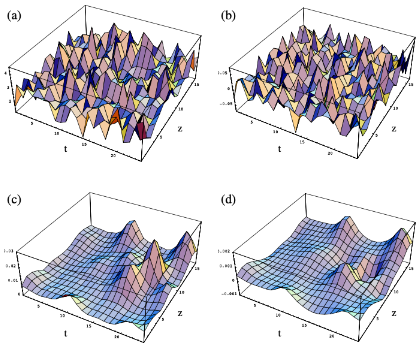
⇒ More convenient in actual calculations

Instanton liquid model



Instantons and anti-instantons are interacting and distributed in space

Schaefer and Shuryak, Rev. Mod. Phys. **70**, 323 (1998).



May be obtained by “cooling” gauge configurations calculated on lattice

Multi-(anti)instanton configurations are good approximation of gauge configurations with quantum effects

Chu et al., Phys. Rev. D **49**, 6039 (1994).

Relatively successful in hadron physics

Ilgenfritz and Mueller-Preussker, Nucl. Phys. B **184**, 443 (1981);
Diakonov and Petrov, Nucl. Phys. B **245**, 259 (1984)

Consistency of our argument with instanton liquid model

Instanton model seems to work well, but

instantons have topological charge, must be unphysical from our conclusion

How to resolve this apparent inconsistency?

Some arguments:

- Multi-(anti)instanton is not an exact solution of (anti)self-duality since self-duality ($x^2 f' - f(1-f) = 0$) is not a linear differential equation.
- Configurations with isolated (uncorrelated) (anti)instantons have topological charge, so they are unphysical. However, correlated multi-(anti)instanton configurations may have zero topological charge, may become physical.
- Instanton size becomes relevant under multi-(anti)instanton configuration. Hadron scale is only introduced by interacting multi-(anti)instanton.

➡ **Multi-(anti)instanton configurations (=instanton liquid model) can be physical!!**

Fermion contribution and chiral Ward identity

Chiral Ward-Takahashi identity

The well-known chiral (or anomalous) Ward-Takahashi identity:

$$\partial_\mu J_5^\mu \equiv \partial_\mu \sum_{\psi}^{N_f} \bar{\psi}\gamma^\mu\gamma_5\psi = -2\sum_{\psi}^{N_f}\left[m_\psi \bar{\psi} i\gamma_5\psi\right] - \frac{N_f\alpha_s}{8\pi} F_{\mu\nu,a}\tilde{F}_a^{\mu\nu}$$

Right-hand side has the topological charge density

Integral of topological charge density is unphysical

What is the unphysical part of the quark?

<u>Atiyah-Singer's theorem:</u>

$$\mathrm{ind}(\not{D}) = -\frac{\alpha_s}{8\pi}\int d^4x\, F_{\mu\nu,a}\tilde{F}_a^{\mu\nu}$$

(number of chiral Dirac zero-modes = topological charge)

Since topological charge is unphysical,

chiral Dirac zero-modes are unphysical !

Physical chiral Ward-Takahashi identity

We may remove the unphysical contribution due to topological charge and chiral Dirac zero-modes from the known chiral Ward-Takahashi identity

$$\sum_{\psi}^{N_f}\left[\partial^\mu \bar{\psi}\gamma_\mu\gamma_5\psi + 2m_\psi\bar{\psi}i\gamma_5\psi\right]_{\lambda\neq 0} = -\frac{N_f\alpha_s}{8\pi}F_{\mu\nu,a}\tilde{F}_a^{\mu\nu}\Big|_{\Delta n=0}$$

Remove only chiral zero-modes (λ=0)

Right-hand side vanishes at zero momentum inflow since no topological charge

(It is just a total divergence, without nontrivial effect at long distance)

⇒ This is the “physical" (observable) chiral WT identity

⇒ Chiral ($U(1)_A$) rotation does not change topological charge !

⇒ $U(1)_A$ symmetry is physical at Lagrangian level !

(of course, up to current quark mass m_ψ)

We anticipate that $U(1)_A$ breaks spontaneously in QCD, like other symmetries

⇒ $U(1)_A$ symmetry should restore at $T > T_c$

Symmetry of massless QCD and its spontaneous breakdown

Symmetry of massless QCD at the Lagrangian level

Global symmetry of the massless quark sector:

$$G^{(\mathrm{quark})} = U(N_f)_L \times U(N_f)_R$$
$$= \frac{SU(N_f)_L \times SU(N_f)_R \times U(1)_L \times U(1)_R}{(\mathbb{Z}_{N_f})_L \times (\mathbb{Z}_{N_f})_R}$$
$$= \frac{SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A}{(\mathbb{Z}_{N_f})_L \times (\mathbb{Z}_{N_f})_R \times \mathbb{Z}_2}$$

<u>**2nd equality:**</u>

We used $U(N_f) = \dfrac{SU(N_f) \times U(1)}{\mathbb{Z}_{N_f}}$

because $\underbrace{e^{\frac{2\pi i}{N_f}}}_{\in\, U(1), \mathbb{Z}_{N_f}} = \underbrace{e^{\frac{2\pi i}{N_f}\mathrm{diag}[1,\cdots,1,1-N_f]}}_{\in\, SU(N_f)}$

<u>**3rd equality:**</u>

We used $U(1)_L \times U(1)_R = \dfrac{U(1)_V \times U(1)_A}{\mathbb{Z}_2}$

because $e^{i\frac{1-\gamma_5}{2}\alpha_L} e^{i\frac{1+\gamma_5}{2}\alpha_R} = \underbrace{e^{i\frac{\alpha_R+\alpha_L}{2}} e^{i\gamma_5\frac{\alpha_R-\alpha_L}{2}}}_{\text{redundancy at } \pi}$

$(e^{i\pi} = e^{i\gamma_5\pi} = -1)$

Y. Tanizaki, JHEP **08** (2018) 171.

Symmetry of QCD with anomalous U(1)$_A$ breaking

Path integral measure is not invariant under U(1)$_A$ transformation

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi \xrightarrow{A} \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{2i\alpha\frac{N_f}{8\pi}\int d^4x\, F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a}$$

Fujikawa, Phys. Rev. Lett. **42**, 1195 (1979)

So far, it was expected that the symmetry is

$$G^{(\mathrm{expected})} = \frac{SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times \boxed{(\mathbb{Z}_{2N_f})_A}}{(\mathbb{Z}_{N_f})_L \times (\mathbb{Z}_{N_f})_R \times \mathbb{Z}_2}$$

However, our finding tells us that the topological charge is unphysical

What will then happen?

The path integral is equivalent to the amplitude, not to its squared

$$\frac{1}{Z}\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A\,\mathcal{O}e^{2i\alpha\frac{N_f}{8\pi}\int d^4x\, F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a + iS_{\mathrm{QCD}}} = \langle 0|\mathcal{O}|0\rangle_A \neq \langle 0|\mathcal{O}|0\rangle$$

The amplitude is not invariant under U(1)$_A$,but

from our derivation based on BRST, the squared amplitude is invariant !

➡ **Observables are invariant under U(1)$_A$!** **(in massless QCD)**

Symmetry of QCD after chiral SSB

What happens to quarks in QCD : spontaneous chiral symmetry breaking

Quarks may simply be regarded as obtaining a mass (NJL, Dyson-Schwinger)

After chiral SSB, symmetries become then

$$\frac{SU(N_f)_L \times SU(N_f)_R}{(\mathbb{Z}_{N_f})_L \times (\mathbb{Z}_{N_f})_R} = \frac{SU(N_f)_V \times SU(N_f)_A}{(\mathbb{Z}_{N_f})_V \times (\mathbb{Z}_{N_f})_A} \quad \to \quad \frac{SU(N_f)_V}{(\mathbb{Z}_{N_f})_V}$$

$$U(1)_A \quad \to \mathbb{Z}_2$$

(due to $e^{i\gamma_5\pi} = e^{i\pi} = -1$)

The final quark level symmetry after chiral SSB becomes

$$G^{(\text{quark SSB})} = \frac{SU(N_f)_V \times U(1)_V}{(\mathbb{Z}_{N_f})_V}$$

➡ This is the well-known symmetry of hadron physics!

$U(1)_A$ is spontaneously broken ⇒ η’ must be NG boson

Large N_c paradox of vacuum energy

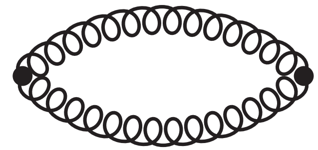
Large N_c paradox and Witten's argument

Analyze vacuum energy (topological susceptibility) in $1/N_c$ expansion

- **Without quarks:**

  At the lowest (2nd) order in θ, we have

$$c_2 = \left.\frac{d^2 E}{d\theta^2}\right|_{\theta=0} = \frac{1}{N_c^2}\frac{\alpha_s^2}{16\pi^2}\int d^4x \langle 0|TF\tilde{F}(x)F\tilde{F}(0)|0\rangle$$

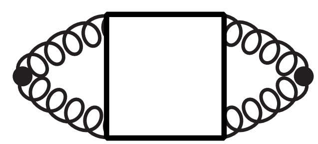


$= O(1)$

  ⇒ O(1) effect in $1/N_c$ expansion

- **With massless quarks: θ is rotated-away (unphysical) by chiral rotation**

  However, effect of quarks is $O(1/N_c)$



$= O(1/N_c)$

Paradox : How can an $O(1/N_c)$ effect cancel O(1) quantity?

⇒ Witten's resolution : η' squared mass is $O(1/N_c)$ and cancels the sum of gluonic modes (glueballs).

(We obtain a relation between η' mass and vacuum energy)

E. Witten, Nucl. Phys. B **156**, 269 (1979).

Our resolution of large N_c paradox

Topological correlators probe the effect of topological sectors to the vacuum

Recall that OPE cannot generate topological charge due to Adler-Bardeen th.

⇒ We lose the information of topological sector for correlated operators

We must therefore isolate operators with infinite distance

$$\langle 0|F_{\mu\nu,a}\tilde{F}_a^{\mu\nu}(x_{n-1})F_{\alpha\beta,b}\tilde{F}_b^{\alpha\beta}(x_{n-2})\cdots F_{\rho\sigma,c}\tilde{F}_c^{\rho\sigma}(0)|0\rangle$$

$$\longrightarrow \sum_{|\Omega\rangle\neq|0\rangle} \underbrace{\langle 0|F_{\mu\nu,a}\tilde{F}_a^{\mu\nu}(x_{n-1})|\Omega\rangle}_{\text{Topological charge!}}\langle\Omega|F_{\alpha\beta,b}\tilde{F}_b^{\alpha\beta}(x_{n-2})\cdots F_{\rho\sigma,c}\tilde{F}_c^{\rho\sigma}(0)|0\rangle$$

in the limit of $|x_{n-1}|, |x_{n-1}-x_j| \to \infty \quad (j=1,2,\cdots,n-2)$

⇒ The correlator has topology change, but it is now unphysical!

⇒ Vacuum energy has no θ-dependence!

Since $O(N_c^0)$ = O(1) terms are unphysical, no need to discuss cancellation of vacuum energy when introducing $O(1/N_c)$ quark effects.

⇒ Large N_c paradox is trivially resolved !

(Constraint on N_c dependence of η' mass does not exist anymore)

Unphysical ’t Hooft vertex

Vacuum tunneling and Dirac zero-modes

Partition function (= vacuum tunneling amplitude):

$$Z = \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{-S_{\rm YM}(A) - \sum_q \int d^4x\, \bar{\psi}[\not{D}(A) + m_q]\psi}$$ (Euclidean)

$$= \int \mathcal{D}A \prod_q \det[\not{D}(A) + m_q] e^{-S_{\rm YM}(A)}$$

$$= \underline{m_q^{n_0} \prod_{q,i} (\lambda_i + m_q)} \int \mathcal{D}A\, e^{-S_{\rm YM}(A)}$$

n_0 : number zero-modes

⇒ Topology changing contribution has quark mass factors due to Dirac zero-mode

In the chiral limit, zero??

⇒ No ! By introducing "quark sources", the zero-modes of the propagator cancel the quark mass factors of the partition function

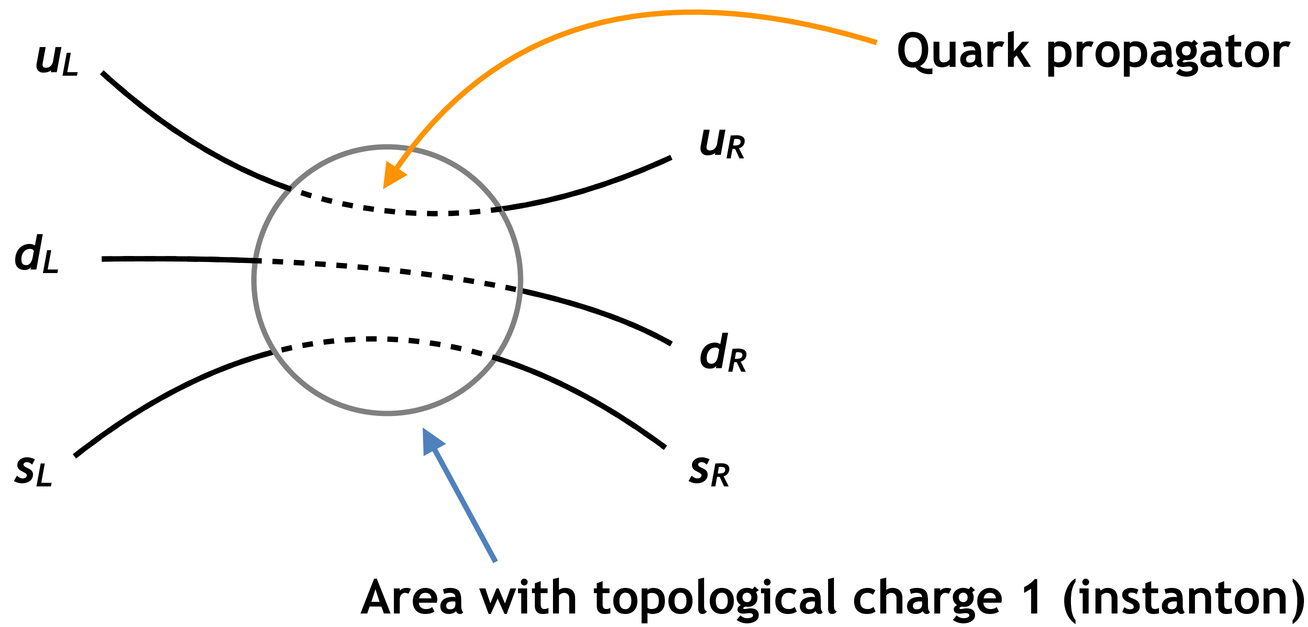
$$S(x,y) = \underline{\frac{\psi_0(x)\psi_0^\dagger(y)}{im}} + \sum_{\lambda \neq 0} \frac{\psi_\lambda(x)\psi_\lambda^\dagger(y)}{\lambda + im}$$

⇒ Quark chiral zero-modes counterbalance the gauge topological charge

G. 't Hooft, Phys. Rev. Lett. **37**, 8 (1976); Phys. Rev. D **14**, 3432 (1976).

't Hooft vertex

Consider the multi-quark amplitude $\langle 0|\bar{u}\Gamma_u u\,\bar{d}\Gamma_d d\,\bar{s}\Gamma_s s\,|0\rangle$ in N_f=3 QCD



Amplitude of this process is finite in the chiral limit if
* all quarks have zero-modes
* all quarks change their chirality from L to R (or vice versa)

This gives rise to a (nonlocal) contact multi-quark interaction ('t Hooft vertex)

In the case of $U(1)_{B+L}$ anomaly, the baryon/lepton number may be generated!

Unobservability of 't Hooft vertex and its consequences

Since we showed that the topological charge is unphysical,
so are chiral Dirac zero-modes

't Hooft vertex is generated by the propagation of zero-modes

⇒ 't Hooft vertex is unphysical !

(it may exist, but does not contribute to observables)

Important phenomenological consequences:

- No instanton induced chiral symmetry breaking contact interaction (so-called Kobayashi-Maskawa-'t Hooft (KMT) interaction)

  ⇒ $U(1)_A$ is spontaneously broken in QCD, not explicitly

- No seed of chiral magnetic effect in relativistic heavy ion collision

- No instanton/sphaleron induced baryogenesis

(we note that KMT interaction of QCD may be generated from chiral SSB,
but the sphaleron induced B+L violation is completely forbidden in the SM)

Nambu-Goldstone nature of η’ meson

η' mass

η' meson (and η) is known to have a large mass due to the mixing of gluonic intermediate states by the chiral anomaly (= topological charge density)

Mass squared matrix in (λ_3,λ_8,λ_0) basis of flavor SU(3) :

$$\begin{pmatrix} 2m_{ud}B_0 & 0 & 0 \\ 0 & \frac{2}{3}(m_{ud}+2m_s)B_0 & \frac{2\sqrt{2}}{3}(m_{ud}-m_s)B_0 \\ 0 & \frac{2\sqrt{2}}{3}(m_{ud}-m_s)B_0 & \frac{2}{3}(2m_{ud}+m_s)B_0+\underline{U_t} \end{pmatrix}$$

$$m_{ud} \equiv \frac{m_u+m_d}{2}$$

$$B_0 = -\frac{\langle 0|\bar{q}q|0\rangle}{f_\pi^2}$$

Topological susceptibility

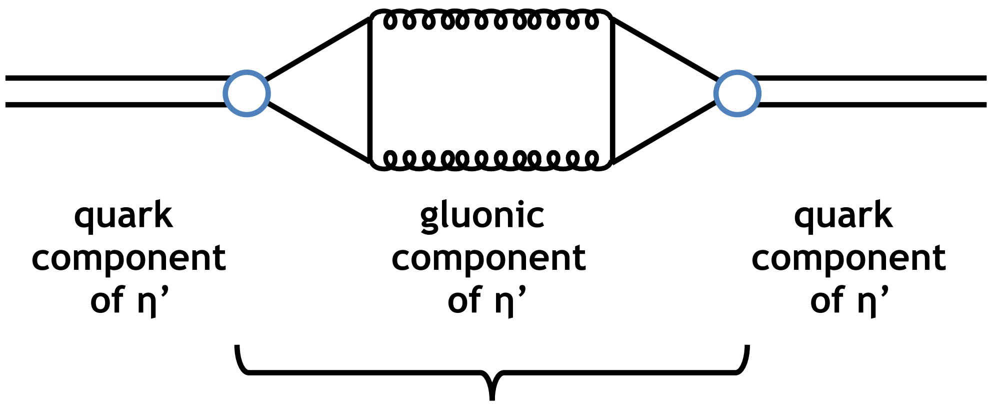
Physical π, η, and η' masses are obtained by diagonalizing this matrix

Large increase of η' (and η) mass due to topological susceptibility

Topological susceptibility must also tend to zero for m_u, m_d, m_s → 0,
if $U(1)_A$ is the symmetry of massless QCD Lagrangian (η' is an NG boson)

η’ and topological susceptibility

η’ mass increased by the topological susceptibility



$$\left(\frac{\alpha_s}{8\pi}\right)^2 \int d^4x\, e^{ik\cdot x} \langle 0|F_{\mu\nu,a}\tilde{F}_a^{\mu\nu}(x)F_{\rho\sigma,b}\tilde{F}_b^{\rho\sigma}(0)|0\rangle$$

$$= \sum_h \langle 0|F\tilde{F}|h(k)\rangle \frac{i}{k^2 - m_h^2} \langle h(k)|F\tilde{F}|0\rangle$$

$\langle 0|F\tilde{F}|h(k)\rangle \propto$ invariant mass, vanishes for zero energy-momentum inflow (property inherited from topological charge, due to nonrenormalizability)

⇒ Topological susceptibility contributes to mass shift since quark components of η’ (and η) are massive, i.e. nonzero momentum inflow

What happens when current quarks become massless?

η' with massless current quarks

Massless quarks form massless NG component, with zero invariant mass

⇒ Zero energy-momentum inflow to topological susceptibility

Two possibilities:

- Massive lightest gluonic intermediate state (*h*)

$$\sum_h \langle 0|F\tilde{F}|h(k)\rangle \frac{i}{k^2 - m_h^2} \langle h(k)|F\tilde{F}|0\rangle \underset{k\to 0}{\to} 0$$

⇒ Topological susceptibility vanishes, η' is massless if quarks are massless

- Existence of massless gluonic intermediate state

$$\langle 0|F\tilde{F}|h(k)\rangle \frac{i}{k^2} \langle h(k)|F\tilde{F}|0\rangle$$ may be finite for k→0

But, massless mode is constant over space-time, proportional to vacuum

$$\langle 0|F\tilde{F}|h(k)\rangle \propto \langle 0|F\tilde{F}|\Omega\rangle$$ ⇒ Unphysical !

⇒ η' (and η) becomes massless with zero current quark masses (NG boson!)

CP phase of quark mass and physics beyond standard model

CP-odd mass of quarks

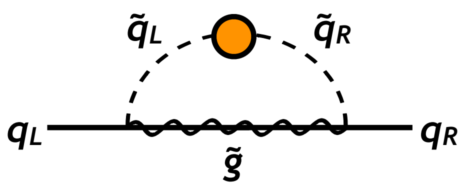
$$\mathcal{L}_{\mathrm{odd}} = -m_{\mathrm{odd}}\bar{\psi} i\gamma_5 \psi$$

⇒ Introduce complex phase to quark mass

Generated in new physics beyond standard model (and also in standard model)

Example of supersymmetry:

Squarks have CP-odd transition



Physical particles are in mass eigenstates : real mass

CP-odd mass may be "rotated away" by $U(1)_A$ transformation:

$$m_{\mathrm{even}}\bar{\psi}\psi + m_{\mathrm{odd}}\bar{\psi} i\gamma_5 \psi \to m\bar{\psi}'\psi'$$

If θ-term is physical, chiral rotation generates θ, cannot erase both θ and m_{odd}

From our discussion, θ-term is unphysical ⇒ Chiral rotation removes m_{odd} !

m_{odd} does not decouple with increasing BSM scale, but it is unphysical:
Our result (unphysical m_{odd}) leads to correct decoupling of BSM scale

Modification of the BSM phenomenology after our work

- Modification procedure:

  Neglect θ-term

  Neglect CP-odd mass of quarks

- Comparison with axion mechanism:

  No axions

  Induced θ-term is unphysical

- Impact on particle physics phenomenology:

  The only source of CP violation of SM is the CP phase of CKM matrix

  BSM contribution all decouples with increasing BSM energy scale (scales as power of $1/\Lambda_{BSM}$)

Summary

Important consequences of our study

- θ-term becomes unphysical : resolution of Strong CP problem
  ⇒ No need for axion mechanism
  (Note : this does not mean that axions are forbidden)
- Instanton becomes unphysical
  ⇒ No instanton induced hadron mass generation
- $U(1)_A$ is a physical symmetry
  ⇒ Spontaneously broken in QCD, η' is an NG boson
- No physical topological phase in hot QCD plasma
  ⇒ No "seeds" of chiral magnetic effect in heavy ion collision
- Sphaleron induced baryon number generation is forbidden
  ⇒ No electroweak baryogenesis
- CKM matrix is the sole source of CP violation in standard model