Separability of Maxwell equation in Rotating black hole spacetime and its Geometric aspects

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Recently, a progress was made about Maxwell field perturbation on Kerr BH spacetime and its separability. We try to find the geometric origin of this bland-new technique.

- Perturbations of Kerr black hole
- Recent breakthrough on separability
- Construction of commuting operators
- Summary
Perturbations of Kerr black hole

• Scalar field, Maxwell field, Metric perturbations on Kerr BH
• Important, but difficult
  Complicated PDE, many physical d.o.f. coupled with each other

• Teukolsky equation based on Newman-Penrose formalism
  [Teukolsky ’72]
  EoM $\rightarrow$ decoupled PDEs that admit separation of variables
  $\rightarrow$ set of ODEs
Teukolsky eq. for Maxwell perturbations on 4D Kerr BH

\[
\begin{align*}
    ds^2 &= \frac{1}{\Sigma} \left\{ -\Delta [dt - a \sin^2 \theta d\phi]^2 + \sin^2 \theta [(r^2 + a^2) d\phi - adt]^2 \right\} + \Sigma \left[ \frac{dr^2}{\Delta} + d\theta^2 \right] \\
    &= -2\ell_{(\mu n_\nu)} + 2m_{(\mu \bar{m}_\nu)} \\
    \left[ \Delta = r^2 + a^2 - 2Mr, \Sigma = r^2 + a^2 \cos^2 \theta \right] \\
    F_{\mu\nu} &= 2 \left[ \phi_1 (n_{[\mu} l_{\nu]} + m_{[\mu} \bar{m}_{\nu]} ) + \phi_2 l_{[\mu} m_{\nu]} + \phi_0 \bar{m}_{[\mu} n_{\nu]} \right] + c.c. \\
    \psi_+ &= \phi_0, \quad \psi_- = \bar{\rho}^2 \phi_2, \quad \psi_s = e^{i\omega t + im\phi} R_s(r) S_s(\theta)
\end{align*}
\]

Maxwell equation \( \nabla^\mu F_{\mu\nu} = 0 \) \( \Rightarrow \) Teukolsky equation

\[
\begin{align*}
    \frac{1}{\Delta^s} \frac{d}{dr} \left[ \Delta^{s+1} \frac{dR_s}{dr} \right] + \left[ \frac{K(K - 2isr) + 2isMK}{\Delta} - 4is\omega r - \Lambda - (a\omega + m)^2 + m^2 \right] R_s &= 0 \\
    \frac{1}{\sin \theta} \frac{d}{d\theta} \left[ \sin \theta \frac{dS_s}{d\theta} \right] + \left[ (a\omega \cos \theta + s)^2 - \frac{(m + s \cos \theta)^2}{\sin^2 \theta} + s(1 - s) + \Lambda \right] S_s &= 0 \\
    \left( s = \pm 1, \rho = r + ia \cos \theta, \ K = (r^2 + a^2) \omega - am \right)
\end{align*}
\]

✓ \( \phi_0 \) and \( \phi_2 \) are solved by Teukolsky eq. (while PDE for \( \phi_1 \) cannot be separated)

✓ Works only in 4D: separation of variables NOT achieved in higher dim.
Recent breakthrough on Separability

◆ Lunin’s new ansatz [Lunin ’17]

\[
\begin{align*}
\ell^\mu A_\mu &= G_+(r)\ell^\mu \partial_\mu \Psi \\
n^\mu A_\mu &= G_-(r)n^\mu \partial_\mu \Psi \\
m^\mu A_\mu &= F_+(\theta)m^\mu \partial_\mu \Psi \\
\bar{m}^\mu A_\mu &= F_-(\theta)\bar{m}^\mu \partial_\mu \Psi
\end{align*}
\]

✓ $G_\pm(r), F_\pm(\theta)$ chosen to achieve separation of variable
✓ Separable equations for all the variables $[\Psi = e^{i\omega t + im\phi} R(r) S(\theta)]$
✓ Works even in higher dimensions

◆ Covariant version of Lunin’s ansatz [Krtouš, Frolov, Kubizňák ’18]

\[
A^\mu = B^{\mu\nu} \nabla_\nu Z \quad \text{with} \quad B^{\mu\nu} = (g_{\mu\nu} - \beta h_{\mu\nu})^{-1}
\]

$h_{\mu\nu}$: Principal tensor = non-degenerate closed conformal Killing-Yano tensor

= “square root” of Killing tensor $K_{\mu\nu} = (\ast h)_{\mu\rho} (\ast h)_{\rho\nu}$

Killing tensor $K_{\mu\nu} (\nabla_\mu K_{\nu\rho} = 0) \approx$ “Hidden symmetry” of spacetime: $K_{\mu\nu} p^\mu p^\nu = \text{(constant of motion)}$

Killing vector $\xi^\mu (\nabla_\mu \xi_\nu = 0) \approx$ Symmetry of spacetime: $\xi^\mu p_\mu = \text{(constant of motion)}$

Teukolsky’s ansatz

\[
\begin{align*}
\ell^\mu A_\mu &= \frac{2ia}{r} l^\mu \partial_\mu [e^{i\omega t + im\phi} g_+(r)f_+(\theta)] \\
n^\mu A_\mu &= \frac{2ia}{r} n^\mu \partial_\mu [e^{i\omega t - im\phi} g_-(r)f_-(\theta)] \\
m^\mu A_\mu &= -\frac{2ia}{i\alpha \cos \theta} m^\mu \partial_\mu [e^{i\omega t + im\phi} f_+(\theta)g_+(r)] \\
\bar{m}^\mu A_\mu &= -\frac{2ia}{i\alpha \cos \theta} \bar{m}^\mu \partial_\mu [e^{i\omega t + im\phi} f_-(\theta)g_-(r)]
\end{align*}
\]
**Covariant ansatz** [Krtouš, Frolov, Kubizňák ’18]

◆ Most-general $2N$ dim. spacetime admitting $h_{\mu\nu} = \text{Kerr-NUT-(A)dS}$

\[
ds^2 = \sum_{\mu = 1}^{N} \left[ \frac{U_\mu}{X_\mu} \, dx_\mu^2 + \frac{X_\mu}{U_\mu} \left( \sum_{j=0}^{N-1} A^{(j)}_\mu \, d\psi_j \right)^2 \right]
\]

- \(x^\mu = \{r, y(\theta), \ldots\}\) nontrivial directions
- \(\psi^i = \{\tau, \phi, \ldots\}\) Killing directions

\[\begin{aligned}
\text{Maxwell equation} & \quad \mathcal{C}_0 Z \equiv \left( \Box + 2\beta \xi_k B^{kn} \nabla_n \right) Z = 0 \\
\text{Lorenz gauge} & \quad \mathcal{C} Z \equiv \nabla_m \left( B^{mn} \nabla_n Z \right) = 0
\end{aligned}\]

\[
\begin{aligned}
&c = \sum_\nu \frac{A^{(\nu)}_\nu}{U_\nu} \tilde{c}_\nu, \quad c_k = \sum_\nu \frac{A^{(\nu)}_\nu}{U_\nu} \tilde{c}_\nu, \quad \mathcal{L}_k = -i \frac{\partial}{\partial \psi_k}, \quad \tilde{c}_\nu = (1 + \beta^2 x_\nu^2) \frac{\partial}{\partial x_\nu} \left[ \frac{X_\nu}{1 + \beta^2 x_\nu^2} \frac{\partial}{\partial x_\nu} \right] - \frac{1}{X_\nu} \bar{c}_\nu^2 + i\beta \frac{1 - \beta^2 x_\nu^2}{1 + \beta^2 x_\nu^2} \beta^{(1-N)} \mathcal{L}
\\
&\text{Both equations given by commuting operators} \quad [\mathcal{C}_k, \mathcal{C}_l] = [\mathcal{C}_k, \mathcal{L}_l] = [\mathcal{L}_k, \mathcal{L}_l] = 0 \\
&\text{→ } Z \text{ is given by simultaneous eigenfunctions of } \mathcal{C}_k, \mathcal{L}_k \\
&\mathcal{C}_k Z = C_k Z, \quad \mathcal{L}_k Z = L_k Z \quad \Rightarrow \quad Z = Z(\beta; C_0, C_1, \ldots, C_{N-1}, L_0, \ldots, L_{N-1})
\end{aligned}\]

? What is the geometric origin & covariant form of the commuting operators?
Construction of commuting operators

1. Express perturbation equations in terms of gauged Laplacian:
   \[(\nabla^\mu - iqA^\mu)(\nabla_\mu - iqA_\mu) + \cdots = 0\]

2. Express it as \(\Box \psi = 0\) by applying the Eisenhart-Duval lift: \(g_{\mu\nu} \rightarrow \hat{g}_{AB}\)
   [Eisenhart 1928, Duval+ 1985]

3. It turns out that the geodesic equation for the lifted metric \(\hat{g}_{AB}\) admits
   separation of variables completely. [Benenti '91]
   Then, there exists the Killing tensors \(\hat{K}_{AB}\) s.t.
   \[\left\{\hat{g}_{AB}p^A p^B, \hat{K}_{AB}p^A p^B\right\} = 0.\]

4. By quantization \(p_\mu \rightarrow -i\nabla_\mu\) it follows
   \[\left[\hat{g}^{AB}\hat{\nabla}_A \hat{\nabla}_B, \hat{\nabla}_A \hat{K}^{AB} \hat{\nabla}_B\right] = \frac{4}{3} \nabla_A \left(\hat{K}_{[A} \hat{R}^{B]C}\right) \hat{\nabla}_B.\]

5. Then, if the anomaly-free condition \(\nabla_A \left(\hat{K}_{C}^{[A} \hat{R}^{B]C}\right) = 0\) is satisfied,
   it follows \[\left[\hat{g}^{AB}\hat{\nabla}_A \hat{\nabla}_B, \hat{\nabla}_A \hat{K}^{AB} \hat{\nabla}_B\right] = 0.\] [Carter 1977]

6. The commuting operators s.t. \([\Box, C_K] = 0\) is then given by \(C_k = \hat{\nabla}_A \hat{K}^{AB} \hat{\nabla}_B.\)
Construction of commuting operators

- Lunin’s equation \((\Box + 2\beta \xi_k B^{kn} \nabla_n) Z = 0\)
- Teukolsky equation \((\Box + f_1^\mu \nabla_\mu + f_2) \psi = 0\)

Both given by gauged wave equation \((\nabla^\mu - iqA^\mu)(\nabla_\mu - iqA_\mu) + \cdots = 0\)

Can be expressed as wave operator in higher dimensions \(\hat{\Box} \psi = 0\)
by lifting the metric \(g_{\mu\nu}\) to a higher-dimensional one \(\hat{g}_{\mu\nu}\) [Eisenhart 1928, Duval+ 1985]

\[
d s^2 = \hat{g}_{AB} dx^A dx^B = g_{\mu\nu} dx^\mu dx^\nu + 2q A_\mu dx^\mu du + 2 du dv - 2V du^2, \quad \hat{\psi}(x^A) = \psi(x^\mu) e^{i A^\mu} \\
\hat{\Box} \hat{\psi} = e^{i u} (\Box_A - 2V) \psi \quad \left[\Box_A = \Box - 2iqA^\mu \nabla_\mu - q^2 A^\nu A_\mu - iq \nabla_\mu A^\mu\right]
\]

- It turns out that the geodesic eq. for the uplifted metric \(\hat{g}_{AB}\) admits separation of variable completely. [Benenti ’91]

Separation of variable for geodesics
  - \(\exists\) Constants of motion
  - \(\exists\) Killing tensor satisfying \(\{\hat{H}, \hat{K}_{AB} p^A p^B\} = 0\)
  - Commuting operators \(\left[\hat{\Box}, \hat{\nabla}_A \left(\hat{K}^{AB} \hat{\nabla}_B\right)\right] = 0\) by quantization \(p_\mu \rightarrow -i \hat{\nabla}_\mu\)
if anomaly-free condition \(\nabla_A \left(\hat{K}_C [A \hat{R}^B]_C\right) = 0\) is satisfied.

\[
\begin{align*}
\{f, g\} &= \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p^i} \frac{\partial g}{\partial x_i} \\
\hat{H} &= \hat{g}_{AB} p^A p^B
\end{align*}
\]
Construction of commuting operators

ex.) Teukolsky eq. for 4D Kerr BH

\[
\left[ \frac{1}{\Delta} \frac{\partial}{\partial r} \left( \Delta^{s-1} \frac{\partial}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{K(K-2isr)+2isMK}{\Delta} - Ais\omega r \\
-(a\omega + m)^2 + m^2 + (a\omega \cos \theta + s)^2 - \frac{(m + s \cos \theta)^2}{\sin^2 \theta} + s(1-s) \right] \psi = 0
\]

\[
\left( \square - 2iqA^\mu \nabla_\mu - q^2 A^\mu A_\mu - iq \nabla_\mu A^\mu - 2V \right) \psi = 0
\]

\[
ds^2 = \hat{g}_{AB}dx^Adx^B = g_{\mu\nu}dx^\mu dx^\nu + 2qA_\mu dx^\mu du + 2du dv - 2V du^2
\]

\[
\begin{align*}
\hat{g}^{\mu\nu} &= g^{\mu\nu} & \hat{g}^{\mu u} &= 0 & \hat{g}^{r u} &= g^{rr} \frac{is(M-r)}{\Delta} \\
\hat{g}^{\theta\nu} &= 0 & \hat{g}^{\phi\nu} &= g^{rr} \frac{isa(M-r)}{\Delta^2} + g^{\theta\theta} \frac{s \cos \theta}{\sin^2 \theta} \\
\hat{g}^{t\nu} &= g^{rr} \frac{is(M-a^2)}{\Delta^2} + g^{\theta\theta} (-sa \cos \theta) \\
\hat{g}^{\nu\nu} &= 1 & \hat{g}^{vv} &= g^{\theta\theta} \frac{s^2 \cot^2 \theta}{\sin^2 \theta}
\end{align*}
\]

\[
\{ \hat{g}^{AB} p_A p_B, \hat{K}_{AB} p_A p_B \} = 0
\]

\[
[\square, \hat{\nabla}_A (\hat{K}^{AB} \hat{\nabla}_B)] = 0 \quad \text{if} \quad \nabla_A (\hat{K}_C [A \hat{R} B]^C) = 0
\]

\[
\nabla_A (\hat{K}_C [A \hat{R} B]^C) = 0 \quad \text{is indeed satisfied, hence} \quad \nabla_A \hat{K}^{AB} \nabla_B \quad \text{becomes a commuting op.}
\]

\[
\text{This procedure works also for 4D Kerr-NUT-AdS spacetime}
\]
Construction of commuting operators

ex.) $D$-dim. Kerr-NUT-AdS spacetime ($D = 2n + \varepsilon$)

$$g^{-1} = \sum_{\mu=1}^{n} (X_\mu \otimes X_\mu + X_\mu' \otimes X_\mu') + \varepsilon X_0 \otimes X_0 = \sum_{\mu=1}^{n} g^{\mu\mu} \left( \frac{\partial}{\partial x_\mu} \right)^2 + \sum_{k,\ell=0}^{n-1+\varepsilon} g^{k\ell} \frac{\partial}{\partial \psi_k} \frac{\partial}{\partial \psi_\ell}$$

$$g^{\mu\mu} = Q_\mu, \quad g^{k\ell} = \sum_{\mu=1}^{n-1+\varepsilon} \zeta_{\mu}^{k\ell}(x_\mu) Q_\mu, \quad \zeta_{\mu}^{k\ell}(x_\mu) = \frac{(-1)^{k+\ell}}{X_\mu^2} \frac{x_\mu^{2(2n-2-k-\ell)}}{c^{2\mu} X_\mu} + \frac{(-1)^{n+1}}{c^{2\mu} X_\mu} \delta_{\kappa\kappa} \delta_{\lambda\lambda} \delta_{\kappa\lambda}$$

Perturbation eq. $(\Box + 2\beta \xi_k B^{kn} \nabla_n) Z = 0 \rightarrow$ gauge field $q A^a = 2i \beta b A^{ba}$

Lifted metric

$$\hat{g}^{\mu\mu} = g^{\mu\mu}, \quad \hat{g}^{k\ell} = g^{k\ell}, \quad \hat{g}^{\mu
u} = -q g^{\mu\mu} A_\mu, \quad \hat{g}^{k\nu} = -q g^{k\ell} A_\ell, \quad \hat{g}^{\nu\nu} = -i q \operatorname{div} A, \quad \hat{g}^{\mu
u} = 1$$

$$\Leftrightarrow \hat{g}^{\mu\mu} = Q_\mu, \quad \hat{g}^{AB} = \sum_{\mu=1}^{n} \zeta_{\mu}^{AB}(x_\mu) \sigma_j(x_\mu) Q_\mu \quad (A = B \neq \mu)$$

By Benenti's construction, the Killing tensor $\hat{K}_{(j)AB}$ of the lifted metric $\hat{g}_{AB}$ is given by

$$\hat{K}_{(j)\mu\nu} = \sigma_j(\hat{x}_\mu) Q_\mu, \quad \hat{K}_{(j)AB} = \sum_{\mu=1}^{n} \zeta_{\mu}^{AB}(x_\mu) \sigma_j(\hat{x}_\mu) Q_\mu \quad (A = B \neq \mu)$$

The anomaly-free condition $\nabla_A \left( \hat{K}_{(j)C}^{[A} \hat{R}_{B]}^{C} \right) = 0$ turns out to be satisfied by $\hat{K}_{(j)AB}$, hence the operator $\hat{\nabla}_A (\hat{K}_{(j)AB} \hat{\nabla}_B)$ commutes with the Laplacian $\hat{\Box}$:

$$\left[ \hat{\Box}, \hat{\nabla}_A (\hat{K}_{(j)AB} \hat{\nabla}_B) \right] = 0$$

The operator $\hat{\nabla}_A (\hat{K}_{(j)AB} \hat{\nabla}_B)$ coincides with the commuting operators $\mathcal{C}_k$ up to (Killing vector)$^\mu \nabla_\mu$
Summary

✓ New ansatz for Maxwell perturbations on Kerr BH

✓ EoMs given by commuting operators → Separability for all variables

◆ Tried to give geometric interpretation to the commuting operators
  • Master eq. = scalar field eq. with gauged wave operator
    = scalar eq. with (non-gauged) wave op. in higher dimensions
  • Uplifted higher-dimensional metric possesses Killing tensors
  • This Killing tensor generates commuting operators $[\Box, \nabla_A (\hat{K}^{AB} \nabla_B)] = 0$
  • Procedure above works for Teukolsky eq. and also Lunin’s eq.

■ Future tasks
  • Uplifted spacetimes corresponding to Teukolsky eq. and Lunin’s eq. are apparently different. What is the essential difference?
  • Can we apply this procedure to gravitational perturbations in higher D?