Stability of Fractional Chern Insulators in the Effective Continuum Limit of $|C| > 1$ Harper-Hofstadter Bands

Bartholomew Andrews & Gunnar Möller

TCM Group, Cavendish Laboratory, University of Cambridge

October 31, 2017

arXiv:1710.09350

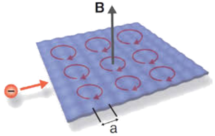
Outline

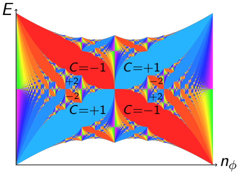
Introduction

- **Fractional Chern Insulators (FCIs)** generalize the FQHE to systems with non-trivial Chern number, C.
- The **Harper-Hofstadter model** has provided some of the first examples of FCIs (Sørensen *et al.*, 2005), and hosts a fractal energy spectrum with any desired Chern number.
- Examine states of the composite fermion (CF) series predicted by Möller & Cooper, 2015.
- Generalize the $n_\phi \to 0$ continuum limit to the **effective continuum limit** at $n_\phi \to 1/|C|$ (Möller & Cooper, 2015).
- Investigate the **stability** (i.e. robustness in the effective continuum limit) of the many-body gap, Δ.

Harper-Hofstadter Model

We consider N spinless particles hopping on an $N_x \times N_y$ square lattice with a constant effective magnetic flux.





magnetic translation-invariant phase

interaction potential

$$H = \sum_{i,j} \left[\boxed{t_{ij}} \boxed{e^{\phi_{ij}}} c_j^\dagger c_i + \text{h.c.} \right] + \boxed{\mathcal{P}_{\text{LB}}} \left[\sum_{i<j} \boxed{V_{ij}} :\rho(\mathbf{r}_i)\rho(\mathbf{r}_j): \right] \mathcal{P}_{\text{LB}}$$

hopping parameter

lowest-band projection operator

- bosons $\Rightarrow$ on-site interactions
- fermions $\Rightarrow$ nearest-neighbour interactions

Composite Fermion Theory

Predicted **filling fraction** from CF theory on the lattice for a well-isolated lowest band (Möller & Cooper, 2015):

$$\nu = \frac{r}{|kC|r + 1} \equiv \frac{r}{s}, \quad \text{where } r \text{ and } s \text{ are co-prime}$$

- C = Chern number of the band
- k = number of flux quanta attached to the particles
- $|r|$ = number of bands filled in the CF spectrum
- $\mathrm{sgn}(r) = \mathrm{sgn}(C^*)$ for the CF band relative to C
- $|s|$ = ground state degeneracy

Scaling & Stability

Aim to consider **2D** isotropic limit $\Rightarrow$ demand $N_x = N_y$.

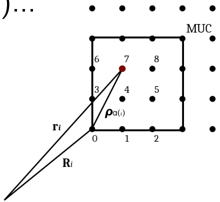
- Note: $\frac{\text{Nbr. Sites}}{\text{Nbr. MUCs}} = q$ is a measure of MUC size.

Scaling relations (Bauer *et al.*, 2016):

$$\Delta \propto q^{-1} \quad \text{for bosons (contact interactions),}$$
$$\Delta \propto q^{-2} \quad \text{for fermions (NN interactions).}$$

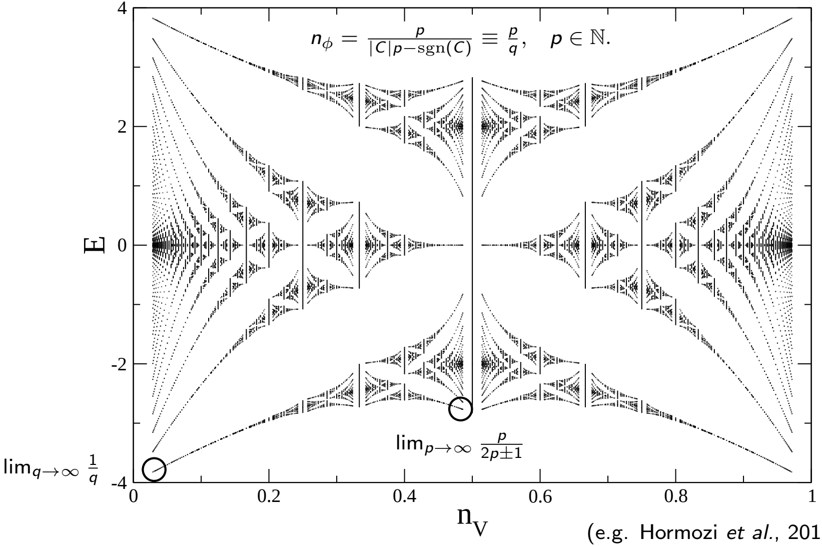
Investigate stability (robustness of many-body gap)...

1. ...in the **effective continuum limit**: $q \to \infty$.
2. ...in the **thermodynamic limit**: $N \to \infty$.



Harper-Hofstadter Model

Approaching the Effective Continuum

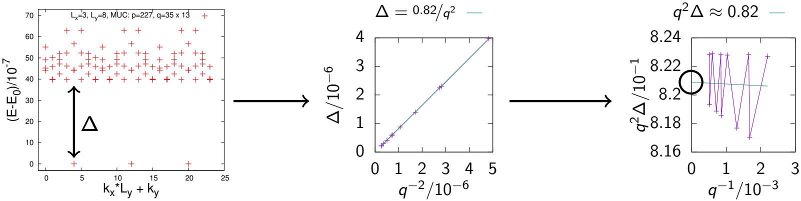


(e.g. Hormozi *et al.*, 2012)

Basic Method

e.g. $N = 8$ fermions in the $|C| = 2$ band at $\nu = 1/3$ filling

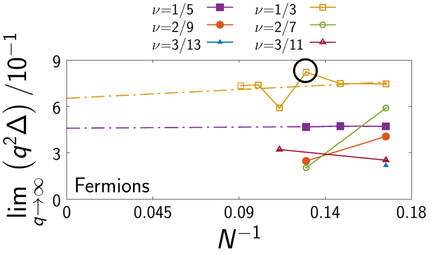
1. Plot the many-body energy spectrum for a particular $\{C, r, N\}$ configuration and for a variety of MUC sizes, q. Identitify the ground states, predicted by CF theory.
2. Read off the many-body gap, Δ, for each energy spectrum.
3. Plot Δ against q. Read off $\lim_{q\to\infty}(q^{(2)}\Delta)$.



Basic Method

e.g. $N = 8$ fermions in the $|C| = 2$ band at $\nu = 1/3$ filling

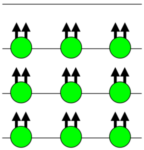
1. Plot the many-body energy spectrum for a particular $\{C, r, N\}$ configuration and for a variety of MUC sizes, q. Identitify the ground states, predicted by CF theory.
2. Read off the many-body gap, Δ, for each energy spectrum.
3. Plot Δ against q. Read off $\lim_{q\to\infty}(q^{(2)}\Delta)$.
4. Plot $\lim_{q\to\infty}(q^{(2)}\Delta)$ against N. Read off $\lim_{N,q\to\infty}(q^{(2)}\Delta)$.



Constraints

1. We are only interested in filled CF levels:

   N must be a multiple of r.



2. $\nu = N/N_{\mathrm{c}} \Rightarrow N = \nu N_{\mathrm{c}}$:

   N_{c} must be a multiple of s.

3. Isolated lowest Chern number C band at

$$n_\phi = \frac{p}{|C|p - \mathrm{sgn}(C)} \equiv \frac{p}{q}, \quad p \in \mathbb{N}.$$

4. Consider **2D systems** $\Rightarrow$ approximately unit aspect ratios:

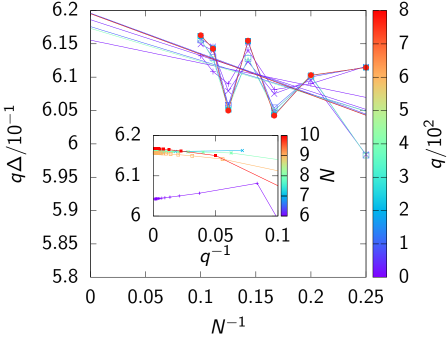
$$\left|1 - \frac{N_x}{N_y}\right| \leq \epsilon, \quad \text{for small } \epsilon.$$

5. Limited computation time:

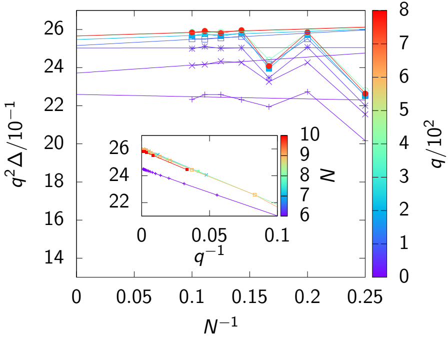
$$\dim\{\mathcal{H}\} < 10^7.$$

Approaching the Thermodynamic Effective Continuum

Q: In which order should we take the $N \to \infty$ and $q \to \infty$ limits?



(a) $\nu = 1/2$ bosons



(b) $\nu = 1/3$ fermions

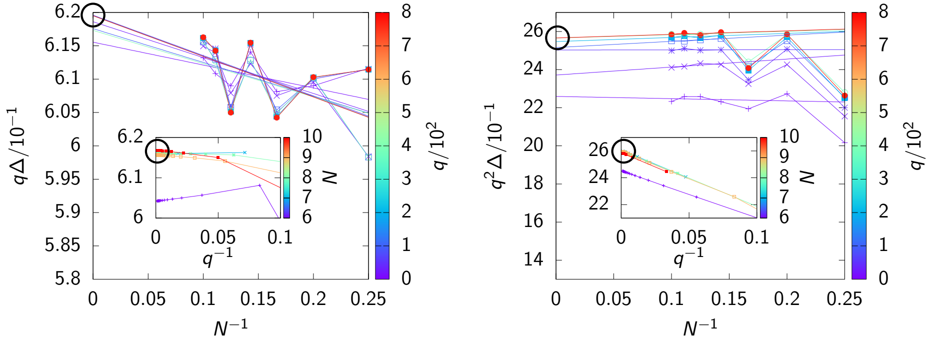
Figure: Finite-size scaling of the gap for Laughlin states

Approaching the Thermodynamic Effective Continuum

Q: In which order should we take the $N \to \infty$ and $q \to \infty$ limits?

A: Doesn't matter. We take the effective continuum limit first.

$N, q \to \infty$ limits commute! (if both limits can be taken)



(a) $\nu = 1/2$ bosons (b) $\nu = 1/3$ fermions

Figure: Finite-size scaling of the gap for Laughlin states

Warm-up: $|C| = 1$ Band

Bosons - Stability in the Continuum

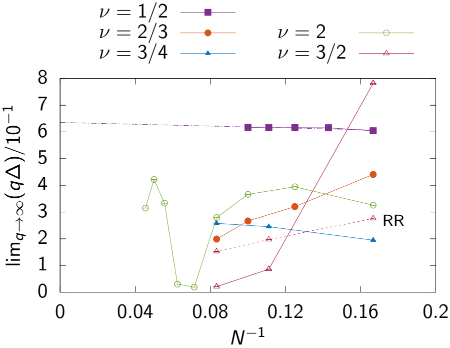


Figure: Finite-size scaling of the gap to the thermodynamic continuum limit at fixed aspect ratio

Warm-up: $|C| = 1$ Band

Bosons - Stability in the Continuum

agrees with Bauer *et al.* ✓

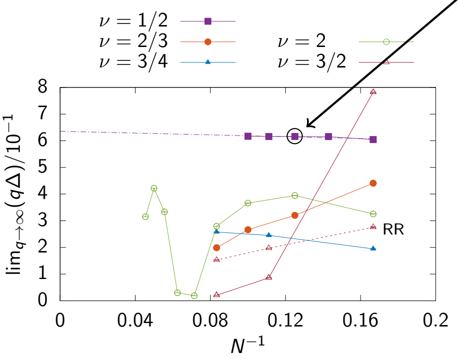


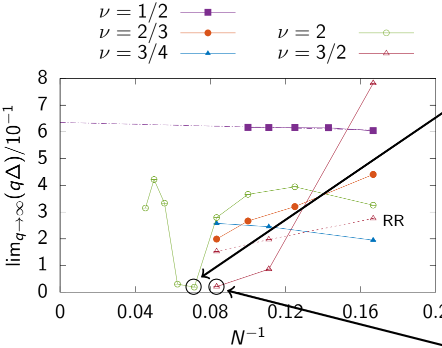
Figure: Finite-size scaling of the gap to the thermodynamic continuum limit at fixed aspect ratio

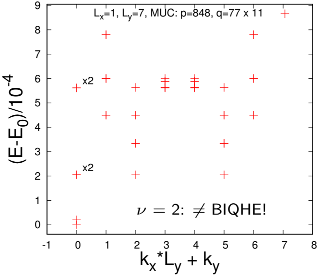
$\nu = 2$ DMRG BIQHE: more than just LLL involved in stabilizing the state (He *et al.*, 2017)

$\nu = 2$ competition expected from continuum results (Cooper & Rezayi, 2007)

Warm-up: $|C| = 1$ Band

Bosons - Stability in the Continuum





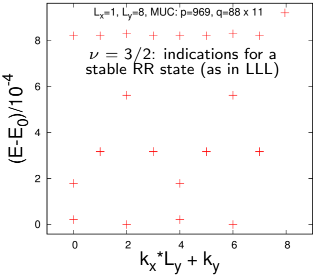


Figure: Finite-size scaling of the gap to the thermodynamic continuum limit at fixed aspect ratio

Warm-up: $|C| = 1$ Band

Bosons - Pair Correlation Functions

no perfect correlation hole at $\nu > 1/2$

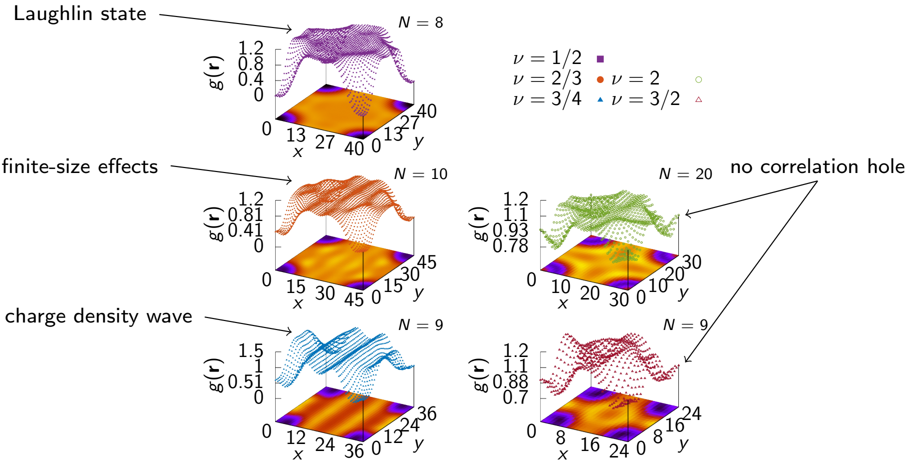


Figure: Pair correlation functions for the lowest-lying ground state in the $(k_x, k_y) = (0, 0)$ momentum sector

Warm-up: $|C| = 1$ Band

Fermions - Stability in the Continuum

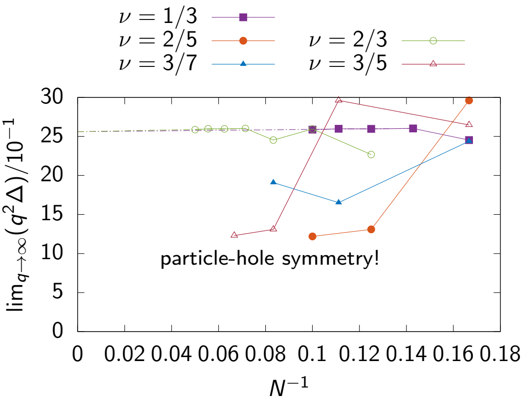


Figure: Finite-size scaling of the gap to the thermodynamic continuum limit at fixed aspect ratio

Warm-up: $|C| = 1$ Band

Fermions - Pair Correlation Functions

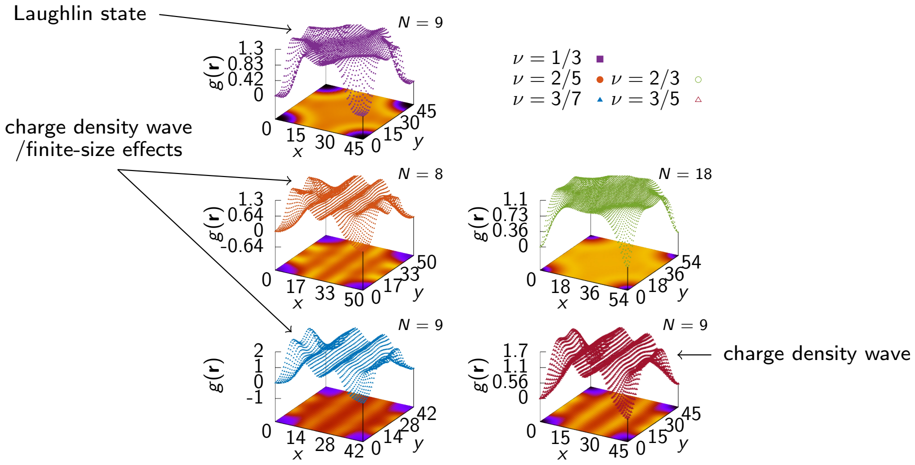
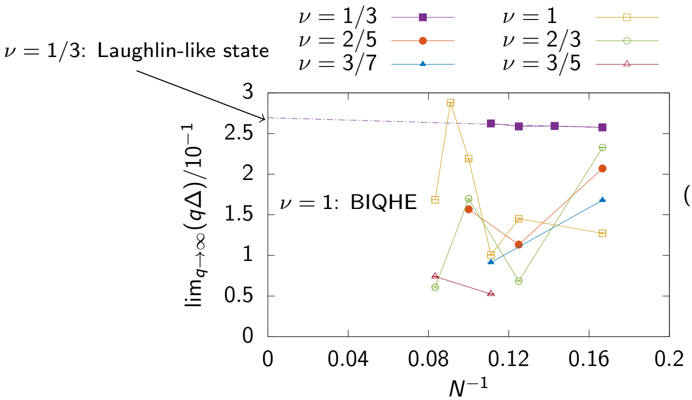


Figure: Pair correlation functions for the lowest-lying ground state in the $(k_x, k_y) = (0, 0)$ momentum sector

$|C| = 2$ Band

Bosons - Stability in the Effective Continuum

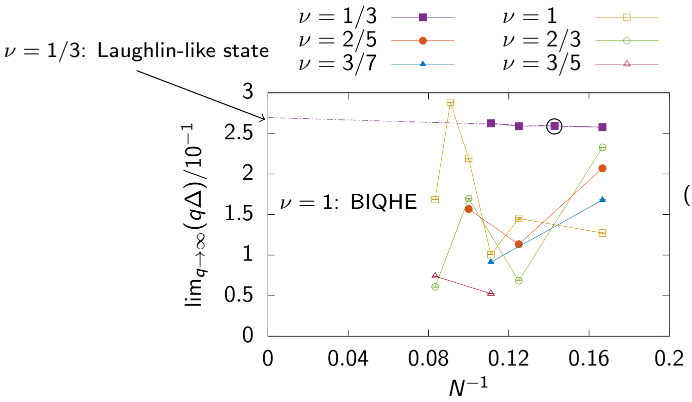


in-line with size dependence of BIQHE in optical flux lattices (Sterdyniak *et al.*, 2015)

Figure: Finite-size scaling of the gap to the thermodynamic effective continuum limit at fixed aspect ratio

$|C| = 2$ Band

Bosons - Stability in the Effective Continuum



in-line with size dependence of BIQHE in optical flux lattices (Sterdyniak *et al.*, 2015)

Figure: Finite-size scaling of the gap to the thermodynamic effective continuum limit at fixed aspect ratio

$|C| = 2$ Band

Bosons - Correlation Function

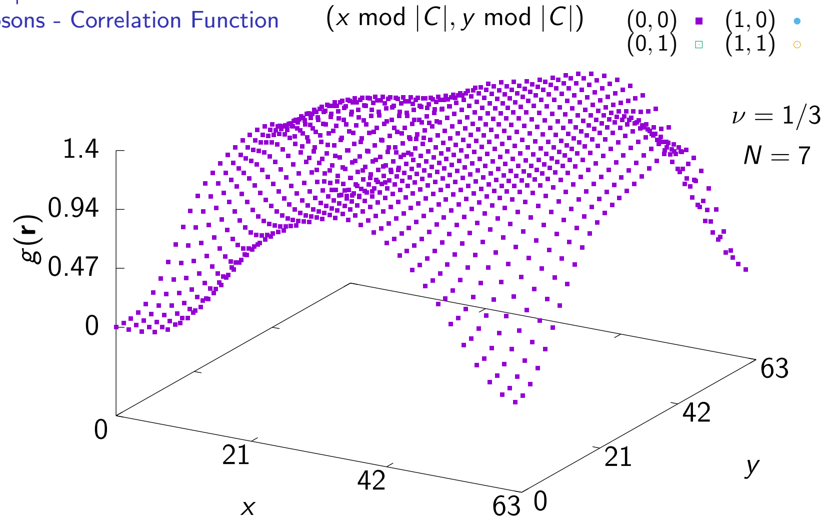


Figure: Pair correlation functions for the lowest-lying ground state in the $(k_x, k_y) = (0,0)$ momentum sector

$|C| = 2$ Band

Bosons - Correlation Function

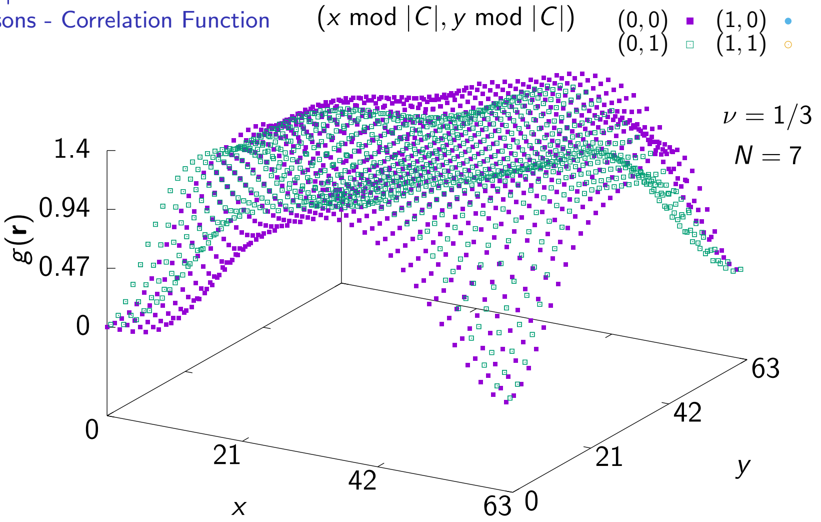


Figure: Pair correlation functions for the lowest-lying ground state in the $(k_x, k_y) = (0,0)$ momentum sector

$|C| = 2$ Band

Bosons - Correlation Function

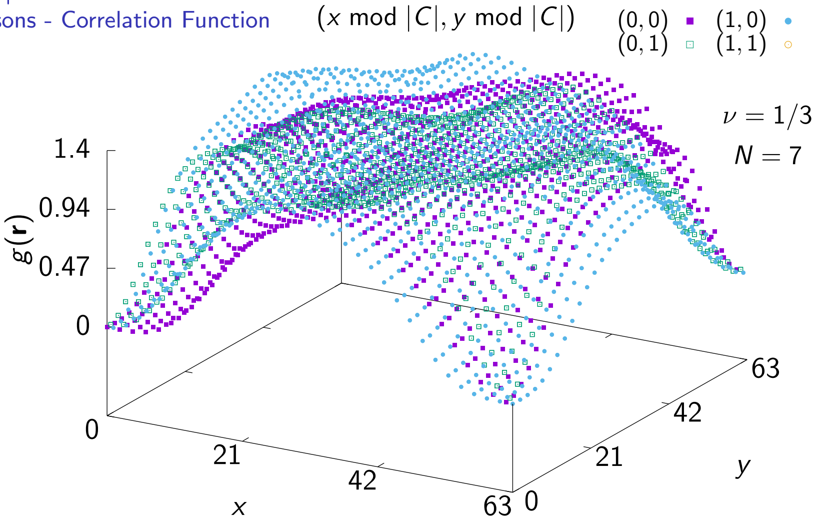


Figure: Pair correlation functions for the lowest-lying ground state in the $(k_x, k_y) = (0,0)$ momentum sector

$|C| = 2$ Band

Bosons - Correlation Function

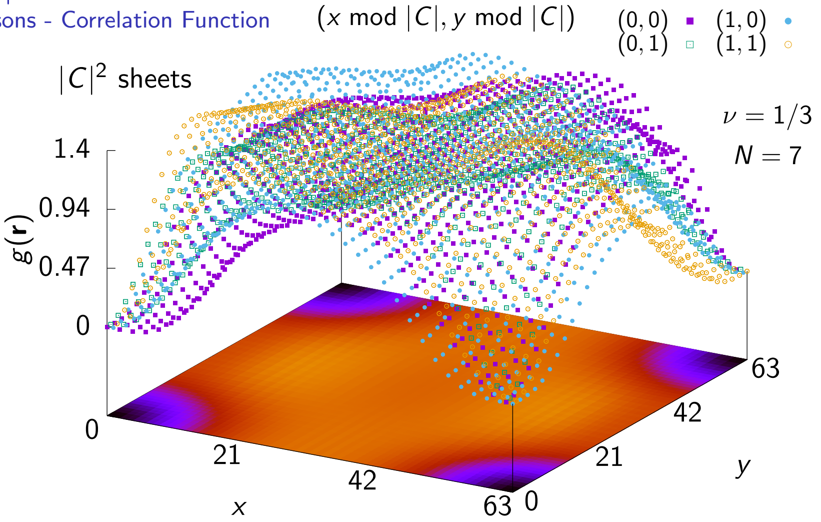


Figure: Pair correlation functions for the lowest-lying ground state in the $(k_x, k_y) = (0, 0)$ momentum sector

$|C| = 2$ Band

Bosons - Correlation Functions & Entanglement Spectra

$$|\Psi\rangle = \sum_{k,n} \lambda_{k,n} |\Psi^A_{k,n}\rangle \otimes |\Psi^B_{k,n}\rangle$$

$$\boxed{\xi \equiv -\ln \lambda^2}$$

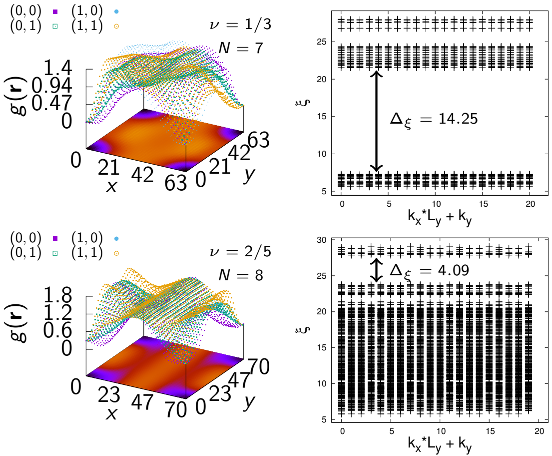


Figure: Pair correlation functions for the lowest-lying ground state in the $(k_x, k_y) = (0, 0)$ momentum sector, and correponding entanglement spectra

$|C| = 2$ Band

Fermions - Stability in the Effective Continuum

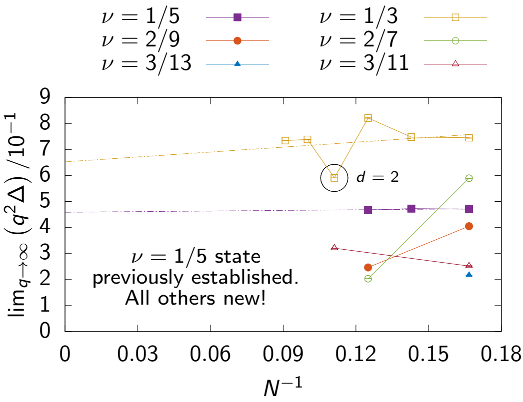


Figure: Finite-size scaling of the gap to the thermodynamic effective continuum limit at fixed aspect ratio

$|C| = 2$ Band

Fermions - Correlation Functions & Entanglement Spectra

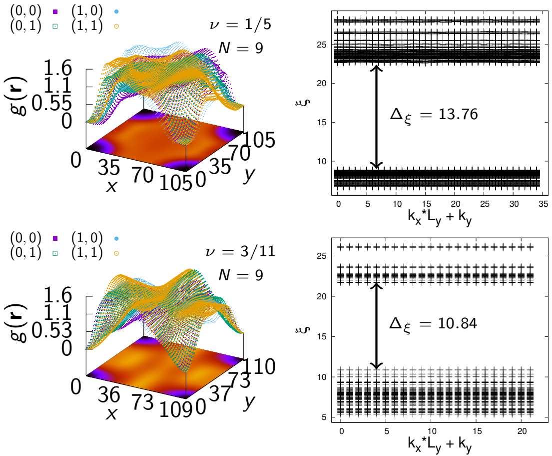


Figure: Pair correlation functions for the lowest-lying ground state in the $(k_x, k_y) = (0, 0)$ momentum sector, and correponding entanglement spectra

$|C| = 3$ Band

Bosons - Stability in the Effective Continuum

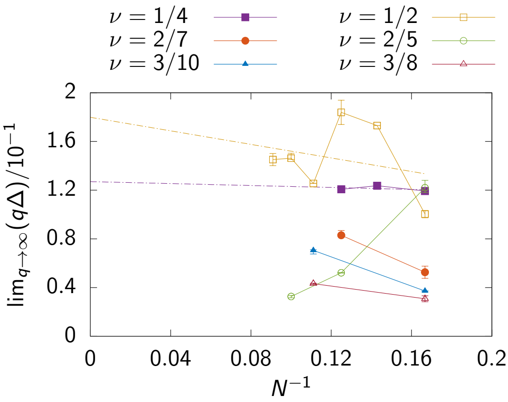


Figure: Finite-size scaling of the gap to the thermodynamic effective continuum limit at fixed aspect ratio

$|C| = 3$ Band

Bosons - Correlation Functions

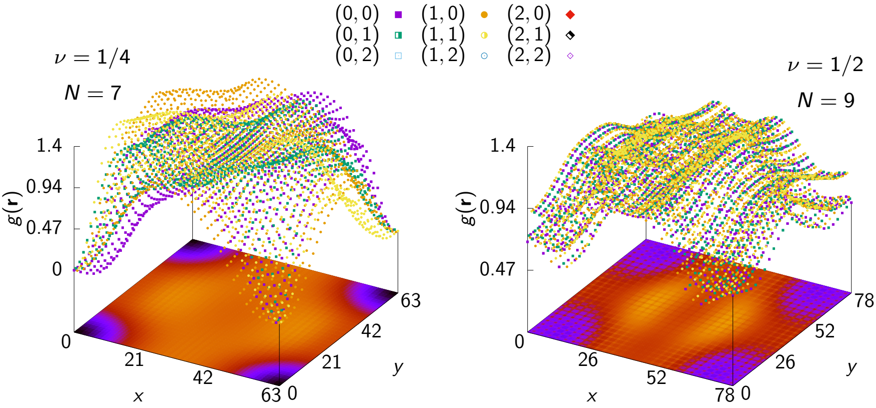
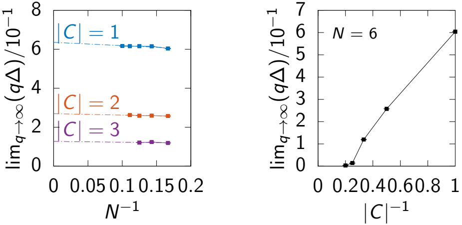


Figure: Pair correlation functions for the lowest-lying ground state in the $(k_x, k_y) = (0, 0)$ momentum sector

Stability in the Effective Continuum

Summary



(a) bosons

Figure: Finite-size scaling of the gap at fixed aspect ratio for bosonic Laughlin states

(b) fermions

$\lvert C\rvert$	r	ν	$\lim_{N,q\to\infty}(q\Delta)$
1	1	1/2	0.64 ± 0.01
2	1	1/3	0.27 ± 0.005
3	1	1/4	0.13 ± 0.01
	-1	1/2	0.18 ± 0.07

$\lvert C\rvert$	r	ν	$\lim_{N,q\to\infty}(q^2\Delta)$
1	1	1/3	2.56 ± 0.02
	-2	2/3	2.56 ± 0.02
2	1	1/5	0.46 ± 0.02
	-1	1/3	0.65 ± 0.16

Table: States with (effective) continuum limits that could be extrapolated to the thermodynamic limit

Conclusion

- Scaling to the effective continuum limit at fixed aspect ratio converges faster than scaling at fixed flux density.
- Vast majority of finite-size spectra produce the ground state degeneracy predicted by CF theory.
- Laughlin-like states with $\nu = 1/(|kC| + 1)$ are the most robust, and yield a clear gap in the effective continuum limit.
- Instability may be caused by competing topological phases, charge density waves, or finite-size effects.
- Stable FCIs found with clear entanglement gaps in $|C| > 1$ bands - largest gaps seen for $|C| = 2$ fermions.
- Pair-correlations are smooth functions modulated by $|C|$ sites along both axes, giving rise to the appearance of $|C|^2$ sheets.

Conclusion

- Scaling to the effective continuum limit at fixed aspect ratio converges faster than scaling at fixed flux density.
- Vast majority of finite-size spectra produce the ground state degeneracy predicted by CF theory.
- Laughlin-like states with $\nu = 1/(|kC| + 1)$ are the most robust, and yield a clear gap in the effective continuum limit.
- Instability may be caused by competing topological phases, charge density waves, or finite-size effects.
- Stable FCIs found with clear entanglement gaps in $|C| > 1$ bands - largest gaps seen for $|C| = 2$ fermions.
- Pair-correlations are smooth functions modulated by $|C|$ sites along both axes, giving rise to the appearance of $|C|^2$ sheets.

ご清聴ありがとうございました！

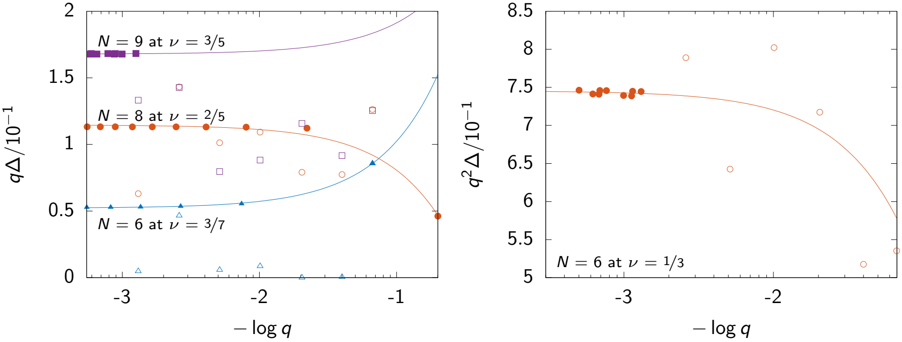
Supplementary Slides

1. Approaching the Effective Continuum
2. Warm-up: $|C| = 1$ Band: Bosons & Fermions - Scaling of the Gap with MUC Size
3. $|C| = 2$ Band: Bosons - Rectangular Geometries
4. $|C| = 3$ Band: Bosons - Rectangular Geometries

Approaching the Effective Continuum

Q: Should we fix n_ϕ or fix aspect ratio?

hollow symbols $\Rightarrow$ fixed n_ϕ
filled symbols $\Rightarrow$ fixed aspect ratio



(a) bosons (b) fermions

Figure: Finite-size scaling of the gap in the $|C| = 2$ band

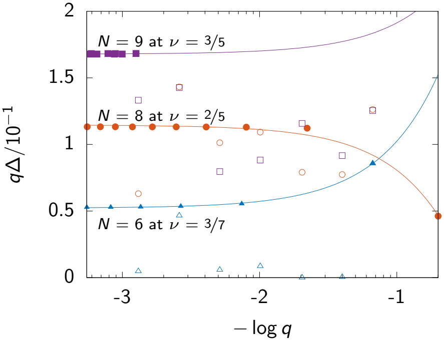
Approaching the Effective Continuum

Q: Should we fix n_ϕ or fix aspect ratio?

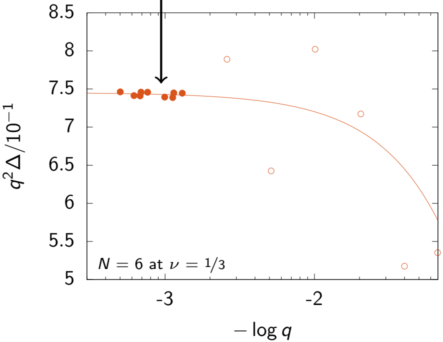
A: Fix aspect ratio

hollow symbols $\Rightarrow$ fixed n_ϕ
filled symbols $\Rightarrow$ fixed aspect ratio

scaling at fixed aspect ratio is more robust!



(a) bosons



(b) fermions

Figure: Finite-size scaling of the gap in the $|C| = 2$ band

Warm-up: $|C| = 1$ Band

Bosons & Fermions - Scaling of the Gap with MUC Size

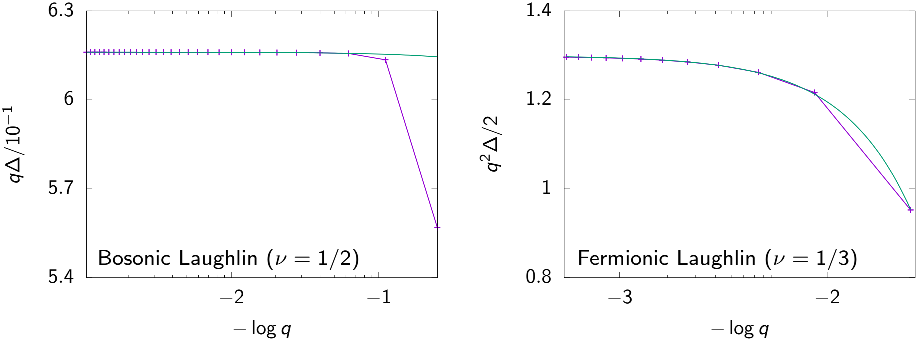


Figure: Finite-size scaling of $q^{(2)}\Delta$ to a constant value in the continuum limit for 8-particle Laughlin states

Warm-up: $|C| = 1$ Band

Bosons & Fermions - Scaling of the Gap with MUC Size

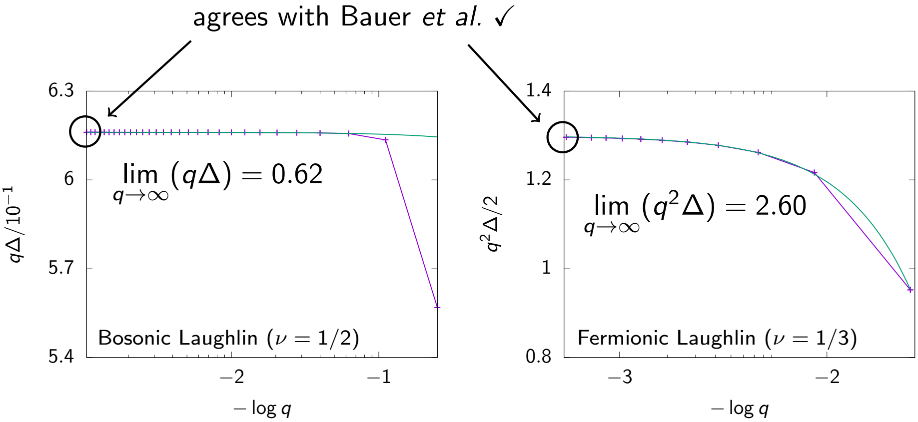


Figure: Finite-size scaling of $q^{(2)}\Delta$ to a constant value in the continuum limit for 8-particle Laughlin states

$|C| = 2$ Band

Bosons - Rectangular Geometries

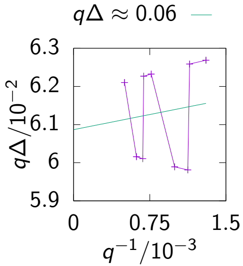
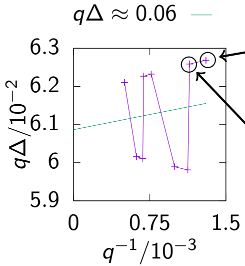
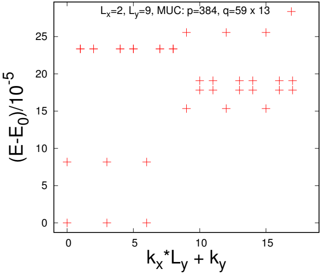


Figure: Magnitude of the gap for the 12-particle state at $\nu = 2/3$

$|C| = 2$ Band

Bosons - Rectangular Geometries





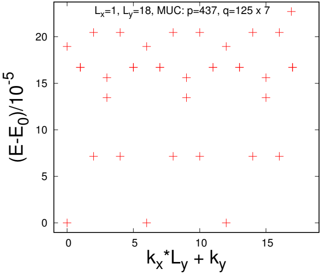


Figure: Magnitude of the gap for the 12-particle state at $\nu = 2/3$

$|C| = 3$ Band

Bosons - Rectangular Geometries

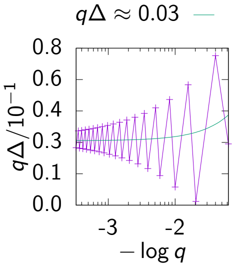
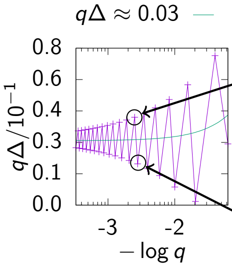


Figure: Magnitude of the gap for the 6-particle state at $\nu = 3/8$

$|C| = 3$ Band

Bosons - Rectangular Geometries



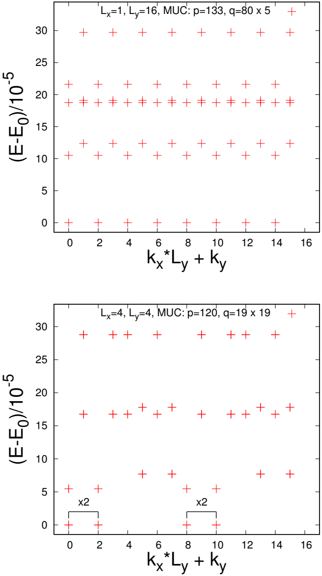
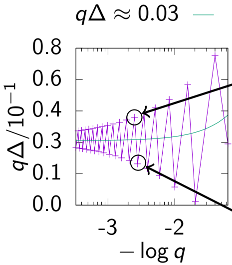
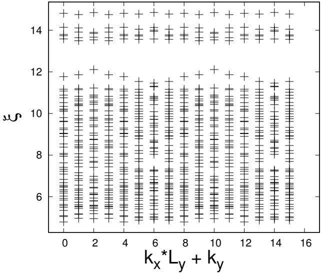


Figure: Magnitude of the gap for the 6-particle state at $\nu = 3/8$

$|C| = 3$ Band

Bosons - Rectangular Geometries





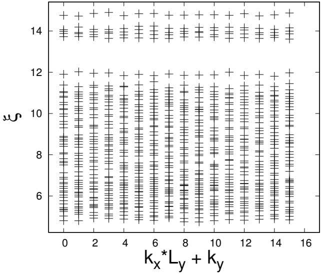


Figure: Magnitude of the gap for the 6-particle state at $\nu = 3/8$