Symmetry Protected Topological Criticality

Linhao Li (ISSP @ U.tokyo)

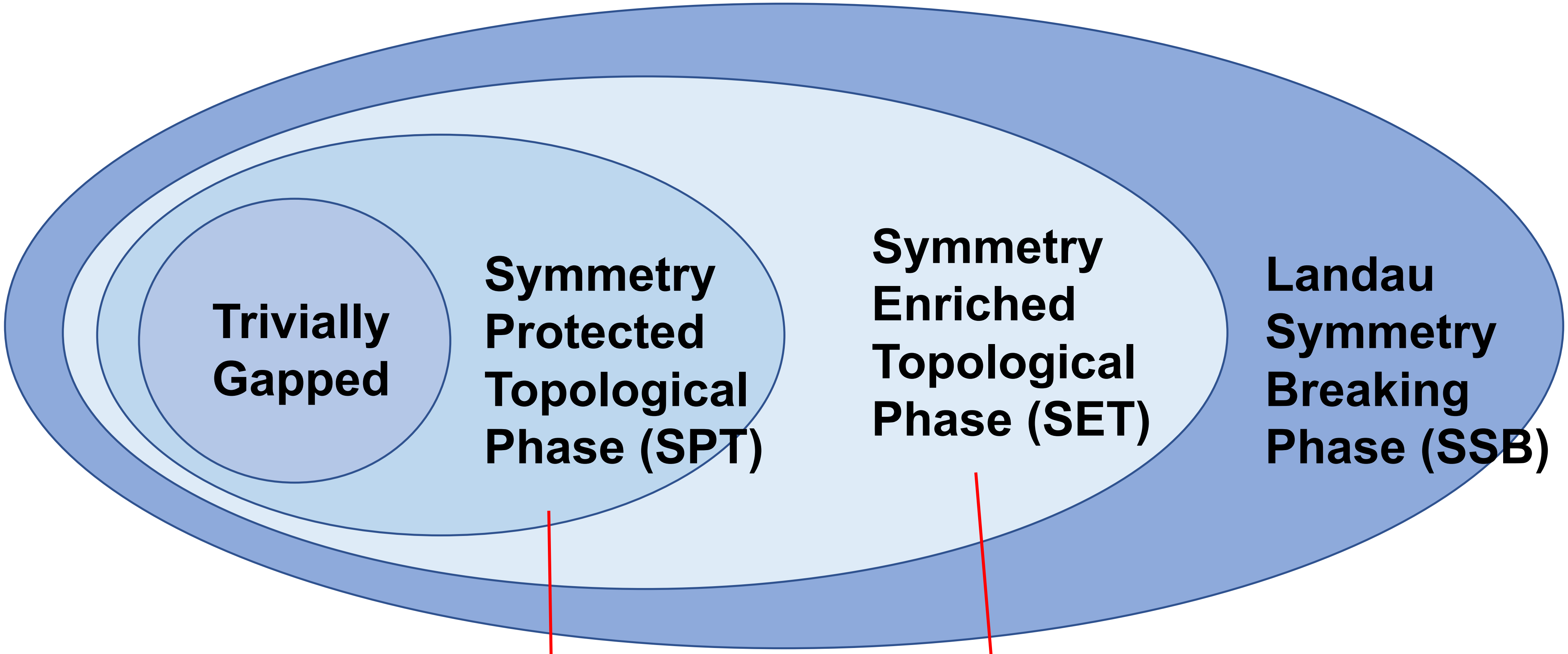
Nov 9^{th} 2022

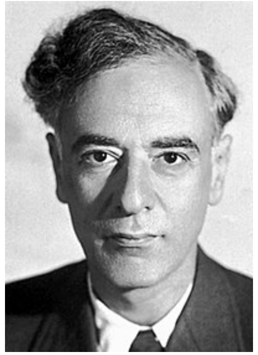
arXiv:2204.03131

with Yunqin Zheng, Masaki Oshikawa

Gapped Quantum Matter

relatively well-understood





't Hooft anomaly (Hep-th)
Integer QH (Cond-mat)

Symmetric TQFT (Hep-th)
Toric code (Cond-mat)

Properties of Gapped SPT Phase

Systems belonging to a gapped SPT phase satisfy:

- Global symmetry Γ is non-anomalous
- Single ground state under **periodic** boundary condition(PBC).
- Degenerate ground states under **open** boundary condition(OBC).
- Ground state under Γ **twisted** boundary condition(TBC) carries Γ charge.

Examples: AKLT chain

- Global symmetry is onsite SO(3) symmetry

- Ground state degeneracy is 1 or 4 under PBC or OBC

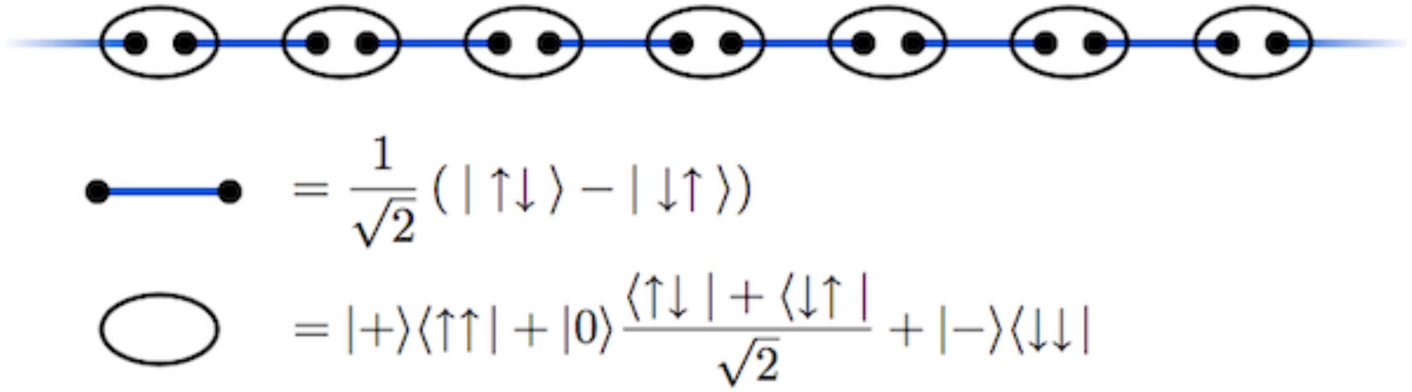


Figure from Wiki

Gapless Quantum Matter

much less well-understood

Gapless quantum matter typically appear in two situations:

1. Critical point/Critical region.
2. Goldstone boson/fermion.

Quantum Criticality and
we focus on it in this talk

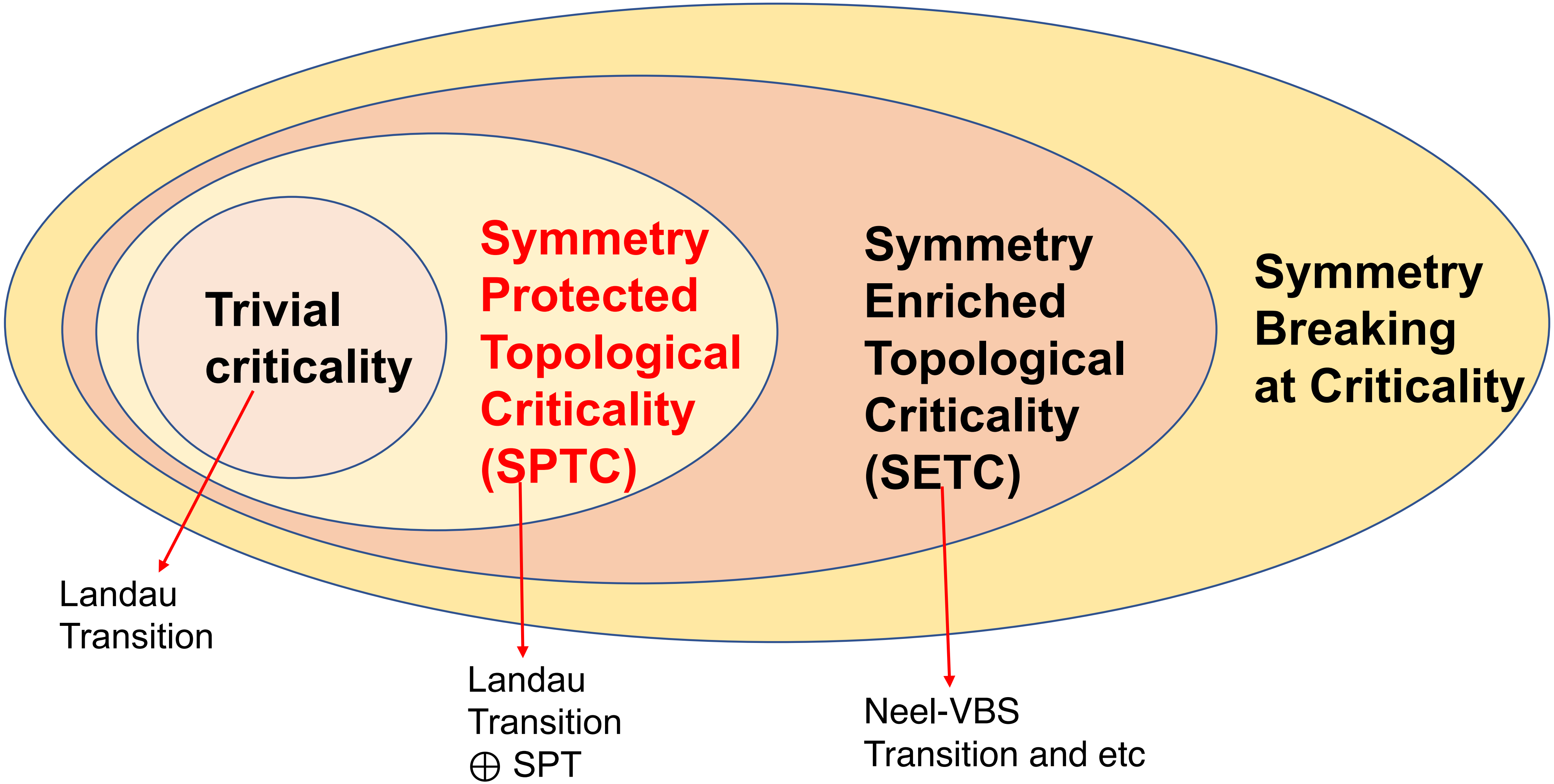
What is the landscape of Quantum criticality?

Not know yet. But there are several examples in the past and we wish to organize them, analogous to the classification of gapped SPT phase.

Symmetry Protected Topological Criticality (SPTC)

Landau Transition $\oplus$ SPT

Landscape of Quantum Criticality - A Dream



Properties of SPTC

By minimally generalizing the properties of gapped SPT, we expect SPTC should satisfy:

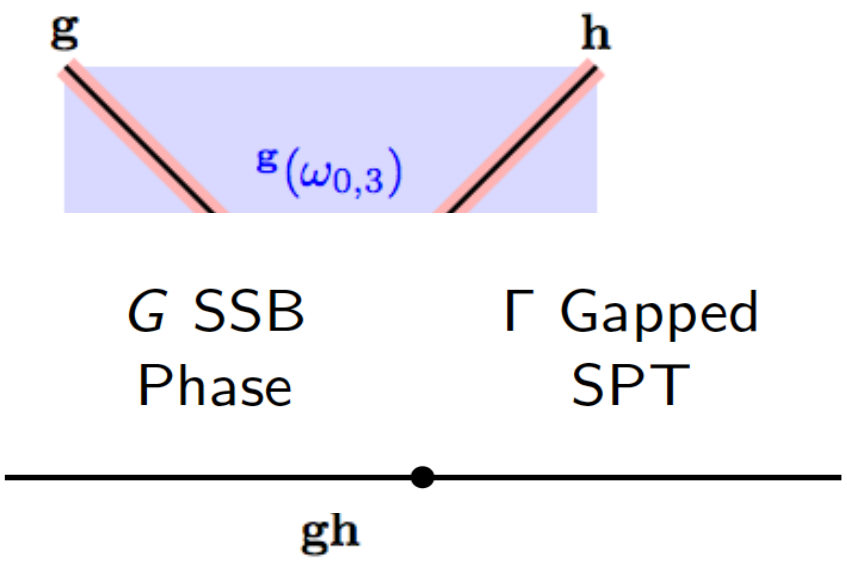
- Global symmetry Γ is non-anomalous
- Single ground state under **PBC** with a polynomial finite size gap.
- Degenerate ground states under **OBC** with a exponential decaying finite size gap.
- Ground state under Γ **TBC** carries Γ charge.

Can we construct a nontrivial SPTC given global symmetry Γ?

Decorated Defect Construction of Gapped SPT

$$1 \to A \to \Gamma \to G \to 1$$

- G SSB phase, decorate the G-defects **consistently**. → Γ is anomaly-free
- Proliferate the G-defect to restore the entire Γ symmetry
- Fully fluctuate the decorated G-defects to a gapped SPT phase.

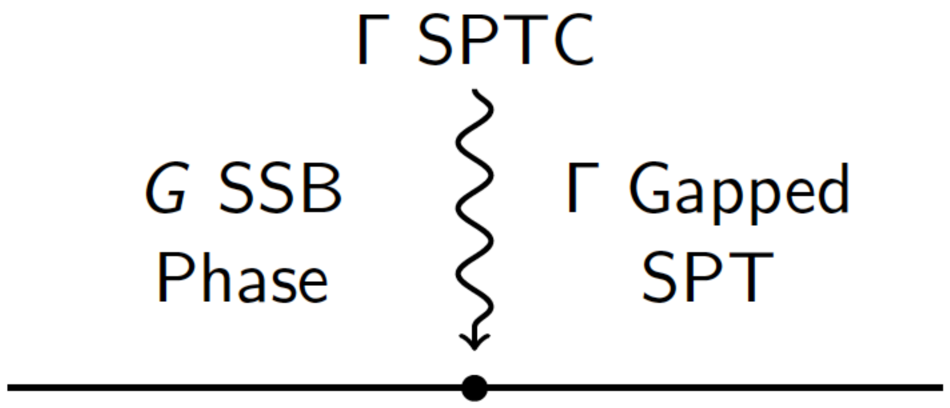


[Chen, Lu, Vishwanath,1303.4301]
[Wang,Ning,Chen,2104.13233]

Decorated Defect Construction of SPTC

$$1 \to A \to \Gamma \to G \to 1$$

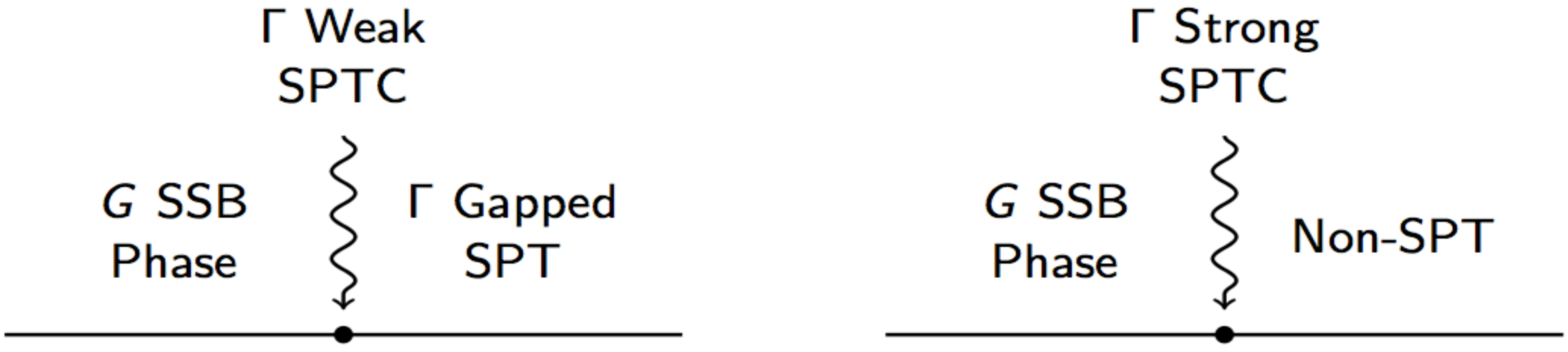
- G SSB phase, decorate the G-defects **consistently**. $\longrightarrow$ Γ is anomaly-free
- Fine-tune the fluctuation of the decorated G-defects to criticality.



[Scadi,Parker,Vasseur,1705.01557]
[**Li**,Oshikawa,YZ,2204.03131]

Decorated Defect Construction of Weak and Strong SPTC

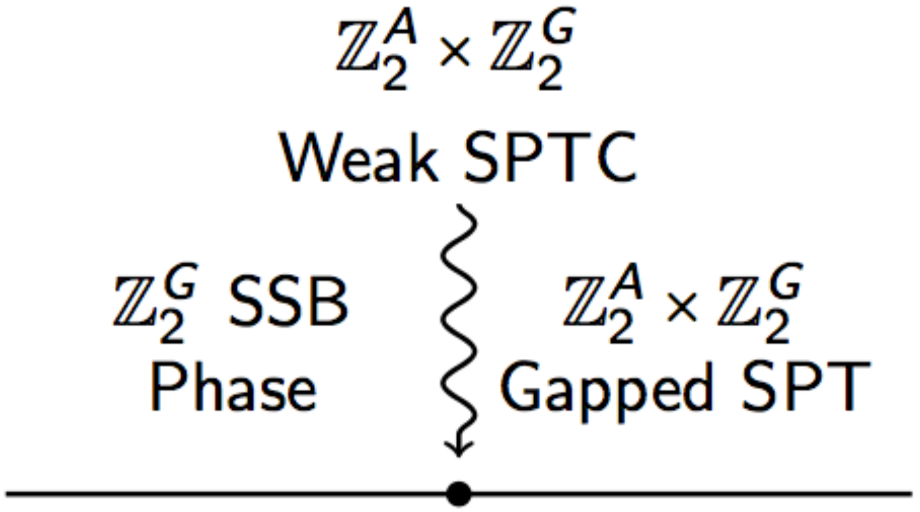
$$1 \to A \to \Gamma \to G \to 1$$



1. G SSB phase with **certain anomaly**, decorate the G-defects **~~consistently~~**. $\Rightarrow \Gamma$ is anomaly free.
2. Fine-tune the fluctuation of the decorated G-defects to criticality.

[Scadi,Parker,Vasseur,1705.01557]
[Thorngren,Vishwanath,Verresen,2008.06638]
[Li,Oshikawa,YZ,2204.03131]

$\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ Weak SPTC in (1+1)d Spin chain



$\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ Weak SPTC in (1+1)d Spin chain

- Hibert space: two types of spin-1/2: τ and σ
- Symmetry operators:

$$U_A = \prod_{i=1}^{L} \tau_{i+\frac{1}{2}}^{x} \qquad\qquad U_G = \prod_{i=1}^{L} \sigma_i^x$$

- $\mathbb{Z}_2^G$ SSB Hamiltonian:

$$H = -\sum_{i=1}^{L} \tau_{i+\frac{1}{2}}^{x} + \sigma_i^z \ \sigma_{i+1}^z$$

- Two ground states:

$$|+\rangle = |\uparrow \rightarrow \uparrow \rightarrow \ldots \uparrow \rightarrow\rangle$$

$$|-\rangle = |\downarrow \rightarrow \downarrow \rightarrow \ldots \downarrow \rightarrow\rangle$$

Domain Wall and Fluctuation (Pre-decoration)

- Flipping a string of σ spin, by acting on the ground state with $\prod_{i=j}^{j+k} \sigma_i^x$, creates one domain wall excitations at each end.

$$|\ldots \uparrow \rightarrow \uparrow \rightarrow \uparrow \rightarrow \uparrow \rightarrow \uparrow \rightarrow \uparrow \rightarrow \uparrow \rightarrow \uparrow \rightarrow \uparrow \rightarrow \uparrow \rightarrow \uparrow \rightarrow \uparrow \rightarrow \uparrow \rightarrow \ldots\rangle$$

$$\Downarrow$$

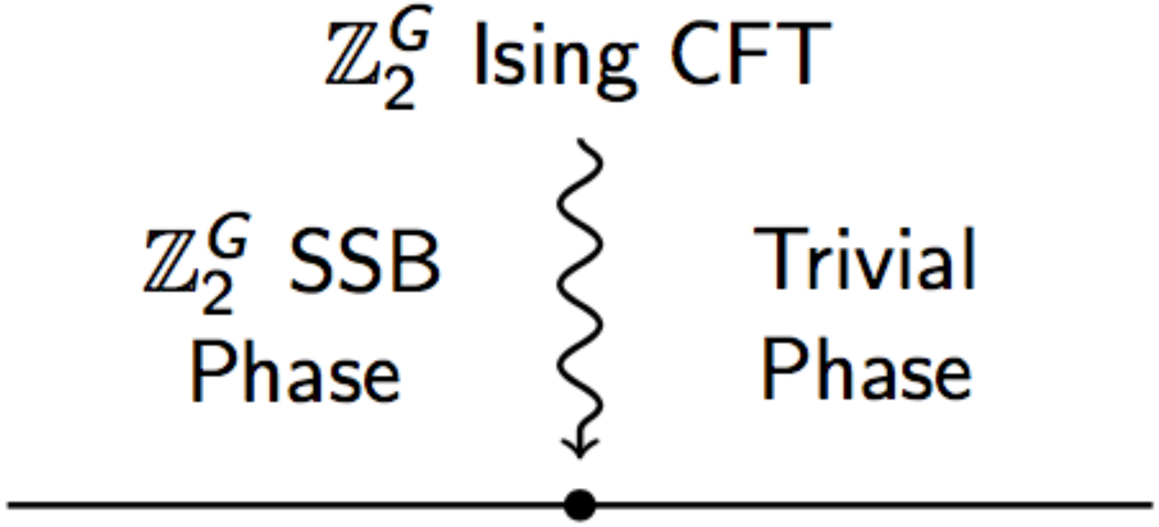
$$|\ldots \uparrow \rightarrow \uparrow \rightarrow \uparrow \rightarrow \downarrow \rightarrow \downarrow \rightarrow \downarrow \rightarrow \downarrow \rightarrow \downarrow \rightarrow \downarrow \rightarrow \uparrow \rightarrow \uparrow \rightarrow \uparrow \rightarrow \uparrow \rightarrow \ldots\rangle$$

- Fluctuating the domain wall achieved by adding to Hamiltonian

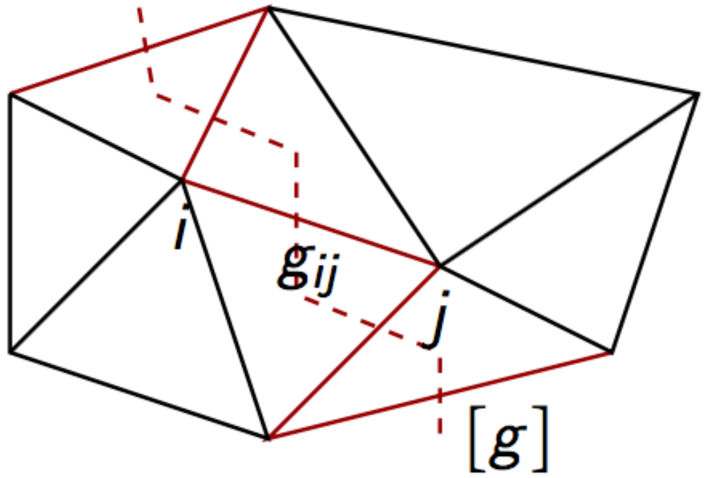
$$-h \sum_{i=1}^{L} \sigma_i^x$$

Domain Wall and Fluctuation (Pre-decoration)

$$H_0 = -\sum_{i=1}^{L}(\tau^{x}_{i+\frac{1}{2}} + \sigma^{z}_{i}\ \sigma^{z}_{i+1} + h\sigma^{x}_{i})$$



Domain Wall Decoration



$$\exp\left(i\pi\int_{[g]} a\right) = \exp\left(i\pi\int_{M_2} a \cup g\right)$$

If there is a $\mathbb{Z}_2^G$ domain wall, we stack a (0+1)d $\mathbb{Z}_2^A$ gapped SPT.

Domain Wall Decoration

If there is a $\mathbb{Z}_2^G$ domain wall, we stack a (0+1)d $\mathbb{Z}_2^A$ gapped SPT.

Between two adjacent and opposite σ spins and τ spin is ↓, we assign the wavefunction an additional − sign.

$$-|\uparrow\downarrow\downarrow\rangle, \qquad -|\downarrow\downarrow\uparrow\rangle$$

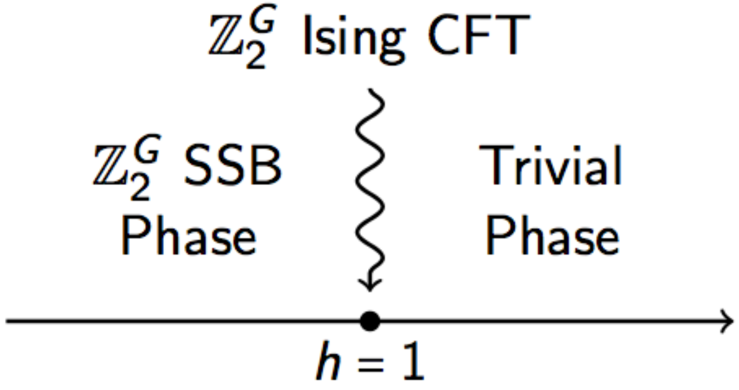
$$|\uparrow\uparrow\uparrow\rangle, \quad |\uparrow\downarrow\uparrow\rangle, \quad |\uparrow\uparrow\downarrow\rangle, \quad |\downarrow\uparrow\uparrow\rangle, \quad |\downarrow\uparrow\downarrow\rangle, \quad |\downarrow\downarrow\downarrow\rangle$$

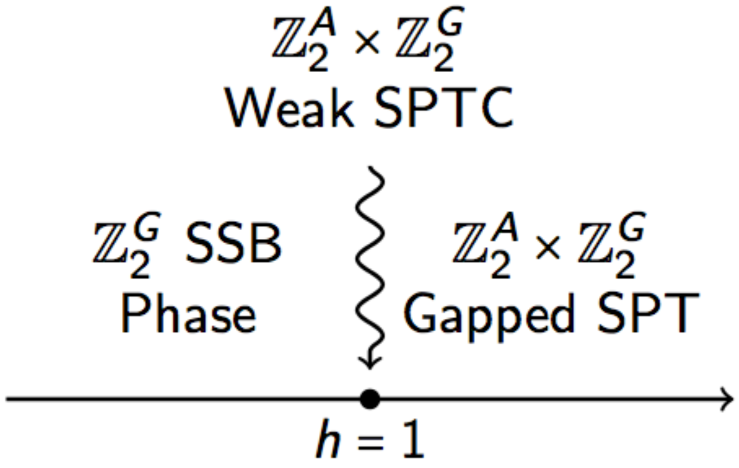
$$U_{DW} = \prod_{i=1}^{L} \exp\left[\frac{\pi i}{4}(1-\sigma_i^z)(1-\tau_{i+\frac{1}{2}}^z)\right] \exp\left[\frac{\pi i}{4}(1-\tau_{i+\frac{1}{2}}^z)1-\sigma_{i+1}^z)\right]$$

Domain Wall Decoration

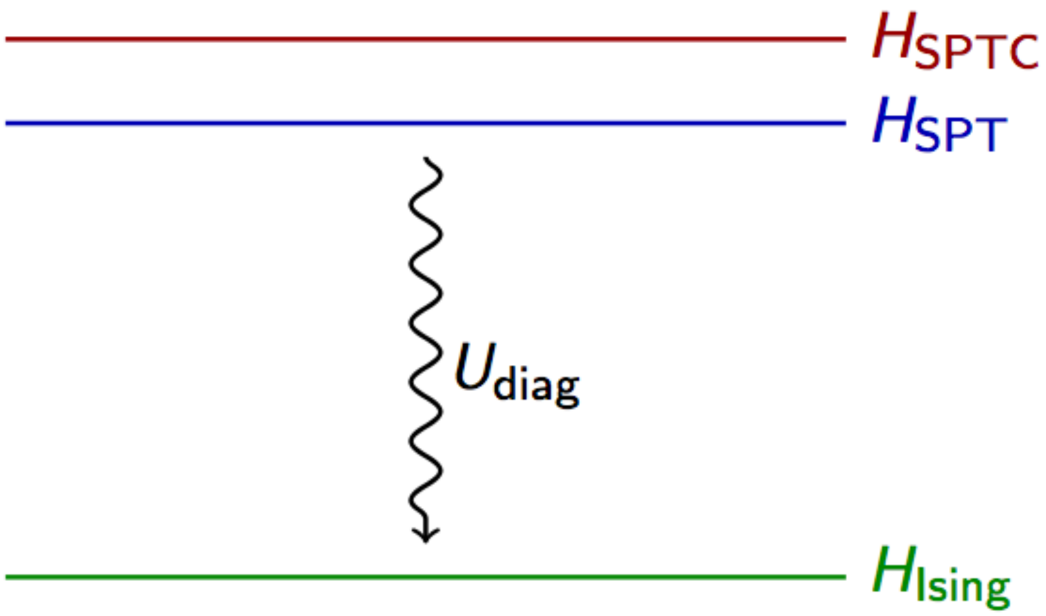
- The Hamiltonian after the domain wall decoration

$$H_1 = U_{DW} H_0 U_{DW}^{+} = -\sum_{i=1}^{L}(\sigma_i^z \tau_{i+\frac{1}{2}}^x \sigma_{i+1}^z + \sigma_i^z\ \sigma_{i+1}^z + h\tau_{i-\frac{1}{2}}^x \sigma_i^x \tau_{i+\frac{1}{2}}^x)$$





Trivializability Upon Stacking Gapped SPTs



It seems to imply that all the nontrivial topological properties of the $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ weak SPTC inherit from gapped SPT.

As far as topological properties are concerned, it seems that there isn't anything new. Is it true?

Signatures under Periodic Boundary Condition

$$H_{SPTC} = -\sum_{i=1}^{L}(\sigma_i^z \tau_{i+\frac{1}{2}}^x \sigma_{i+1}^z + \sigma_i^z \ \sigma_{i+1}^z + \tau_{i-\frac{1}{2}}^z \sigma_i^x \tau_{i+\frac{1}{2}}^z)$$

- Ground state degeneracy is 1.

- $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ charge of the ground state is (0, 0).

Same as the gapped SPT 

Signatures under Twisted Boundary Condition

Use $\mathbb{Z}_2^A$ to twist the boundary condition of τ spins

$$H_{SPTC}^{\mathbb{Z}_2^A} = -\sum_{i=1}^{L}(\sigma_i^z \tau_{i+\frac{1}{2}}^x \sigma_{i+1}^z + \sigma_i^z\ \sigma_{i+1}^z) - \sum_{i=1}^{L-1} \tau_{i-\frac{1}{2}}^z \sigma_i^x \tau_{i+\frac{1}{2}}^z + \tau_{L-\frac{1}{2}}^z \sigma_i^x \tau_{L+\frac{1}{2}}^z$$

- Ground state degeneracy is 1.

- $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ charge of the ground state is (0, 1).

Likewise, one can use $\mathbb{Z}_2^G$ to twist the boundary condition, and the ground state charge is instead (1,0)

Same as the gapped SPT 

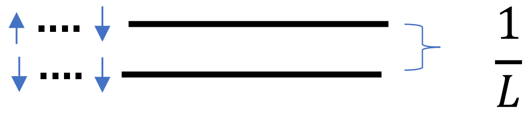
Signatures under open Boundary Condition

$$H_{SPTC}^{OBC} = -\sum_{i=1}^{L}(\sigma_i^z \tau_{i+\frac{1}{2}}^x \sigma_{i+1}^z + \sigma_i^z\ \sigma_{i+1}^z) - \sum_{i=2}^{L} \tau_{i-\frac{1}{2}}^z \sigma_i^x \tau_{i+\frac{1}{2}}^z - \tau_{L-\frac{1}{2}}^x$$

- Besides symmetry operators U_A, U_G, there are additional operators localized on the boundary which commutes with the Hamiltonian.

$$\left\{\sigma_1^z, \sigma_L^z \tau_{L+\frac{1}{2}}^x, U_A, U_G\right\}$$

- The dimension of irreducible representation is 2.
- Exact ground state degeneracy is 2.



This should be contrasted with GSD= 4 for gapped SPT 

Summary of Signatures of ZA × ZG Weak SPTC

		$\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ Weak SPTC	$\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ Gapped SPT
PBC:	GSD	1	1
	$\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ Charge	$(0,0)$	$(0,0)$
$\mathbb{Z}_2^A$-TBC:	GSD	1	1
	$\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ Charge	$(0,1)$	$(0,1)$
$\mathbb{Z}_2^G$-TBC:	GSD	1	1
	$\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ Charge	$(1,0)$	$(1,0)$
OBC:	GSD	2	4

Stability of Signatures

- Are the above signatures stable? Or just an artifact of the our model?

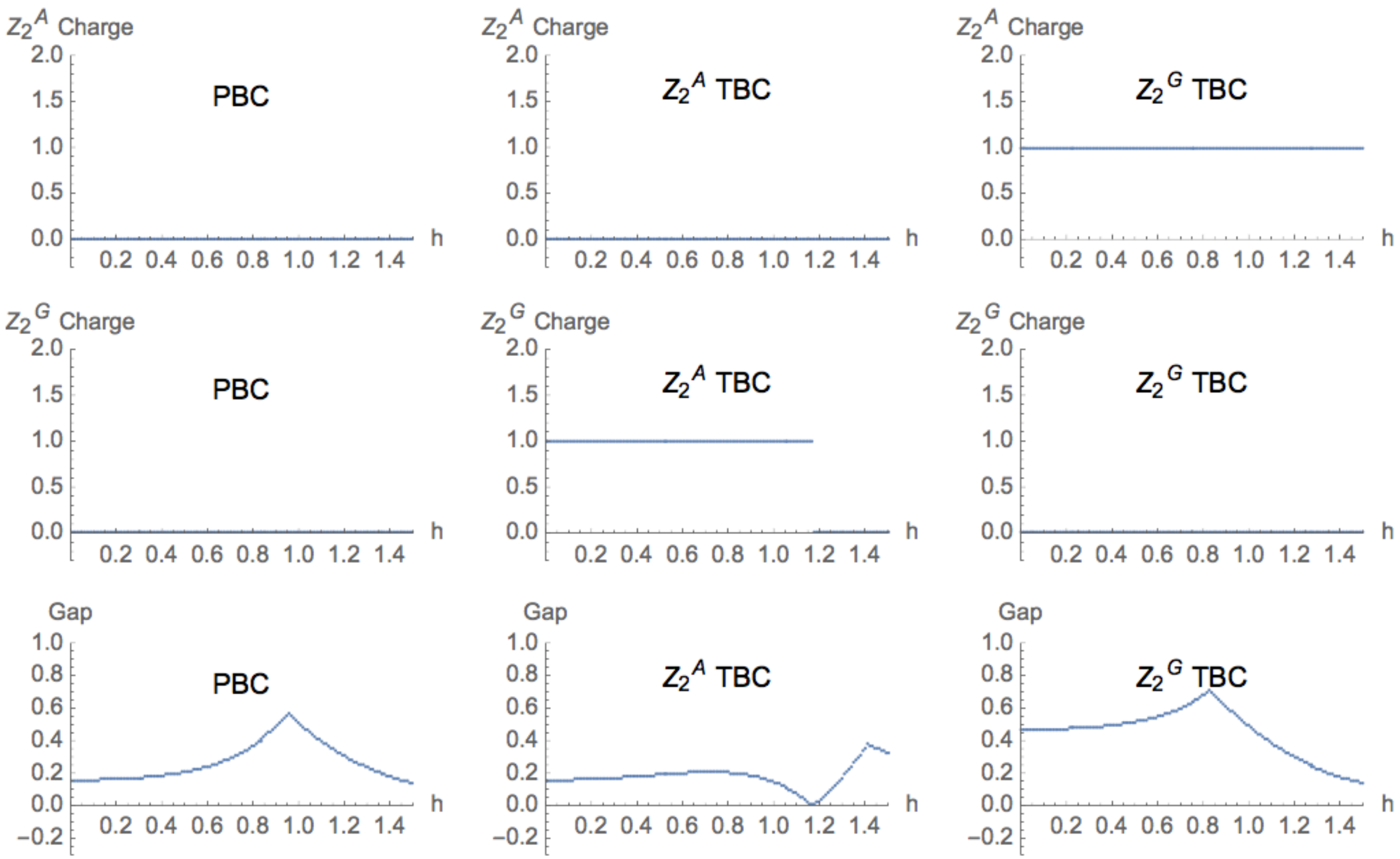
- Add $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ symmetric perturbation, the exact boundary degeneracy under OBC lifted by **exponentially decaying gap**. 

$$V = -h \sum_{i=2}^{L} (\tau^x_{i-\frac{1}{2}} \tau^x_{i+\frac{1}{2}} + \sigma^x_1 + \sigma^x_L)$$

- Add the same perturbation, check the ground state charge under TBC.

$$V = -h \sum_{i=1}^{L} \tau^x_{i-\frac{1}{2}} \tau^x_{i+\frac{1}{2}}$$

Stability of Signatures

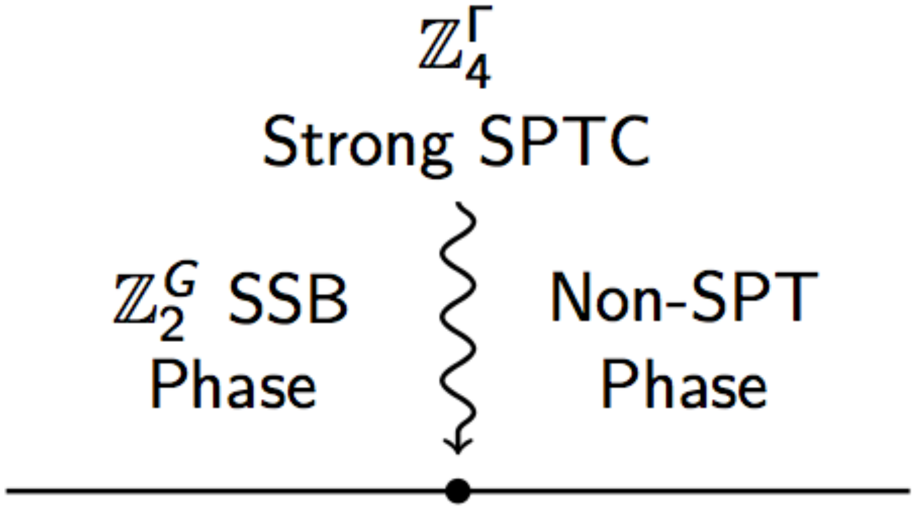


Stability of Signatures

- Exact ground state degeneracy under **OBC** lifted by an exponential gap under generic symmetric perturbation

- Non-trivial ground state charge under **TBC** is stable under a symmetric and small enough perturbation

Z_4^{Γ} Strong SPTC in (1 + 1)d Spin Chain



Symmetry Extension

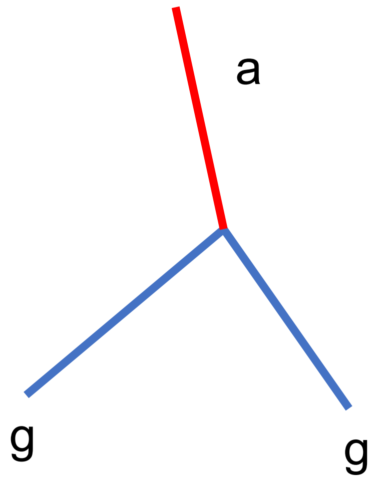
- Symmetry fits into nontrivial extension

$$1 \to \mathbb{Z}_2^A \to \mathbb{Z}_4^\Gamma \to \mathbb{Z}_2^G \to 1$$

Nontrivial extension means

Flatness of Z_4 gauge field

$$\delta a = \mathrm{Bock}(g) := \frac{1}{2}\delta\tilde{g} \mod 2, \qquad \delta g = 0 \mod 2.$$



Domain Wall Decoration

- We still start with a $\mathbb{Z}_2^G$ SSB phase, and decorated the $\mathbb{Z}_2^G$ domain wall by （0+1） d $\mathbb{Z}_2^A$ gapped SPT, as before

$$\exp\left(i\pi\int_{[g]} a\right) = \exp\left(i\pi\int_{M_2} a \cup g\right) = \exp\left(i\pi\int_{M_3} g \cup \mathrm{Bock}(g)\right).$$

$$\Downarrow$$

Domain wall decoration induces a nontrivial $\mathbb{Z}_2^G$ anomaly.

Domain Wall Decoration

- The anomaly needs to be compensated by the same anomaly

$$\exp\left(i\pi \int_{M_3} g \cup \mathrm{Bock}(g)\right)$$

  in the $\mathbb{Z}_2^G$ SSB phase.

- The entire system after domain wall decoration is anomaly free.

- One further fluctuate the domain walls to the critical point.

Domain Wall Decoration: The Algorithm

- Start with a critical point with $\mathbb{Z}_2^G$ anomaly. (Levin-Gu model is a well-known example.)

- Decorated the $\mathbb{Z}_2^G$ domain wall by $\mathbb{Z}_2^A$ (0+1)d gapped SPT . (Realized by conjugating U_{DW} .)

Z_4^Γ Strong SPTC

- Hamiltonian for Z_4^Γ Strong SPTC

$$H_{SPTC} = -\sum_{i=1}^{L}(\sigma_i^z \tau_{i+\frac{1}{2}}^x \sigma_{i+1}^z + \tau_{i-\frac{1}{2}}^y \sigma_i^x \tau_{i+\frac{1}{2}}^y + \tau_{i-\frac{1}{2}}^z \sigma_i^x \tau_{i+\frac{1}{2}}^z)$$

- It is invariant under Z_4^Γ symmetry,

$$U_\Gamma = \prod_{i=1}^{L} \sigma_i^x \exp\left[\frac{\pi i}{4}(1 - \tau_{i+\frac{1}{2}}^x)\right]$$

Domain Wall Un-decoration

Let's check that it comes from Levin-Gu model by decoration.

- Undecorated Hamiltonian

$$U_{DW} H_{SPTC} U_{DW}^{+} = -\sum_{i=1}^{L} (\tau_{i+\frac{1}{2}}^{x} - \sigma_{i-1}^{z} \tau_{i-\frac{1}{2}}^{x} \sigma_{i}^{x} \tau_{i+\frac{1}{2}}^{x} \sigma_{i+1}^{x} + \sigma_{i}^{x})$$

$$\xrightarrow{\text{Low energy}} -\sum_{i=1}^{L} (-\sigma_{i-1}^{z} \sigma_{i}^{x} \sigma_{i+1}^{x} + \sigma_{i}^{x}) \longrightarrow \text{Levin-Gu model}$$

- Undecorated Symmetry

$$U_{DW} U_{\Gamma} U_{DW}^{+} = \prod_{i=1}^{L} \sigma_{i}^{x} \exp\left[\frac{\pi i}{4} (1 - \sigma_{i}^{z} \tau_{i+\frac{1}{2}}^{x} \sigma_{i+1}^{z})\right]$$

$$\xrightarrow{\text{Low energy}} \prod_{i=1}^{L} \sigma_{i}^{x} \exp\left[\frac{\pi i}{4} (1 - \sigma_{i}^{z} \sigma_{i+1}^{z})\right] \longrightarrow \text{Anomalous } \mathbb{Z}_2^G \text{ symmetry}$$

Signatures under Periodic Boundary Condition

- Ground state degeneracy is

$$\mathrm{GSD}_L = \begin{cases} 2, & L = 2 \mod 4 \\ 1, & \text{otherwise} \end{cases}$$

- Z_4^Γ charge of the ground state is

$$q = \begin{cases} 0, & L = 0,1,7 \mod 8 \\ 2, & L = 3,4,5 \mod 8 \\ 0\&2, & L = 2,6 \mod 8 \end{cases}$$

Signatures under Twisted Boundary Condition

- Use Z_2^A to twist the boundary condition of τ spins.

$$H_{SPTC}^{Z_2^A} = -\sum_{i=1}^{L-1}(\sigma_i^z \tau_{i+\frac{1}{2}}^x \sigma_{i+1}^z + \tau_{i-\frac{1}{2}}^y \sigma_i^x \tau_{i+\frac{1}{2}}^y + \tau_{i-\frac{1}{2}}^z \sigma_i^x \tau_{i+\frac{1}{2}}^z)$$
$$-\sigma_L^z \tau_{\frac{1}{2}}^x \sigma_1^z + \tau_{L-\frac{1}{2}}^z \sigma_L^x \tau_{\frac{1}{2}}^z + \tau_{L-\frac{1}{2}}^y \sigma_L^x \tau_{\frac{1}{2}}^y$$

- Ground state degeneracy is 1.

- Relative Z_4^Γ charge of the ground state is 2.

Signatures under Twisted Boundary Condition

- Use Z_4^{Γ} to twist the boundary condition.

$$H_{SPTC}^{Z_4^{\Gamma}} = -\sum_{i=1}^{L-1}(\sigma_i^z \tau_{i+\frac{1}{2}}^x \sigma_{i+1}^z + \tau_{i-\frac{1}{2}}^y \sigma_i^x \tau_{i+\frac{1}{2}}^y + \tau_{i-\frac{1}{2}}^z \sigma_i^x \tau_{i+\frac{1}{2}}^z)$$
$$+\sigma_L^z \tau_{\frac{1}{2}}^x \sigma_1^z \; -\tau_{L-\frac{1}{2}}^z \sigma_L^x \tau_{\frac{1}{2}}^y + \tau_{L-\frac{1}{2}}^y \sigma_L^x \tau_{\frac{1}{2}}^z$$

- Ground state degeneracy is 2 or 4.

- Relative Z_2^A charge of the ground state is 1.

Signatures under open Boundary Condition

$$H_{SPTC}^{OBC} = -\sum_{i=2}^{L}\left(\tau^{y}_{i-\frac{1}{2}}\sigma^{x}_{i}\tau^{y}_{i+\frac{1}{2}} + \tau^{z}_{i-\frac{1}{2}}\sigma^{x}_{i}\tau^{z}_{i+\frac{1}{2}}\right) + \sum_{i=1}^{L-1} \sigma^{z}_{i}\tau^{x}_{i+\frac{1}{2}}\sigma^{z}_{i+1}$$

- Besides symmetry operators U_Γ, there are additional operators localized on the boundary which commutes with the Hamiltonian.

$$\left\{\sigma^{z}_{1}, \sigma^{z}_{L}\tau^{x}_{L+\frac{1}{2}}, U_\Gamma \right\}$$

- The dimension of irreducible representation is 2.
- Exact ground state degeneracy is 2 or 4.

Comparing with Z_4^Γ Landau Transition

- One can use these signatures to distinguish the Z_4^Γ strong SPTC from Landau transition:

		$\mathbb{Z}_4^\Gamma$ Strong SPTC	$\mathbb{Z}_4^\Gamma$ Landau Transition
PBC:	GSD	1	1
	GSD	1	1
$\mathbb{Z}_2^A$-TBC:	$\mathbb{Z}_2^A$ Charge	0	0
	$\mathbb{Z}_4^\Gamma$ Charge	2	0
	GSD	2, 4	1
$\mathbb{Z}_4^\Gamma$-TBC:	$\mathbb{Z}_2^A$ Charge	1	0
	$\mathbb{Z}_4^\Gamma$ Charge	1 or 3	0
OBC:	GSD	≥ 2	1

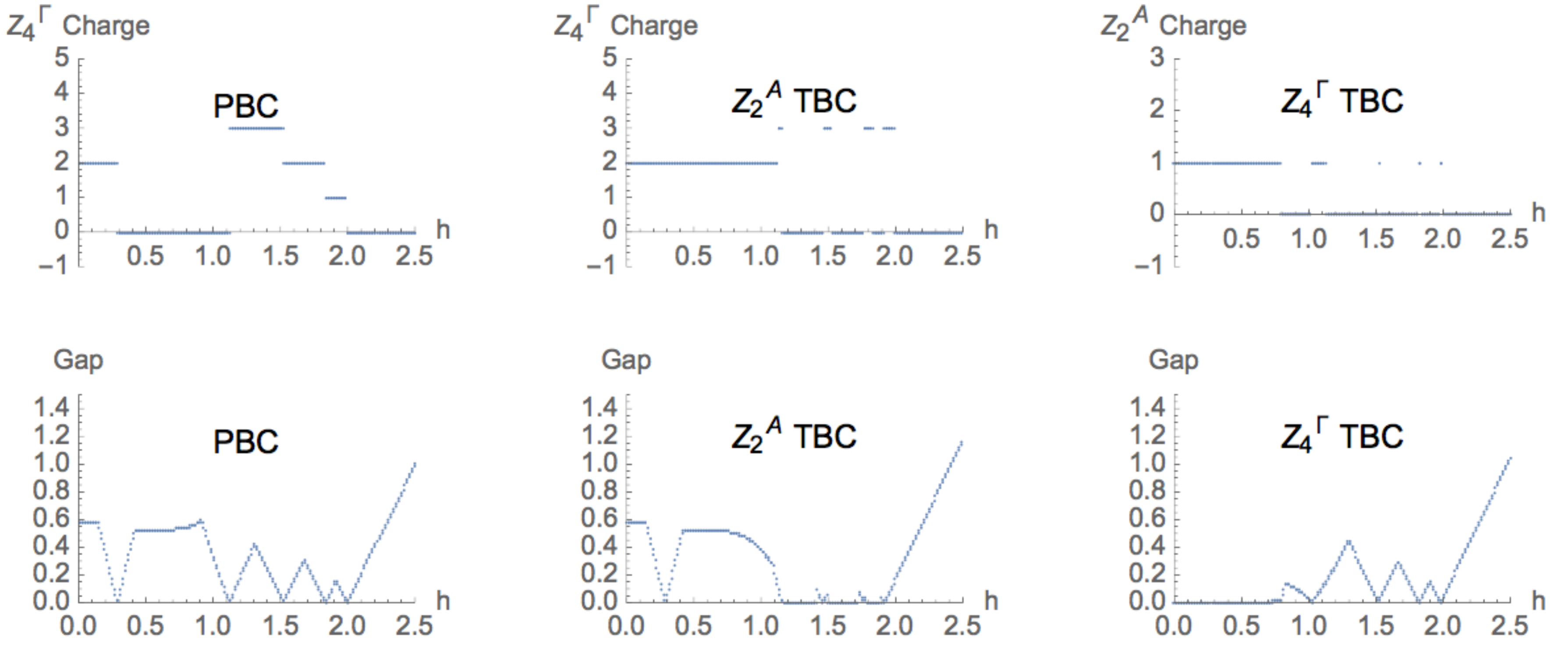
Stability of Strong SPTC

Is the Z_4^{Γ} strong SPTC stable upon perturbing to a gapped phase with a single ground state?

- There is no nontrivial Z4 gapped SPT.
- Since Γ is not anomalous, it should be possible to deform the theory to a trivially gapped phase, if allow **strong enough** perturbation.
- Possible for **arbitrarily small** perturbation?

Stability of Strong SPTC

$$V = -h\sum_{i=1}^{L} \sigma_i^x + \tau_{i+\frac{1}{2}}^x$$



Stability of Strong SPTC

- For finite size, need to pass a critical strength to enter trivially gapped phase. $\Rightarrow$ stable under perturbation.

- Weak SPTC: Signatures coincide with gapped SPT.
- Strong SPTC: Signatures distinct from gapped SPT.

Strong SPTC is more stable than weak SPTC.

Summary

- There is a natural notion of SPT for quantum criticality – symmetry protected topological criticality.
- Decorated defect construction is a powerful tool to systematically construct SPTC.
- Symmetry charge of the ground state under TBC is a physical observable to probe the nontrivial SPTCs. But ground state degeneracy under OBC isn't.
- Strong SPTC is more stable than weak SPTC against perturbing to gapped phase with one ground state.
- Future directions: Classification? Continuous symmetry? Anomaly inflow picture? Entanglement feature?

Thanks for listening!