Non-Hermitian topological band structures protected by inversion symmetry

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Outline

1. Introduction

    Non-Hermitian skin effect

2. Non-Hermitian band topology with generalized inversion symmetry

3. Higher-order skin effect protected by inversion symmetry

Non-Hermitian (NH) physics

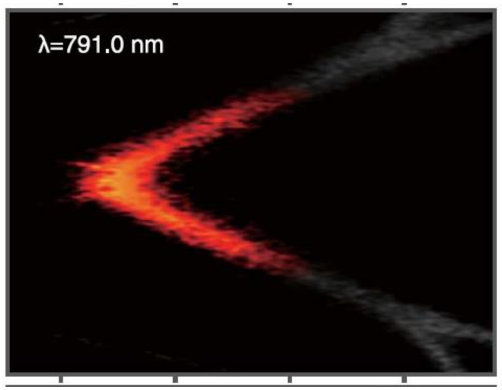
NH physics can appear in

- Open quantum systems
- Photonic systems
- Electronic circuits
- Active matter
- Robotic metamaterials
- Bosonic Bogoliubov-de Gennes systems
- Strongly correlated systems

⋮

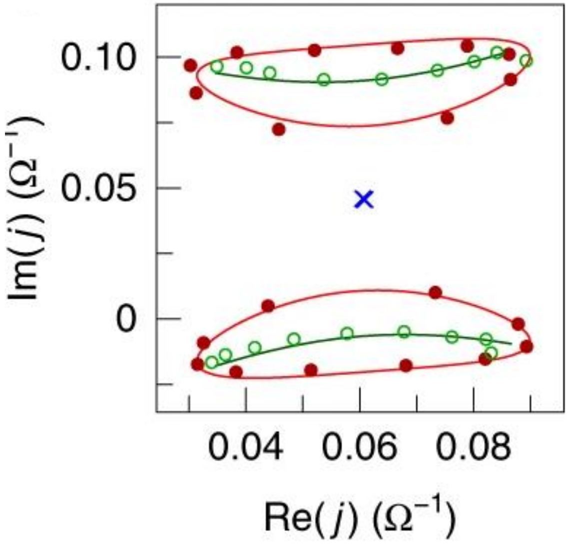
They can be described as effective non-Hermitian Hamiltonians with the band structures.

Photonic system



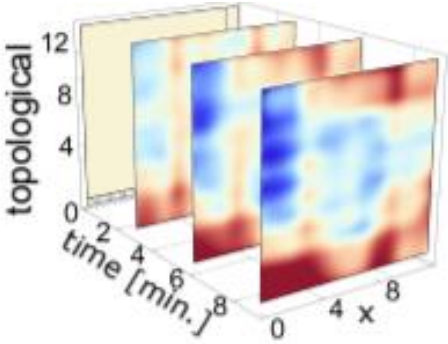
H. Zhou, et al.,
Science, **359**, 1009 (2018)

Electronic circuit



T. Helbig et al.,
Nat. Phys. **16**, 757 (2020)

Active matter



L. S. Palacious et al,
Nat. Commun. 12 4691 (2021)

Non-Hermitian band topology

Non-Hermiticity can realize novel topological band structures that is not allowed to happen in Hermitian physics.

- Non-Hermitian skin effect
- Exceptional point
- Exceptional topological insulator

⋮

NHSEs have been experimentally observed in ultracold atoms, electric circuits, active matter etc.

Q. Liang et al., PRL 129 070401 (2022)

T. Helbig et al., Nat. Phys. **16**, 757 (2020)

L. S. Palacious et al, Nat. Commun. 12 4691 (2021)

Non-Hermitian skin effect (NHSE)

In non-Hermitian systems, energy spectra strongly depend on boundary conditions (even in the thermodynamic limit).

The strong dependence is called a non-Hermitian skin effect.

Ex. Hatano-Nelson model $H = \sum_i (t_L c_{i-1}^\dagger c_i + t_R c_{i+1}^\dagger c_i)$

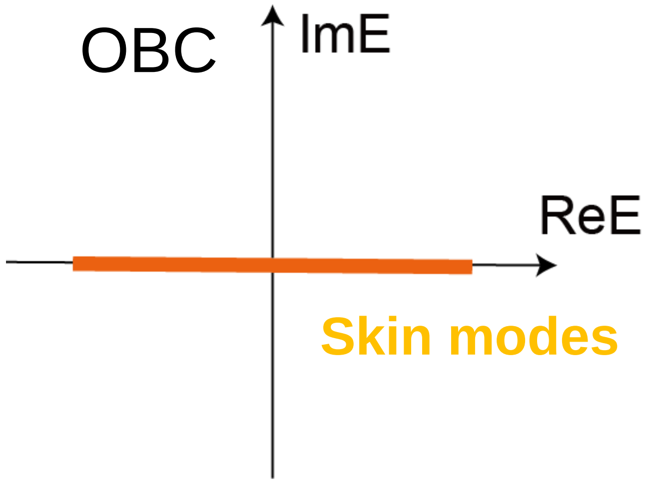
Periodic boundary condition (PBC)

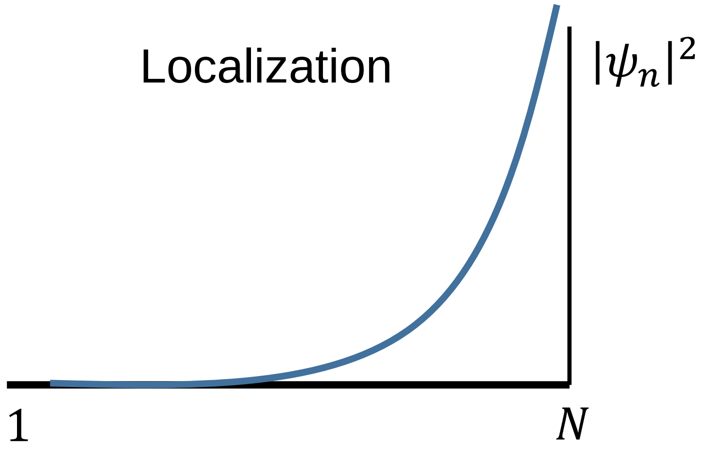
Open boundary condition (OBC)

Non-Hermitian skin effect (NHSE)

When the NHSE happens, many eigenmodes (skin modes) are localized at the edge in the finite system.

$$H = \sum_i (t_L c_{i-1}^\dagger c_i + t_R c_{i+1}^\dagger c_i)$$





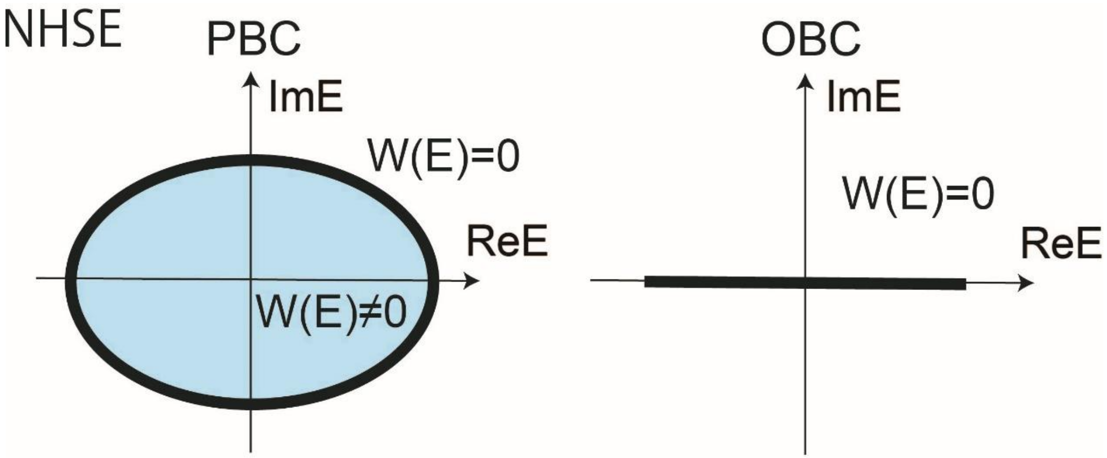
The NHSE is important to bulk-boundary correspondence in non-Hermitian systems.

Topology of the NHSE

The topological origin of the NHSE is a winding number of the band structure under the PBC due to non-Hermiticity.

N. Okuma et al., PRL **124**, 086801 (2020)
K. Zhang et al., PRL **125**, 126402 (2020)

One-band case



The winding number is described as $W(E) = \frac{1}{2\pi i}\int_{-\pi}^{\pi} dk \frac{d}{dk} \log\det[H(k) - E]$.

The NHSE can happen without any symmetries.

Motivation

・How do crystal symmetries affect skin effects?

There are some difficulties in numerical calculations for non-Hermitian Hamiltonians.

・It is difficult to identify skin effects in high-dimensional systems under the full open BC.

・Skin modes should be calculated in a open system, which gives rise to large numerical errors.

➡ We provide a topological approach using inversion symmetry.

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Why winding number?

Topological characterization of NH Hamiltonians

Z. Gong, et al., PRX (2018)

Point-gap topology at a reference energy E

Non-Hermitian Hamiltonian H

Point-gap

$$\det(H-E) \neq 0$$

Topological correspondence

Extended Hermitian Hamiltonian

$$\tilde{H} = \begin{pmatrix} 0 & H-E \\ H^\dagger - E^* & 0 \end{pmatrix}$$

Band topology of H at E corresponds to that of the Hermitian Hamiltonian $\tilde{H}$ with chiral symmetry Γ.

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

➡ Both H and $\tilde{H}$ can be characterized by the same winding number in 1D.

Topological characterization

Non-Hermitian Hamiltonians have the same topology as chiral-symmetric Hermitian Hamiltonians.

- Hermitian Hamiltonians in 1D class AIII

$$w = \frac{1}{2\pi i}\int_{-\pi}^{\pi} dk \frac{d}{dk}\log\det[q(k)] \qquad \widetilde{H} = \begin{pmatrix} 0 & q \\ q^{\dagger} & 0 \end{pmatrix}$$

- Non-Hermitian Hamiltonians without any symmetries

$$W(E) = \frac{1}{2\pi i}\int_{-\pi}^{\pi} dk \frac{d}{dk}\log\det[H(k) - E] \qquad \widetilde{H} = \begin{pmatrix} 0 & H - E \\ H^{\dagger} - E^{*} & 0 \end{pmatrix}$$

How does inversion symmetry affect the topology?

Inversion symmetry and topological invariant

One can easily know whether the system is topological by using crystal symmetry in Hermitian systems.

Ex. Fu-Kane-Mele formula for inversion-symmetric Z2 topological insulators

$$(-1)^{\nu} = \prod_{\boldsymbol{\Gamma}} (-1)^{\frac{n_-(\boldsymbol{\Gamma})}{2}}$$

Γ: time-reversal-invariant momenta (TRIM)

$$\Gamma \equiv -\Gamma \pmod{\mathbf{G}}$$

$n_-(\Gamma)$: the number of occupied states with odd-parity eigenvalues at TRIM

Can we simplify a winding number for non-Hermitian systems using crystal symmetry?

Inversion symmetry and winding number

T. Hughes et al., PRB **83**, 245132 (2011)

Inversion symmetry gives a simple formula of the topological invariant in 1D class AIII.

	Expression	Parity formula
1D class AIII $\tilde{H} = \begin{pmatrix} 0 & q \\ q^\dagger & 0 \end{pmatrix}$	$w = \frac{1}{2\pi i}\int_{-\pi}^{\pi} dk \frac{d}{dk} \log\det[q(k)]$	$(-1)^w = \prod_{\Gamma=0,\pi} (-1)^{n_-(\Gamma)}$

$n_-(\Gamma)$: the number of occupied states with negative parity eigenvalues

Γ: inversion-invariant momenta $\Gamma \equiv -\Gamma \pmod{\mathbf{G}}$

We generalize this parity formula to calculate $W(E)$ for non-Hermitian systems.

Conventional inversion symmetry

When does inversion symmetry appear in an extended Hermitian Hamiltonian?

Conventional inversion symmetry

$$PH(\boldsymbol{k})P^{-1} = H(-\boldsymbol{k}) \qquad PP^{\dagger} = P^2 = 1$$

However, because the winding number is always trivial, the skin effect does not happen…

$$W(E) = \frac{1}{2\pi i}\int_{-\pi}^{\pi} dk \frac{d}{dk} \log\det[H(k) - E] = \frac{1}{2\pi i}\int_{-\pi}^{\pi} dk \frac{d}{dk} \log\det[H(-k) - E] = 0$$

(This inversion symmetry is incompatible with localized modes at one edge in 1D.)

Generalized inversion symmetry

Extended Hermitian Hamiltonians can acquire some symmetries which the original non-Hermitian Hamiltonians do not have.

Generalized inversion symmetry

$$U_I H^\dagger(\boldsymbol{k}) U_I{}^{-1} = H(-\boldsymbol{k}) \qquad U_I U_I^\dagger = U_I^2 = 1$$

In the presence of generalized inversion symmetry,
the extended Hermitian Hamiltonian with real E can obtain inversion symmetry.

$$\widetilde{H} = \begin{pmatrix} 0 & H - E \\ H^\dagger - E & 0 \end{pmatrix} \quad \tilde{I} = \begin{pmatrix} 0 & U_I \\ U_I & 0 \end{pmatrix} \qquad \Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Parity formula for NH systems

Reformulate the parity formula $(-1)^w = (-1)^{n_-(0)-n_-(\pi)}$ for non-Hermitian Hamiltonians H

$$\tilde{H}(\Gamma) = \begin{pmatrix} 0 & H(\Gamma)-E \\ H^\dagger(\Gamma)-E & 0 \end{pmatrix} = \begin{pmatrix} 0 & H(\Gamma)-E \\ U_I H(\Gamma) U_I^{-1} - E & 0 \end{pmatrix} \qquad (\Gamma = 0, \pi)$$

$n_-(\Gamma)$: the number of occupied states with negative parity eigenvalues

⬇

$n_-(\Gamma) = N_+(\Gamma)$: # of positive eigenvalues of $U_I(H(\Gamma)-E)$

This formula can connect non-Hermitian band topology with Hermitian band topology.

$$(-1)^{W(E)} = (-1)^{N_+(0)-N_+(\pi)} = \prod_{\Gamma=0,\pi} \operatorname{sgn} \det[H(\Gamma)-E]$$

Model calculation

Ex. Generalized Hatano-Nelson model

$$H = \sum_i \sum_{m=1}^{M} [t_R^{(m)} c_{i+m}^\dagger c_i + t_L^{(m)} c_i^\dagger c_{i+m}]$$

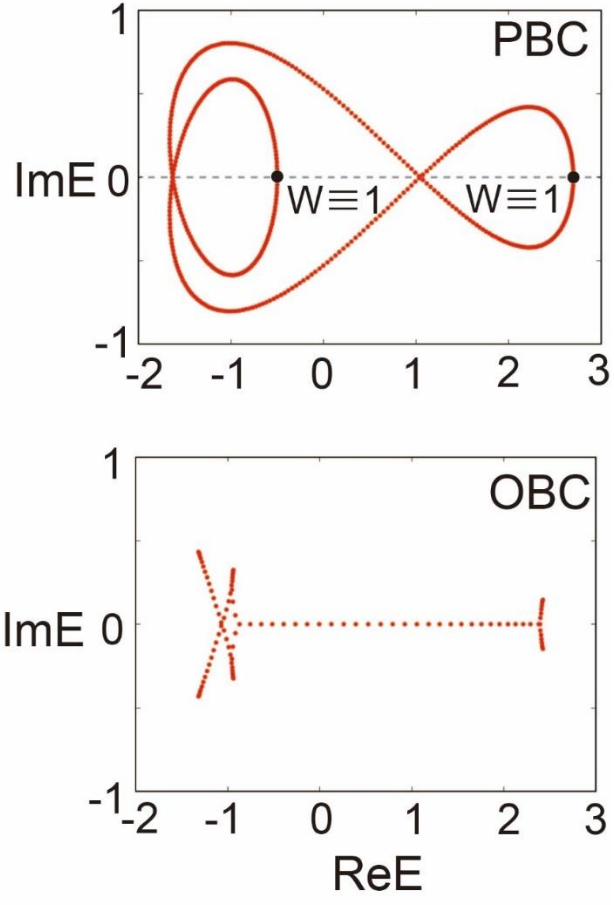
The Hamiltonian under the PBC

$$h(k) = \sum_{m=1}^{M} [t_R^{(m)} e^{-ikm} + t_L^{(m)} e^{ikm}] \qquad h^\dagger(-k) = h(k), U_I = 1$$

The parity formula

$$(-1)^{W(E)} = \mathrm{sgn}\left(\sum_{m=1}^{M} [t_L^{(m)} + t_R^{(m)}] - E\right) \mathrm{sgn}\left(\sum_{m=1}^{M} (-1)^m [t_L^{(m)} + t_R^{(m)}] - E\right)$$

$M = 3$



Summary (1)

Phys. Rev. B **103**, 205205 (2021)

If generalized inversion symmetry is present, the parity of the winding number for the NHSE can be evaluated from the eigenvalues at the inversion-symmetric points.

generalized inversion symmetry

$$U_I H^\dagger(\boldsymbol{k}) U_I^{-1} = H(-\boldsymbol{k}) \qquad U_I U_I^\dagger = U_I^2 = 1$$

$$(-1)^{W(E)} = (-1)^{N_+(0)-N_+(\pi)} = \prod_{\Gamma=0,\pi} \mathrm{sgn}\,\det[H(\Gamma) - E]$$

$N_+(\Gamma)$: # of positive eigenvalues of $U_I(H(\Gamma) - E)$

(This idea can be generalized to a winding number for exceptional points in the momentum space.)

Phys. Rev. B **105**, 085109 (2022)

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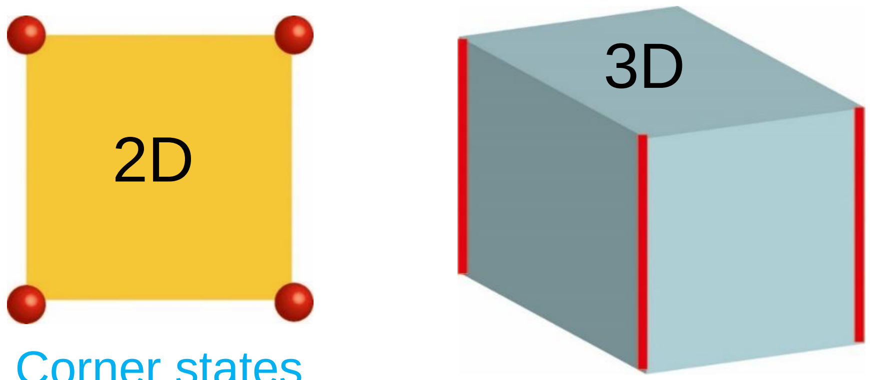
Phys. Rev. B **102**, 241202(R) (2020)
Phys. Rev. B **103**, 205205 (2021)

K. Kawabata, M. Sato, and K. Shiozaki
PRB 102, 205118 (2020)

Higher-order topological phases (HOTPs)

Higher-order topological phases have robust corner and hinge modes.

Higher-order topology



Corner states

Hinge states

2nd-order

Can higher-order topology coexist with non-Hermitian band topology?

Higher-order topology and NH topology

A NH Hamiltonian can inherit band topology of a chiral-symmetric Hermitian Hamiltonian.

Chiral-symmetric Hermitian Hamiltonian
with second-order topology

$$\tilde{H}_{HOTP} = \begin{pmatrix} 0 & H \\ H^\dagger & 0 \end{pmatrix}$$

Non-Hermitian Hamiltonian

$$H \; (\neq H^\dagger)$$

Can we realize a NHSE due to second-order topology?

HOTP protected by inversion symmetry

Hermitian higher-order topology can be realized in 2D class AIII with inversion symmetry, where corner zero modes appear.

E. Khalaf, PRB **97** 205136 (2017)

A. Matsugatani and H. Watanabe, PRB **98** 205129 (2018)

R. Takahashi et al., PRR **2**, 013300 (2020)

Topological invariants in the systems

$$\mu_x = n_-(0,0) - n_-(\pi,\pi) + n_-(\pi,0) - n_-(0,\pi)$$
$$\mu_y = n_-(0,0) - n_-(\pi,\pi) - n_-(\pi,0) + n_-(0,\pi)$$

$n_-(\Gamma)$: the number of occupied states with negative parity eigenvalues

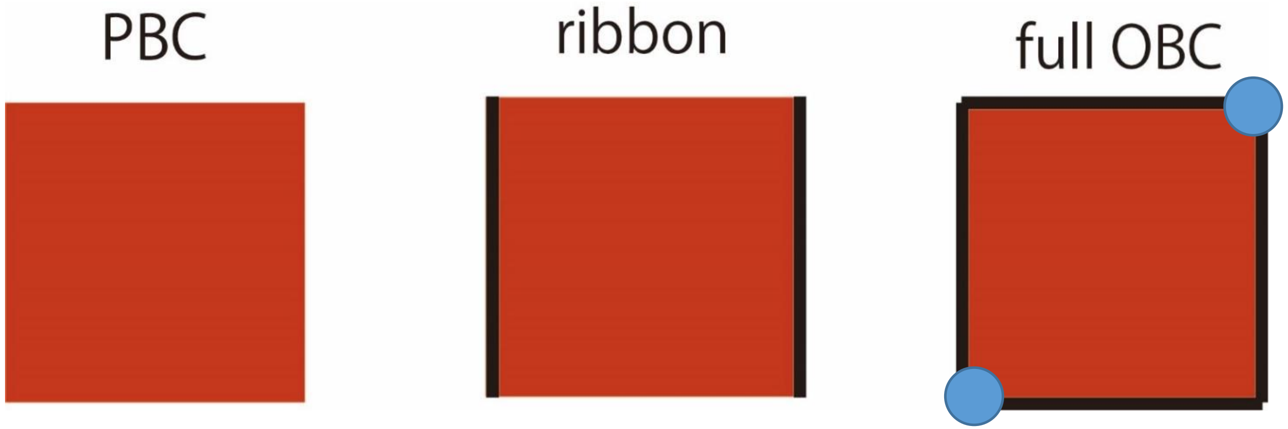
When $\mu_{x(y)} = 2 \pmod 4$, the system is second-order topological.

The HOTP and winding numbers

R. Takahashi et al., PRR **2**, 013300 (2020)

The topological invariants $\mu_{x,y}$ are related to winding numbers for a ribbon geometry as long as the ribbon retains inversion symmetry.

$$w_{x(y)-OBC} \equiv \frac{\mu_{x(y)}}{2} \pmod 2$$



$\mu_x \equiv 2\ (mod\ 4)$ $\quad w_{x-OBC} \equiv 1\ (mod\ 2)$ $\quad$ Corner modes

We can apply the topological invariant to a NHSE due to second-order topology!

Second-order NHSE

If a 2D NH Hamiltonian gives an extended-Hermitian Hamiltonian with the nontrivial $\mu_{x(y)} = 2 \pmod 4$, the NH system shows a skin effect under the full OBC.

$$U_I H^\dagger(\boldsymbol{k}) U_I^{-1} = H(-\boldsymbol{k}) \quad \Longrightarrow \quad \widetilde{H}_{HOTP} = \begin{pmatrix} 0 & H \\ H^\dagger & 0 \end{pmatrix} \quad \tilde{I} = \begin{pmatrix} 0 & U_I \\ U_I & 0 \end{pmatrix}$$

$$w_{x(y)-OBC} \equiv \frac{\mu_{x(y)}}{2} \pmod 2$$

Because H reflects second-order topology with the nonzero winding number,
the NH Hamiltonian can be characterized by the same winding number with real E.

$$W_{x(y)-OBC}(E) \equiv \frac{\mu_{x,y}(E)}{2} \pmod 2$$

Second-order NHSE

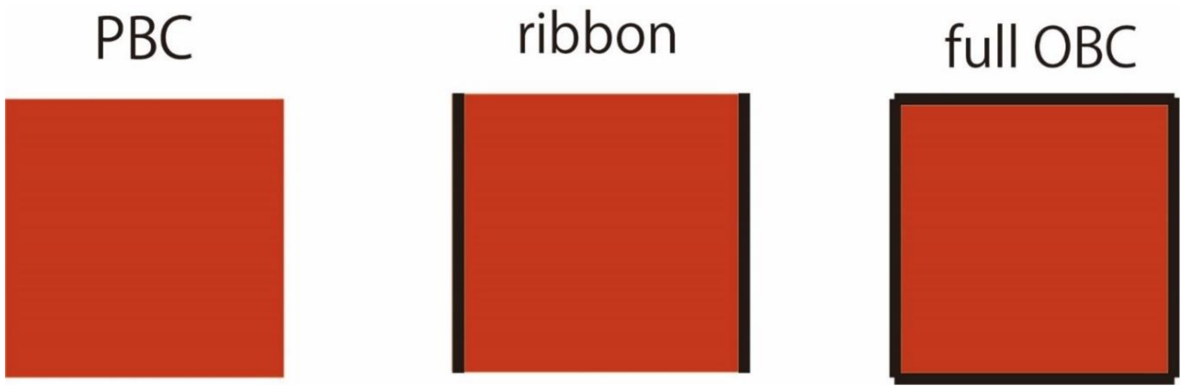
We can calculate topological invariants $\mu_{x(y)}$ from eigenvalues of the non-Hermitian Hamiltonian.

$$n_-(\Gamma) = N_+(\Gamma) \text{ : # of positive eigenvalues of } U_I(H(\Gamma) - E)$$

Winding number for ribbon geometry

$$W_{x(y)-OBC}(E) \equiv \frac{\mu_{x(y)}(E)}{2} \pmod 2$$

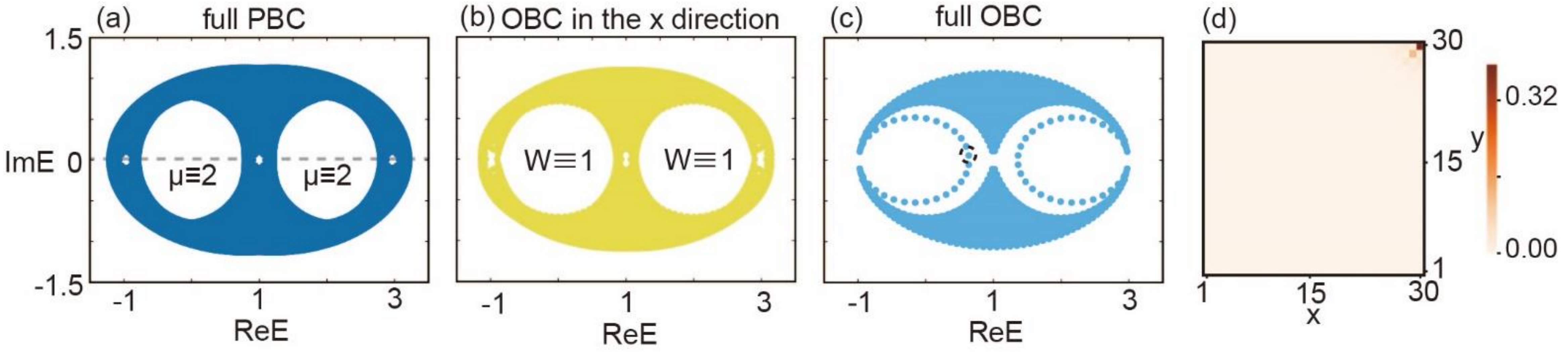
$$\mu_x = N_+(0,0) - N_+(\pi,\pi) + N_+(\pi,0) - N_+(0,\pi)$$
$$\mu_y = N_+(0,0) - N_+(\pi,\pi) - N_+(\pi,0) + N_+(0,\pi)$$



$\mu_x \equiv 2\ (mod\ 4)$ $\quad W_{x-OBC} \equiv 1\ (mod\ 2)$ $\quad$ Skin effect!

Model calculation

$$H_{2D}(\boldsymbol{k}) = (m - c\cos k_x - c\cos k_y)s_0 + it\sin k_y\, s_x + it\sin k_x\, s_y - B_x s_x\ - B_y s_y$$



$m = c = 1.0, t = 0.8, B_x = B_y = 0.15$

Summary (2)

A NHSE can be realized from second-order topology protected by inversion symmetry.

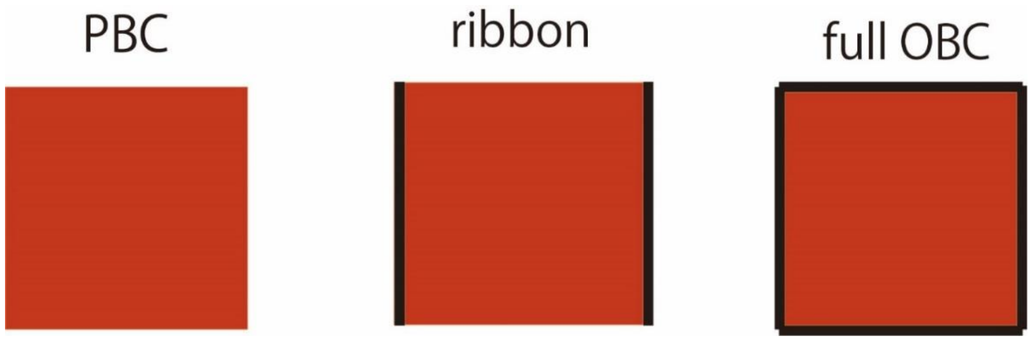
Topological invariants for the second-order NHSE

$$\mu_x = N_+(0{,}0) - N_+(\pi,\pi) + N_+(\pi,0) - N_+(0,\pi)$$

$$\mu_y = N_+(0{,}0) - N_+(\pi,\pi) - N_+(\pi,0) + N_+(0,\pi)$$

$N_+(\Gamma)$: # of positive eigenvalues of $U_I(H(\Gamma) - E)$

When the topological invariants $\mu_{x(y)} = 2 \pmod 4$,
the NHSE can happen under the full OBC.



$\mu_x \equiv 2$ $\quad W_{x-OBC} \equiv 1$ $\quad$ Skin effect!

$$W_{x(y)-OBC}(E) \equiv \frac{\mu_{x,y}(E)}{2} \pmod 2$$

Conclusion

PRB **102**, 241202(R) (2020) PRB **103**, 205205 (2021)

We derived simple formulas to evaluate winding numbers for NHSEs
if generalized inversion symmetry is present.

$$U_I H^\dagger(\boldsymbol{k}) {U_I}^{-1} = H(-\boldsymbol{k}) \qquad U_I U_I^\dagger = U_I^2 = 1$$

・1D skin effect

$$(-1)^{W(E)} = (-1)^{N_+(0)-N_+(\pi)} = \prod_{\Gamma=0,\pi} \mathrm{sgn}\,\det[H(\Gamma) - E]$$

$N_+(\Gamma)$: # of positive eigenvalues of $U_I(H(\Gamma) - E)$

・2D second-order skin effect

$$W_{x(y)-OBC}(E) \equiv \frac{\mu_{x(y)}(E)}{2} \pmod 2$$

$$\mu_x = N_+(0,0) - N_+(\pi,\pi) + N_+(\pi,0) - N_+(0,\pi)$$

$$\mu_y = N_+(0,0) - N_+(\pi,\pi) - N_+(\pi,0) + N_+(0,\pi)$$