Breakdown of the Meissner effect at the exceptional point

in the non-Hermitian two-band BCS model

Novel Quantum States in Condensed Matter 2022

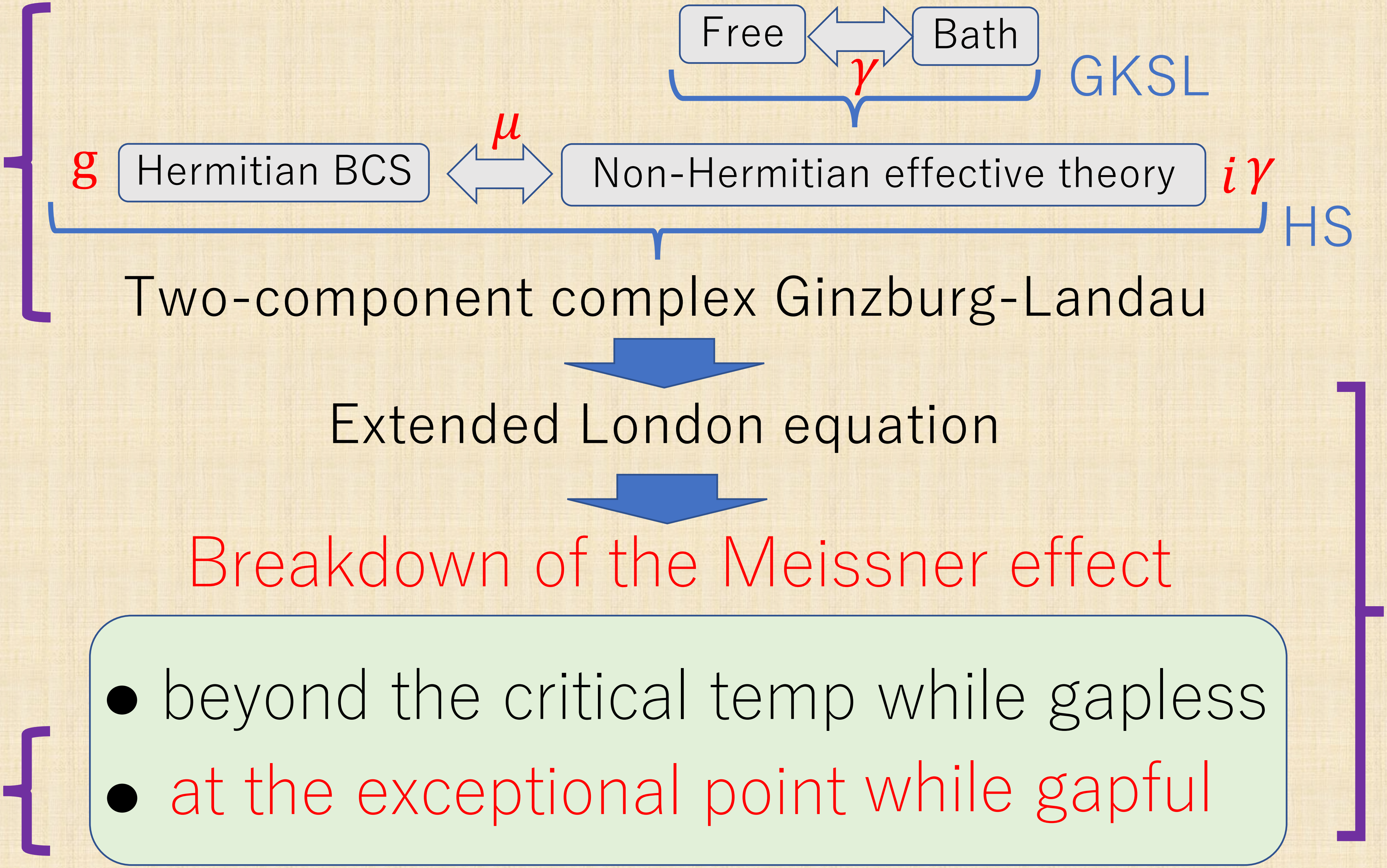
東大生研[A]　平良（たいら）敬信[A]

U-Tokyo IIS[A]　Takanobu Taira[A]

Based on: 2211.11422

Room: Y208

taira904@iis.u-tokyo.ac.jp


Free
Bath
γ
GKSL
μ
g
Hermitian BCS
Non-Hermitian effective theory
$i\gamma$
HS
Two-component complex Ginzburg-Landau
Extended London equation
Breakdown of the Meissner effect
● beyond the critical temp while gapless
● at the exceptional point while gapful

Introduction / Result

[Kato, Perturbation theory for linear operators(1980)]

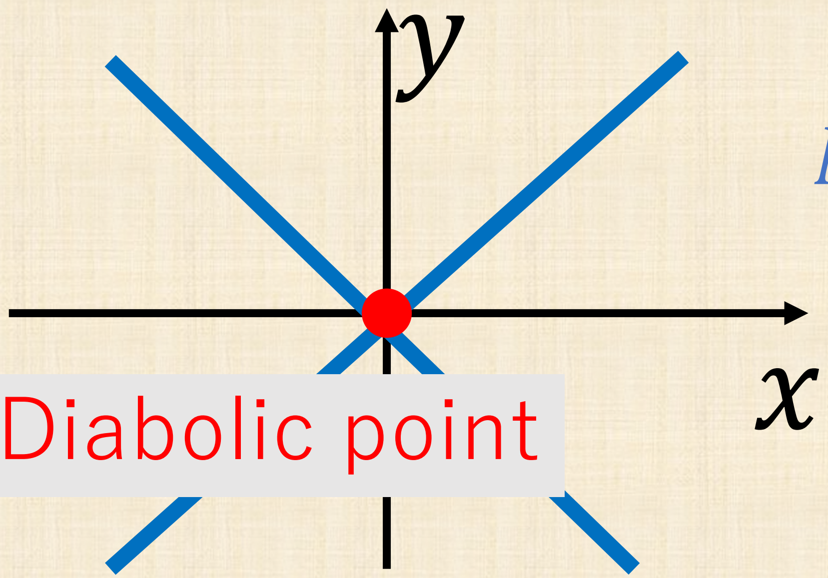
[Heiss, Sannino, J. Phys A, (1990).]

$$M = \begin{pmatrix} x & iy \\ iy & -x \end{pmatrix} \quad \lambda_\pm = \pm\sqrt{x^2 - y^2}\; e^{i\theta/2}$$

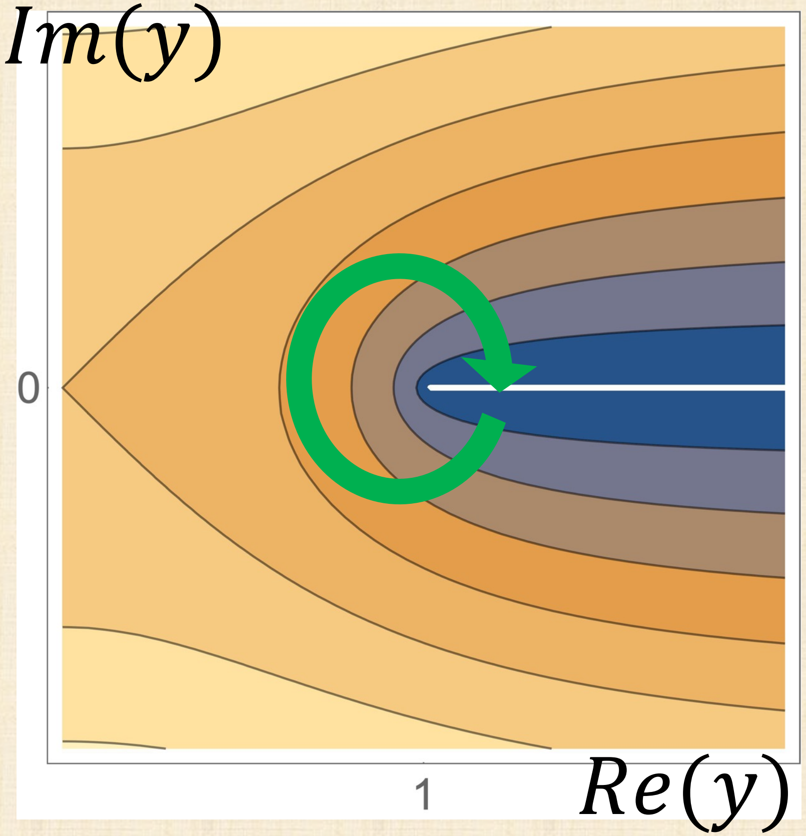
Exceptional point

$$M_{DP} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$M_{EP} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$



$x = 1$



Introd... Interesting Physics at EP

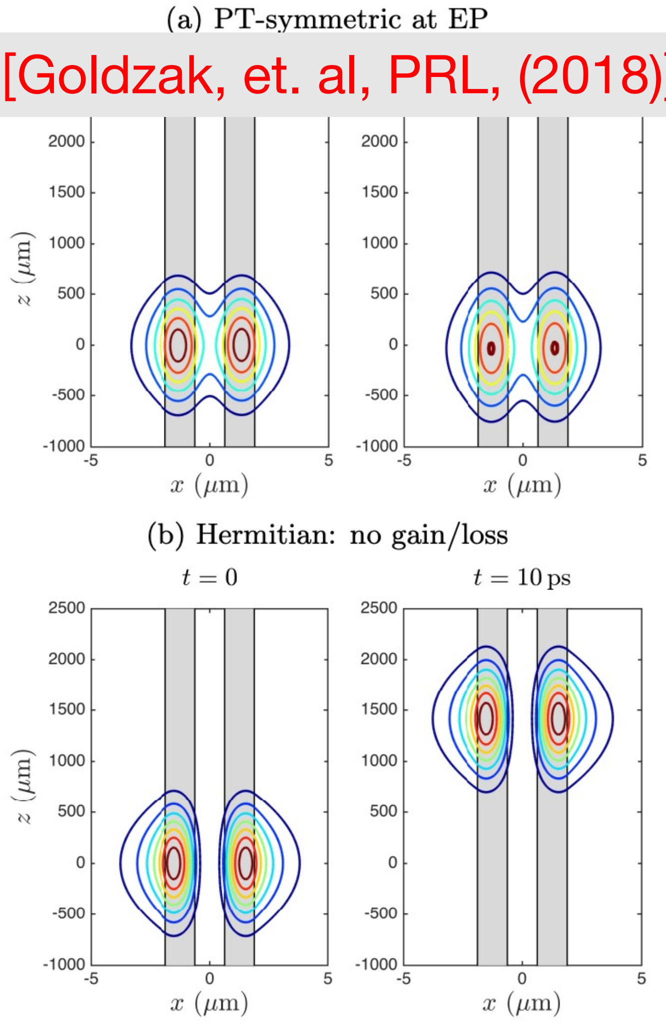


FIG. 4. Contour plot of the light power $|\phi(x,z,t)|^2$ for the Gaussian wave packets at the initial time $t = 0$ and the final time $t = 10\,ps$. In both plots the pulse has the mean propagating constant $\beta = 0.851\mu m^{-1}$ with standard devia-

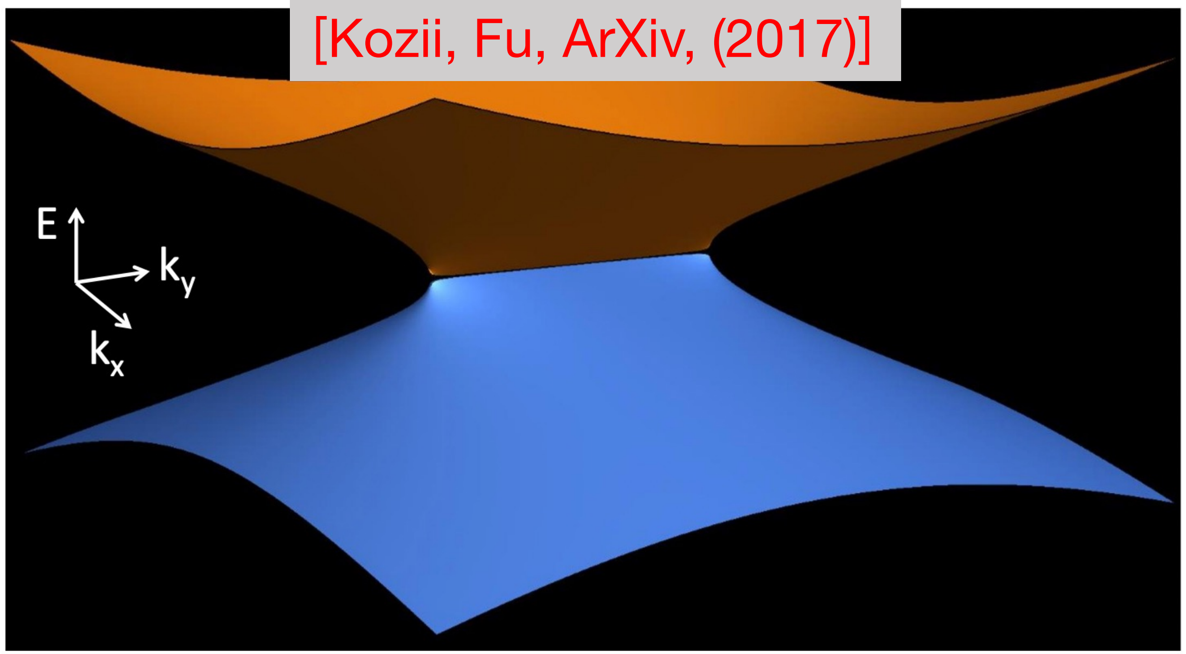


FIG. 1. Quasiparticle energy dispersion of a two-dimensional Dirac semimetal reshaped by the two-lifetime self-energy, see Eq.(10). Instead of touching at the Dirac point, quasiparticle conduction and valence bands stick on a Fermi arc that ends at two topological exceptional points.

Intro... Interesting Physics at EP

[Hanai, et al., PRL (2019)]

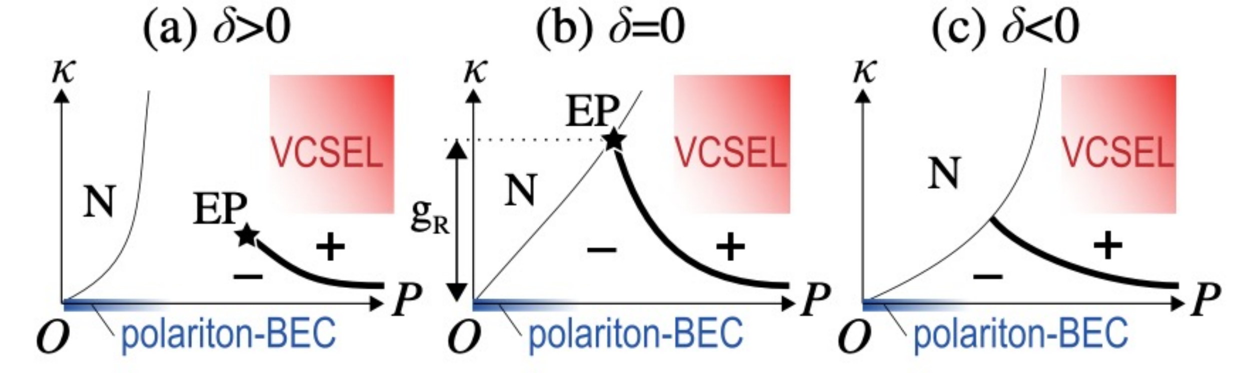


FIG. 1. Proposed phase diagram of a driven-dissipative electron-hole-photon gas, in terms of the photon decay rate κ and the pump power P. (a) Blue detuning. (b) On resonance. (c) Red detuning. "−(+)" represents the "−(+)"-solution phase, "N" represents the normal phase, "EP" is the exceptional point, and g_R is the Rabi splitting. The thick (thin) solid line represents the phase boundary in the condensed phase (between the normal and the condensed phase).

[Dembowski, et al., PRL, (2003)]

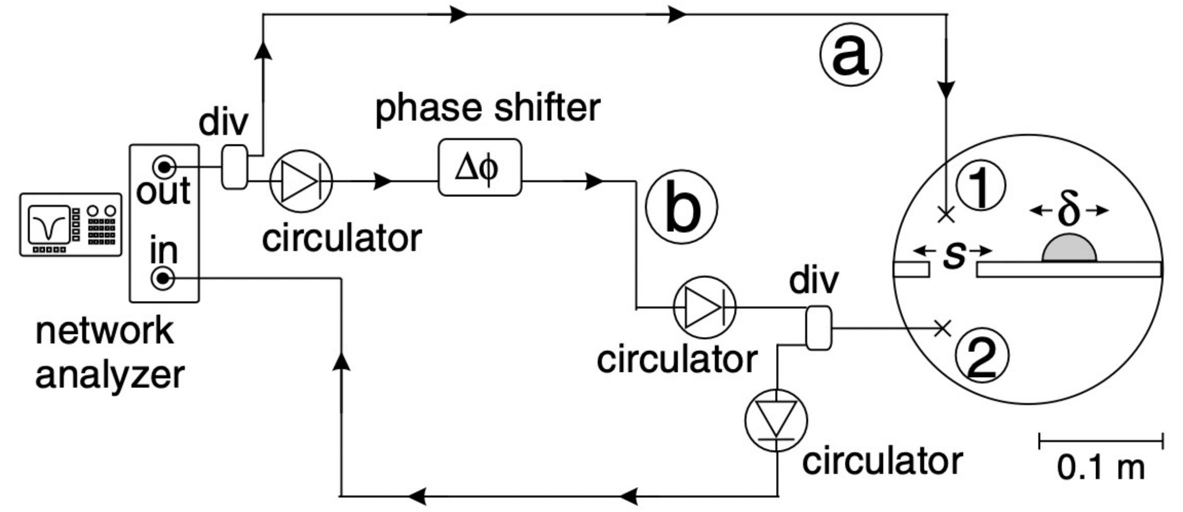


FIG. 1. Experimental setup to measure the phase shift between two different positions in the cavity. The geometry of the resonator can be changed by adjusting the widths s of a slit between the two halves [labeled (1) and (2), respectively] and the position δ of a Teflon stub. Microwave power is coupled into the resonator via path (a) and antenna 1 into part (1) and, with a tunable phase shift $\Delta\phi$, via antenna 2 into part (2),

Experiment in microwave cavity

Introdu…

Interesting Physics at EP

[Fring, Taira, EPJP, (2022)] [Fring, Taira, PLB, (2022)]

Particle physics

Complex two component Ginzburg-Landau type model

Task 1

Microscopic theory in open system

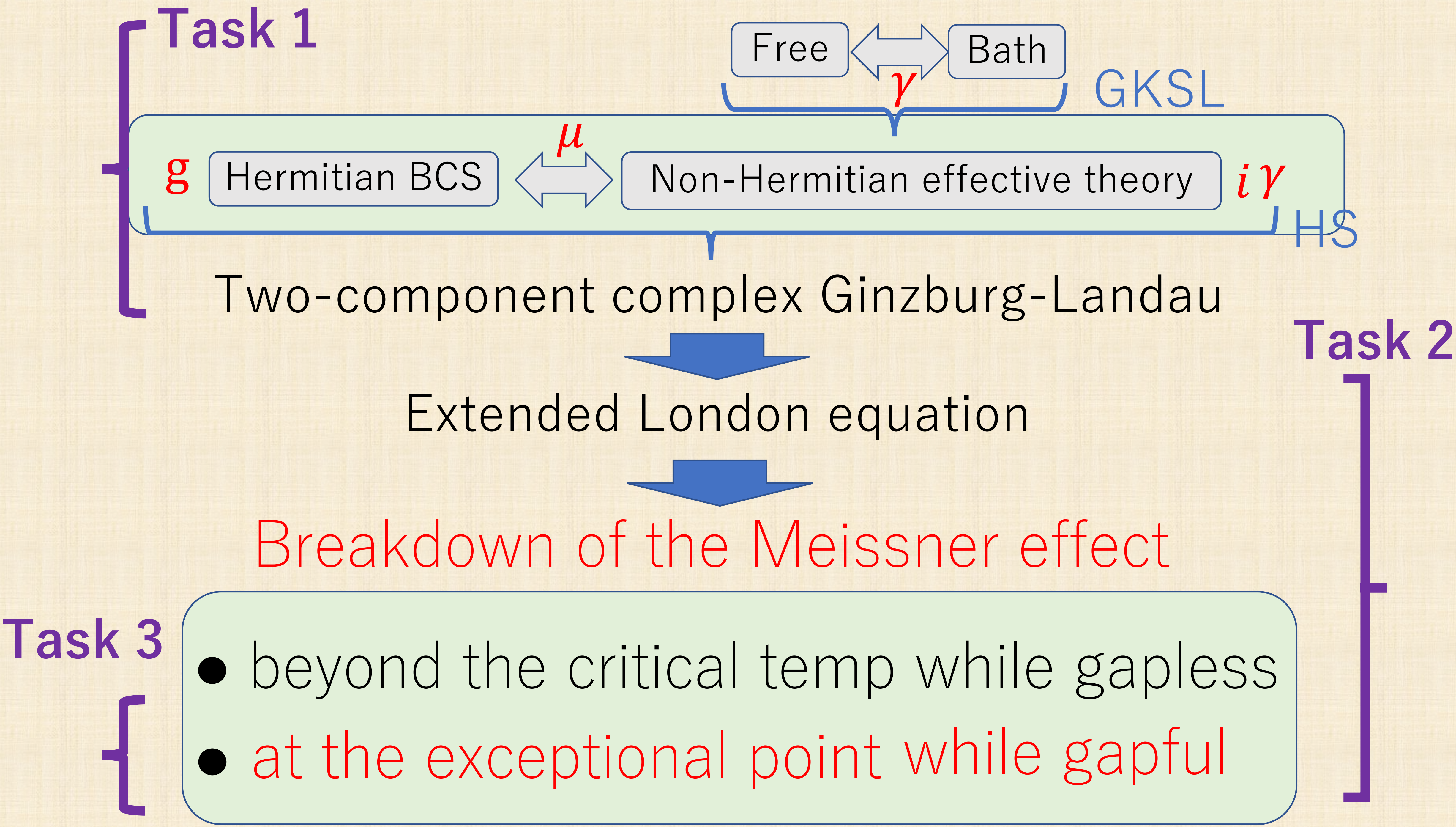
Task 2

Breakdown of Higgs mechanism while finite vacuum

Breakdown of Meissner effect while gapfull

Task 3

at exceptional point



Task 1
Free
Bath
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Hermitian BCS
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iγ
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Task 3
• beyond the critical temp while gapless
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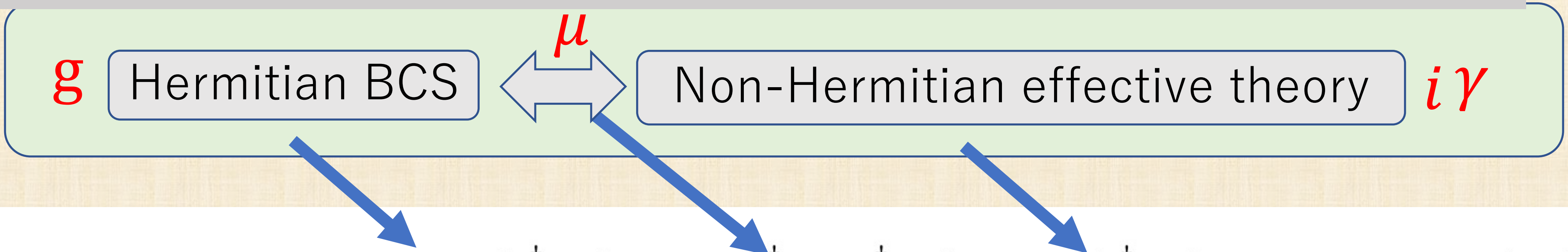
Task 1



$$H_{\mathrm{Tot}} = H_{\mathrm{BCS}}[c_1^\dagger, c_1] + H_{\mathrm{Int}}[c_1^\dagger, c_2, c_1^\dagger, c_2] + H_{\mathrm{NH}}[c_2^\dagger, c_2] \tag{1.1}$$

$$H_{\mathrm{BCS}} = \int d^3r \sum_{\sigma=\uparrow,\downarrow} c_{1\sigma}^\dagger(\vec{r}) \left[-\frac{1}{2m_1}\left(\nabla - ie\vec{A}\right)^2 - \mu_1\right] c_{1\sigma}(\vec{r})$$

$$-g c_{1\uparrow}^\dagger(\vec{r}) c_{1\downarrow}^\dagger(\vec{r}) c_{1\downarrow}(\vec{r}) c_{1\uparrow}(\vec{r}), \tag{1.2}$$

$$H_{\mathrm{NH}} = \int d^3r \sum_{\sigma=\uparrow,\downarrow} c_{2\sigma}^\dagger(\vec{r}) \left[-\frac{1}{2m_2}\left(\nabla - ie\vec{A}\right)^2 - \mu_2\right] c_{2\sigma}(\vec{r})$$

$$-i\gamma c_{2\uparrow}^\dagger(\vec{r}) c_{2\downarrow}^\dagger(\vec{r}) c_{2\downarrow}(\vec{r}) c_{2\uparrow}(\vec{r}), \tag{1.3}$$

$$H_{\mathrm{int}} = -\mu \int d^3r \; c_{1\uparrow}^\dagger c_{1\downarrow}^\dagger c_{2\uparrow} c_{2\downarrow} + c_{2\uparrow}^\dagger c_{2\downarrow}^\dagger c_{1\uparrow} c_{1\downarrow}. \tag{1.4}$$

Task 1

$$-i\gamma c^{\dagger}_{2\uparrow}(\vec{r})c^{\dagger}_{2\downarrow}(\vec{r})c_{2\downarrow}(\vec{r})c_{2\uparrow}(\vec{r}),$$

Effective theory of ultra-cold atom

[Yamamoto, et al., PRL123, (2019)]

The two-body loss is described by $L_i = c_{i\downarrow}c_{i\uparrow}$, giving a NH BCS Hamiltonian

$$H_{\text{eff}} = \sum_{\boldsymbol{k}\sigma} \xi_{\boldsymbol{k}} c^{\dagger}_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} - U \sum_{i} c^{\dagger}_{i\uparrow} c^{\dagger}_{i\downarrow} c_{i\downarrow} c_{i\uparrow}, \qquad (2)$$

with a *complex-valued* interaction $U = U_1 + i\gamma/2$, where $U_1, \gamma > 0$. Here, $\xi_{\boldsymbol{k}} \equiv \epsilon_{\boldsymbol{k}} - \mu$, $\epsilon_{\boldsymbol{k}}$ is the energy dispersion,

[Liu, et al., ArXiv:2209.10427, (2022)]

with the non-Hermitian effective Hamiltonian

$$\hat{H}_{\text{eff}} = H - \frac{i\gamma_b}{2\Omega} \sum_{\mathbf{k},\mathbf{k}',\mathbf{p}} \hat{a}^{\dagger}_{\mathbf{k}+\mathbf{p}} \hat{a}^{\dagger}_{\mathbf{k}'-\mathbf{p}} \hat{a}_{\mathbf{k}'} \hat{a}_{\mathbf{k}}, \qquad (7)$$

[Dürr, et al., PARA 79 (2009)]

with an effective Hamiltonian that is not Hermitian and turn out to be the analytic continuation of H_0

$$H_{\text{eff}} = H_0 - i\frac{\text{Im}(g_{3\text{D}})}{2}\int d^3x\, \Psi^{\dagger 2}(\mathbf{x})\Psi^2(\mathbf{x}) \qquad (1$$

[Takasu, et al,. PTEP, (2020)]

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}(\hat{H}_{\text{eff}}\hat{\rho} - \hat{\rho}\hat{H}^{\dagger}_{\text{eff}}) + \int dx \frac{|\Omega(\mathbf{x})|^2}{\Gamma} \hat{\Psi}(\mathbf{x})\hat{\Psi}(\mathbf{x})\hat{\rho}\hat{\Psi}^{\dagger}(\mathbf{x})\hat{\Psi}^{\dagger}(\mathbf{x}),$$

$$\hat{H}_{\text{eff}} = \int d\mathbf{x} \hat{\Psi}^{\dagger}(\mathbf{x}) \left(-\frac{\hbar^2\nabla^2}{2m} + U(\mathbf{x})\right) \hat{\Psi}(\mathbf{x}) - \int d\mathbf{x} \frac{i|\Omega(\mathbf{x})|^2}{2\Gamma} \hat{\Psi}^{\dagger}(\mathbf{x})\hat{\Psi}^{\dagger}(\mathbf{x})\hat{\Psi}(\mathbf{x})\hat{\Psi}(\mathbf{x}).$$

Task 1

$H_{\mathrm{Tot}}[c_i, c_i^\dagger]$ ⇒ $S_{\mathrm{Tot}}[\psi_i, \psi_i^\dagger]$ ⇒ $\tilde{S}[\psi_i, \psi_i^\dagger \Delta_i, \bar{\Delta}_i]$

Coherent state path integral

[Yamamoto, et al., PRL, (2019)]

Non-Hermitian Mean field theory

$\Delta_1^\dagger \neq \bar{\Delta}_1, \Delta_2^\dagger \neq \bar{\Delta}_2$: Auxiliary fields in MFT

$$\frac{\delta}{\delta \bar{\Delta}_1} \tilde{S}[\psi_i, \psi_i^\dagger \Delta_i, \bar{\Delta}_i] = 0$$

$$\frac{1}{g}\left(\Delta_1 + \frac{\mu}{i\gamma}\Delta_2\right) = \langle \psi_{1\downarrow}\psi_{1\uparrow}\rangle + \frac{\mu}{g}\langle \psi_{2\downarrow}\psi_{2\uparrow}\rangle,$$

$$\frac{1}{i\gamma}\left(\Delta_2 + \frac{\mu}{g}\Delta_1\right) = \langle \psi_{2\downarrow}\psi_{2\uparrow}\rangle + \frac{\mu}{i\gamma}\langle \psi_{1\downarrow}\psi_{1\uparrow}\rangle,$$

$$\frac{1}{g}\left(\bar{\Delta}_1 + \frac{\mu}{i\gamma}\bar{\Delta}_2\right) = \langle \psi_{1\uparrow}^\dagger\psi_{1\downarrow}^\dagger\rangle + \frac{\mu}{g}\langle \psi_{2\uparrow}^\dagger\psi_{2\downarrow}^\dagger\rangle,$$

$$\frac{1}{i\gamma}\left(\bar{\Delta}_2 + \frac{\mu}{g}\bar{\Delta}_1\right) = \langle \psi_{2\uparrow}^\dagger\psi_{2\downarrow}^\dagger\rangle + \frac{\mu}{i\gamma}\langle \psi_{1\uparrow}^\dagger\psi_{1\downarrow}^\dagger\rangle,$$

Task 1

$\tilde{S}[\psi_i, \psi_i^\dagger \Delta_i, \bar{\Delta}_i]$ ⇨ $S_{\text{eff}}[\Delta_i, \bar{\Delta}_i]$

Integrate out $\{\psi_i, \psi_i^\dagger\}$

Want to expand trace log in $\Delta_i, \bar{\Delta}_i$

$$S_{\text{eff}} = -\text{Tr}\log\begin{pmatrix} i\omega_n + \epsilon^{(1)}_{\vec{k}} & \Delta_1 + \frac{\mu}{i\gamma}\Delta_2 \\ \overline{\Delta}_1 + \frac{\mu}{i\gamma}\overline{\Delta}_2 & i\omega_n - \epsilon^{(1)}_{\vec{k}} \end{pmatrix} \quad (5)$$
$$-\text{Tr}\log\begin{pmatrix} i\omega_n + \epsilon^{(2)}_{\vec{k}} & \Delta_2 + \frac{\mu}{g_1}\Delta_1 \\ \overline{\Delta}_2 + \frac{\mu}{g_1}\overline{\Delta}_1 & i\omega_n - \epsilon^{(2)}_{\vec{k}} \end{pmatrix}$$
$$+\frac{\mu}{i\gamma g}(\overline{\Delta}_1\Delta_2 + \overline{\Delta}_2\Delta_1) + \frac{1}{g}\overline{\Delta}_1\Delta_1 + \frac{1}{i\gamma}\overline{\Delta}_2\Delta_2,$$

$$\frac{\delta S_{\text{eff}}}{\delta \bar{\Delta}_i} = 0$$

⇩

Gap equations

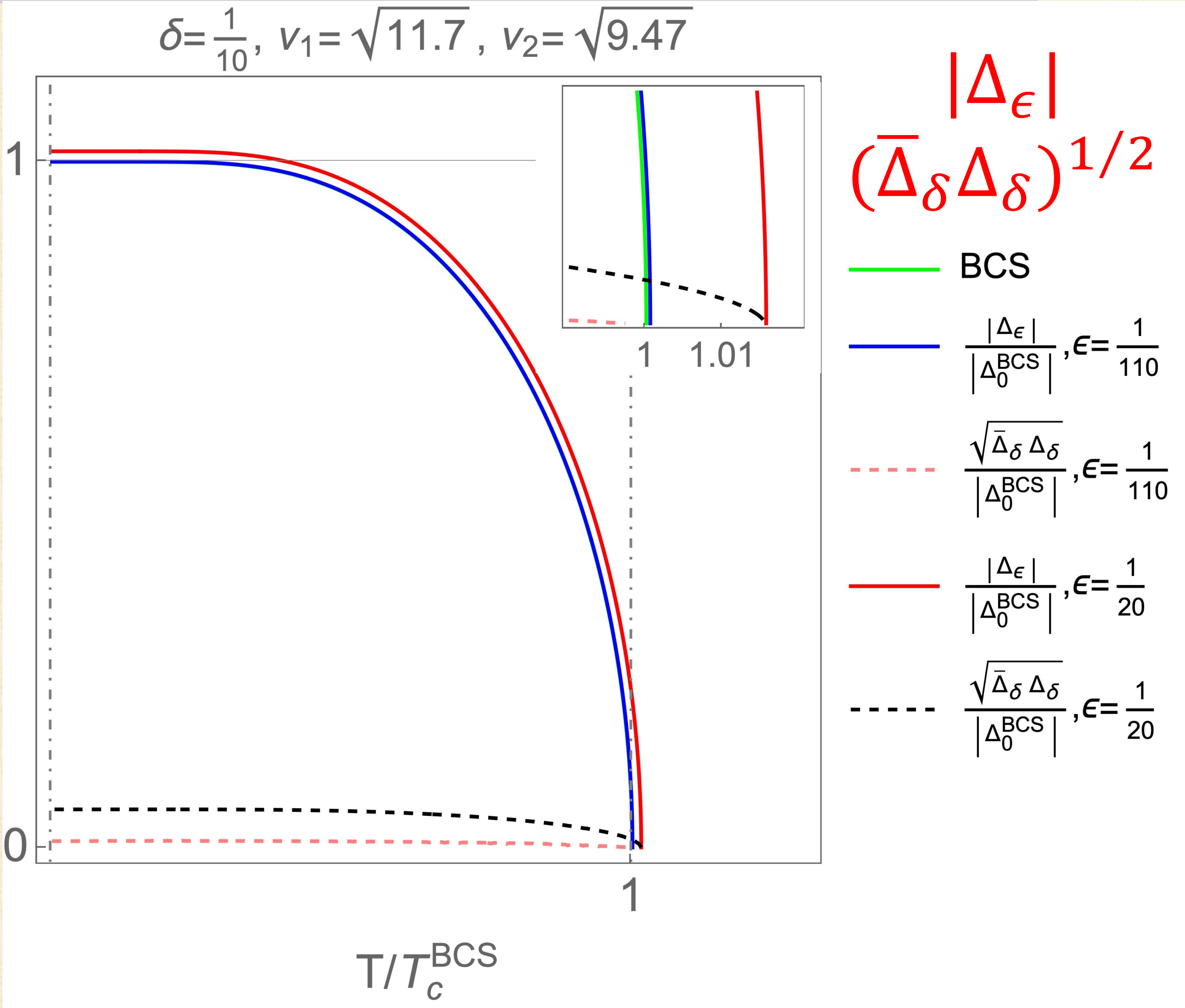
Task 1

Gap equations

Dropping $\mathcal{O}(\epsilon^3)$ from gap eq

$$\Delta_\epsilon \equiv \Delta_1 - i\epsilon\Delta_2$$
$$\Delta_\delta \equiv \Delta_2 + \delta\Delta_1$$
$$\epsilon = \frac{\mu}{\gamma}, \delta = \frac{\mu}{g}$$

g — Hermitian BCS theory ⟺ μ ⟺ γ — Non-Hermitian theory



Task 1

$$
\begin{aligned}
S_{\text{eff}} \;=\; & -\operatorname{Tr}\log\begin{pmatrix} i\omega_n + \epsilon^{(1)}_{\vec{k}} & \Delta_1 + \frac{\mu}{i\gamma}\Delta_2 \\ \overline{\Delta}_1 + \frac{\mu}{i\gamma}\overline{\Delta}_2 & i\omega_n - \epsilon^{(1)}_{\vec{k}} \end{pmatrix} \qquad (5)\\
& -\operatorname{Tr}\log\begin{pmatrix} i\omega_n + \epsilon^{(2)}_{\vec{k}} & \Delta_2 + \frac{\mu}{g_1}\Delta_1 \\ \overline{\Delta}_2 + \frac{\mu}{g_1}\overline{\Delta}_1 & i\omega_n - \epsilon^{(2)}_{\vec{k}} \end{pmatrix} \\
& + \frac{\mu}{i\gamma g}(\overline{\Delta}_1\Delta_2 + \overline{\Delta}_2\Delta_1) + \frac{1}{g}\overline{\Delta}_1\Delta_1 + \frac{1}{i\gamma}\overline{\Delta}_2\Delta_2,
\end{aligned}
$$

Two-component complex Ginzburg-Landau type model

$$
\begin{aligned}
S_{\text{eff}} \;=\; & \int d^3r\; \alpha_1 \nabla_i \overline{\Delta}_1 \nabla_i \Delta_1 \qquad (6)\\
& + \left(r_1 - \frac{1}{g} + r_2\delta^2\right)\overline{\Delta}_1\Delta_1 + u_1\left(\overline{\Delta}_1\Delta_1\right)^2 \\
& + \alpha_2 \nabla_i \overline{\Delta}_2 \nabla_i \Delta_2 + \left(r_2 - \frac{1}{g}\frac{\epsilon}{i\delta}\right)\overline{\Delta}_2\Delta_2 \\
& + \left(-i\epsilon r_1 + \delta\left\{r_2 - \frac{1}{g}\frac{\epsilon}{i\delta}\right\}\right)\left(\overline{\Delta}_1\Delta_2 + \overline{\Delta}_2\Delta_1\right),
\end{aligned}
$$

Task 1

Free ⇔ Bath

γ

GKSL

μ

g Hermitian BCS ⇔ Non-Hermitian effective theory $i\gamma$

HS

Two-component complex Ginzburg-Landau

Task 2

Extended London equation

Breakdown of the Meissner effect

Task 3

- beyond the critical temp while gapless
- at the exceptional point while gapful

Task 2

$$
\begin{aligned}
S_{\text{eff}} \;=\; & \int d^3r \; \alpha_1 \nabla_i \overline{\Delta}_1 \nabla_i \Delta_1 \\
& + \left(r_1 - \frac{1}{g} + r_2 \delta^2 \right) \overline{\Delta}_1 \Delta_1 + u_1 \left(\overline{\Delta}_1 \Delta_1 \right)^2 \\
& + \alpha_2 \nabla_i \overline{\Delta}_2 \nabla_i \Delta_2 + \left(r_2 - \frac{1}{g} \frac{\epsilon}{i\delta} \right) \overline{\Delta}_2 \Delta_2 \\
& + \left(-i\epsilon r_1 + \delta \left\{ r_2 - \frac{1}{g} \frac{\epsilon}{i\delta} \right\} \right) \left(\overline{\Delta}_1 \Delta_2 + \overline{\Delta}_2 \Delta_1 \right),
\end{aligned}
\qquad (6)
$$

Minimal coupling

$\nabla \to \nabla + eA$

$$\nabla \times \left. \frac{\delta S_{eff}}{\delta A_i} \right|_{A=A_0} = 0$$

London equation

Task 2

$$\nabla\times \left.\frac{\delta S_{eff}}{\delta A_i}\right|_{A=A_0} = 0 \qquad \nabla\times A \equiv B$$

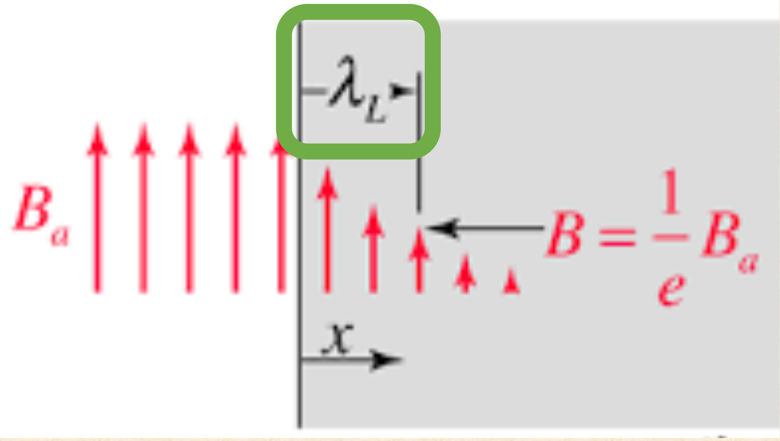
Inverse of London penetration depth

Extended London equation

$$\nabla^2\vec{B} = \frac{2e^2\alpha_2}{\kappa}\left(\overline{\Delta}_1^{\rm sol}\Delta_1^{\rm sol}\right)\left[1+\nu\left(-\delta+\frac{\epsilon}{\gamma}\frac{r_1}{r_2^2+\frac{1}{\gamma^2}}+i\epsilon\frac{r_1 r_2}{r_2^2+\frac{1}{\gamma^2}}\right)^2\right]\vec{B}.$$

$= 0$ Gapless

$= 0$ Gapless ?



$$r_i \equiv \frac{T}{V}\sum_{n,\vec{p}} 1/\left(\omega_n^2+\left(\epsilon_{\vec{p}}^{(i)}\right)^2\right) = \frac{\nu_i}{2}\int_0^{\omega_D/2T} dx\,\frac{\tanh(x)}{x}$$

No!!!

Task 2

$$\left[1+\nu\left(-\delta+\frac{\epsilon}{\gamma}\frac{r_1}{r_2^2+\frac{1}{\gamma^2}}+i\epsilon\frac{r_1 r_2}{r_2^2+\frac{1}{\gamma^2}}\right)^2\right]$$

$$\frac{1}{\nu_1}\frac{T}{V}\sum_{n,\vec{p}}\frac{1}{\omega_n^2+(\epsilon^{(1)}_{\vec{p}})^2}=\frac{1}{g\nu_1}\left(\frac{\epsilon}{\delta^2}\frac{1}{\nu\sqrt{\nu}}\right).$$

If and only if

$$\frac{T_{\mathrm{EP}}}{T_c^{\mathrm{BCS}}}=e^{\frac{1}{g\nu_1}-\frac{1}{g\nu_1}\left(\frac{\epsilon}{\delta^2}\frac{1}{\nu\sqrt{\nu}}\right)},$$

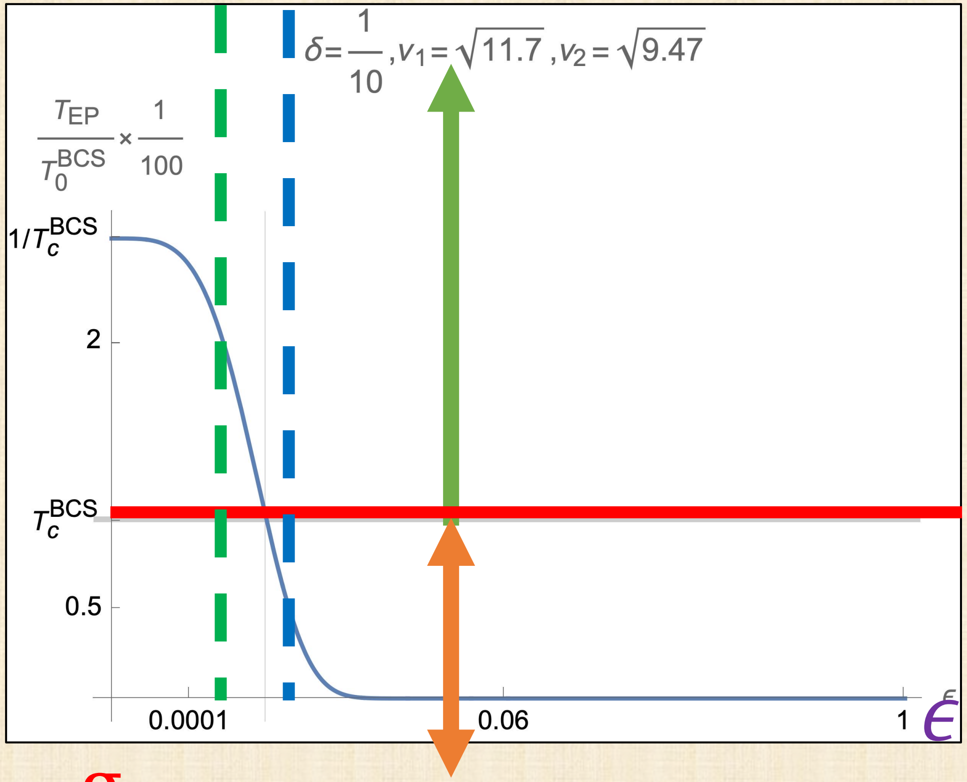
Meissner effect breaks at this temp

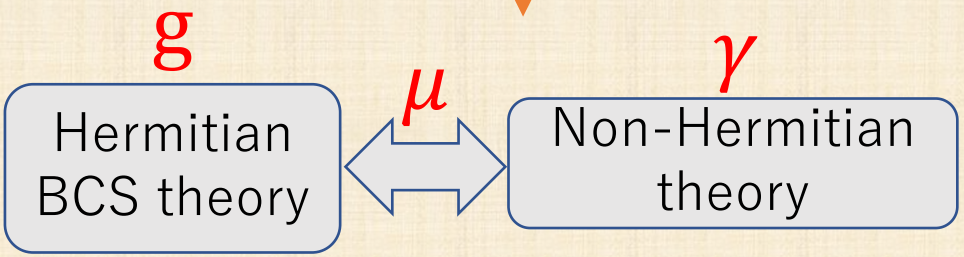
$\nu \equiv \nu_2/\nu_1$

Gapful!!

The breakdown of Meissner effect

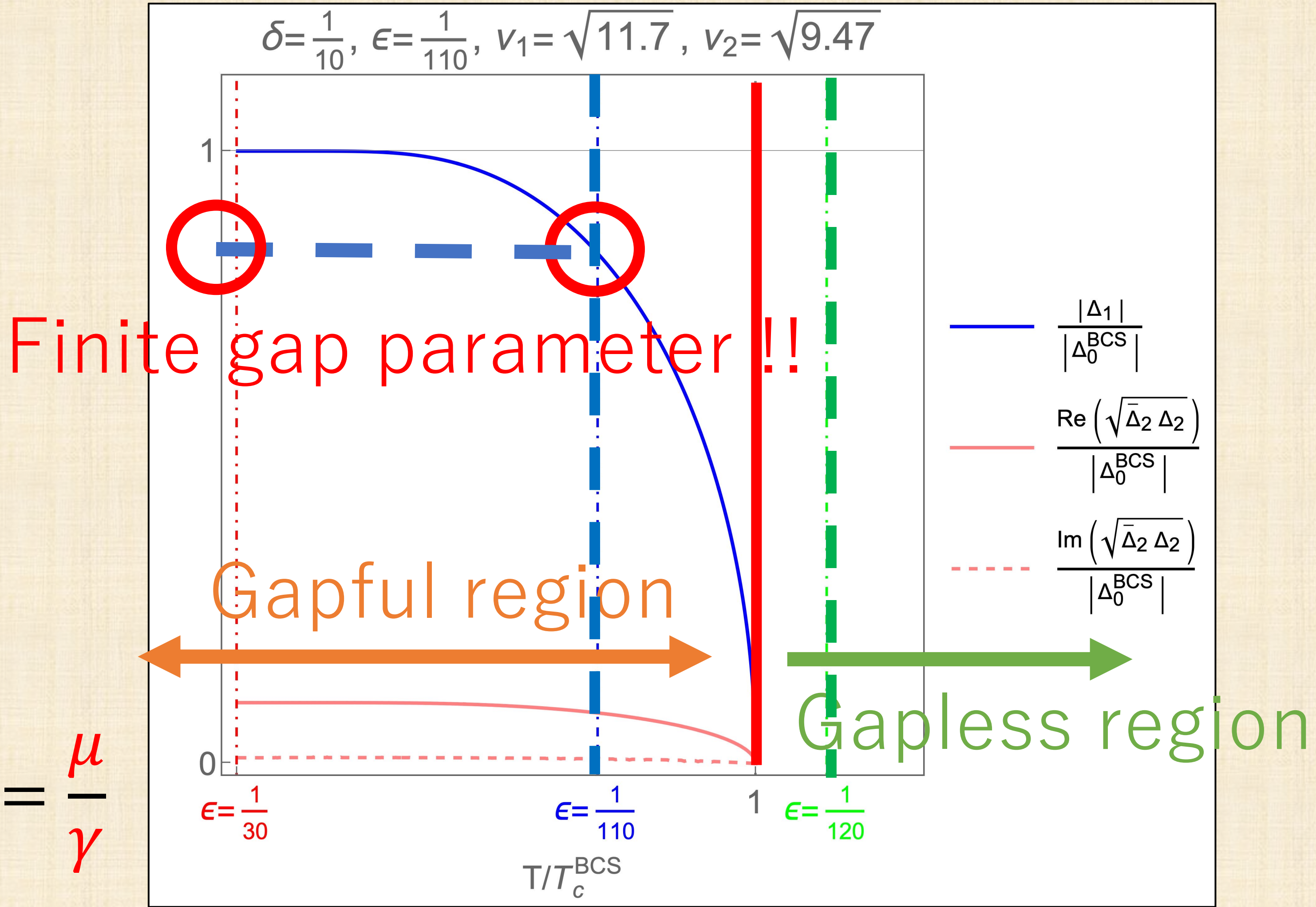
Exceptional temperature





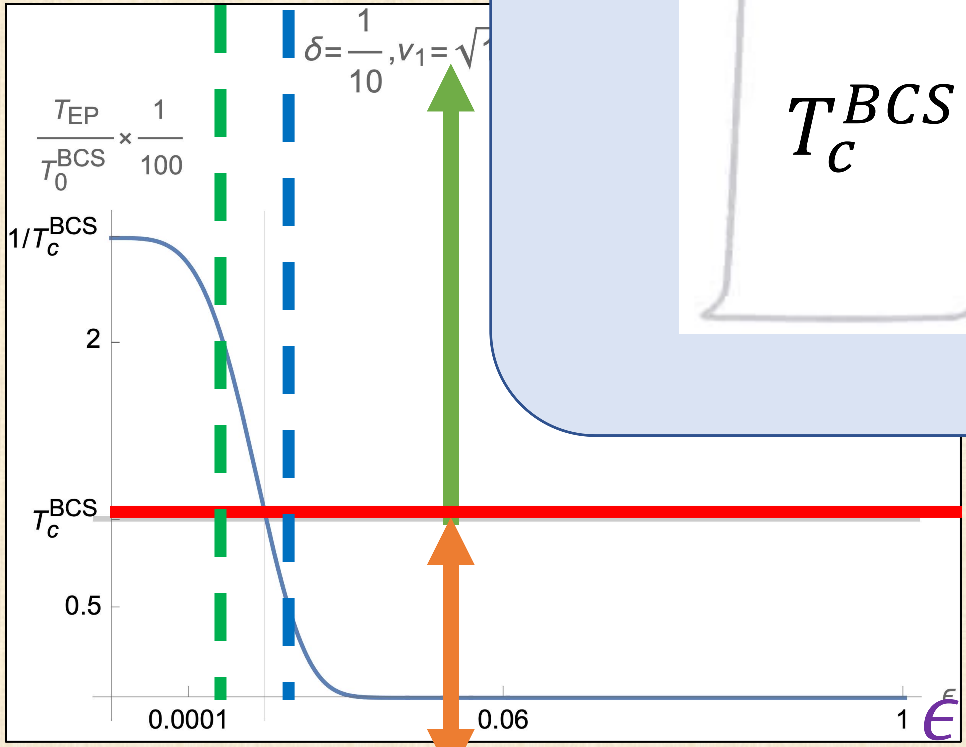
$\epsilon = \frac{\mu}{\gamma}$

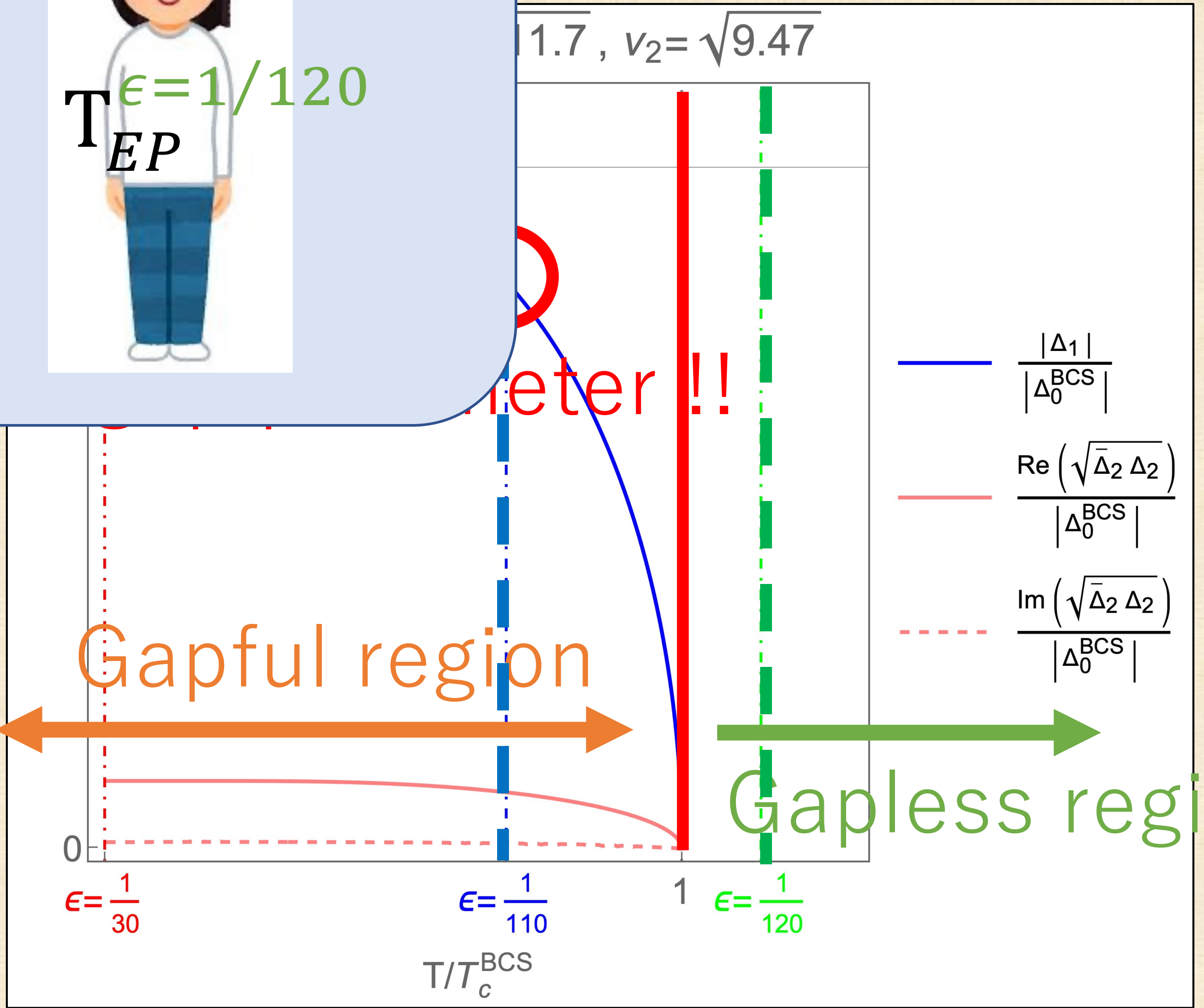
Gap parameters Δ_1 and Δ_2



The breakdown of Meissner effect

Exceptional ... ers Δ_1 and Δ_2





g

Hermitian BCS theory

μ

γ

Non-Hermitian theory

$$\epsilon = \frac{\mu}{\gamma}$$

Task 1

たいへんよくできました

Free ⇔ Bath (γ) — GKSL

g Hermitian BCS ⇔ (μ) Non-Hermitian effective theory $i\gamma$ — HS

Two-component complex Ginzburg-Landau

⬇

Extended London equation

⬇

Task 2

たいへんよくできました

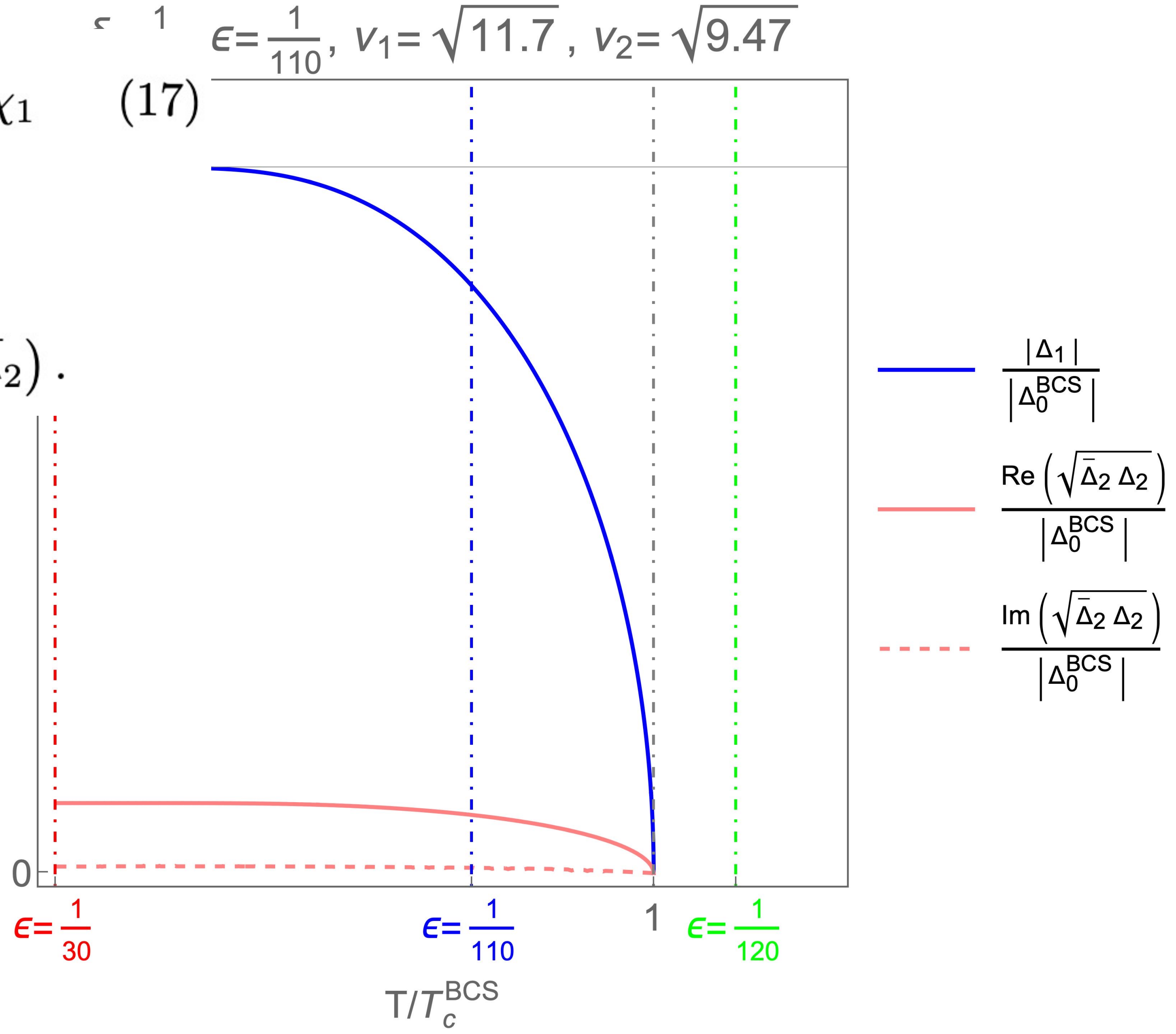
Breakdown of the Meissner effect

Task 3

- beyond the critical temp while gapless
- at the exceptional point while gapful

Task 3

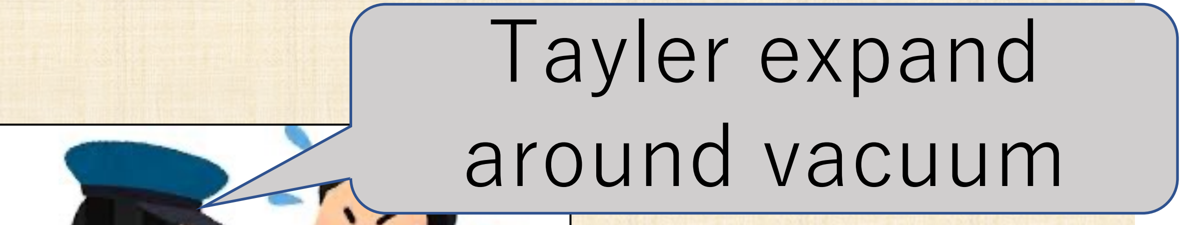
$$
\begin{aligned}
S_{\text{eff}} &= \int d^3r\ \alpha_1 \partial_\mu \phi_1 \partial_\mu \phi_1 + \alpha_1 \partial_\mu \chi_1 \partial_\mu \chi_1 \qquad (17)\\
&+\alpha_2 \partial_\mu \overline{\Delta}_2 \partial_\mu \Delta_2 + a_2 \overline{\Delta}_2 \Delta_2 \\
&+a_1\left(\phi_1^2+\chi_1^2\right) + u_1\left(\phi_1^2+\chi_1^2\right)^2 \\
&+b\phi_1\left(\Delta_2+\overline{\Delta}_2\right) + ib\chi_1\left(\Delta_2-\overline{\Delta}_2\right).
\end{aligned}
$$



$$\Delta_1^\dagger = \overline{\Delta}_1 \qquad \Delta_2^\dagger \neq \overline{\Delta}_2$$

$$\Delta_1 \equiv \phi_1 + i\,\chi_1, \Delta_1^\dagger \equiv \phi_1 - i\,\chi_1$$

Action is U(1) symmetric !



Task 3

$\Delta_1 \equiv \phi_1 + i\,\chi_1, \Delta_1^\dagger \equiv \phi_1 - i\,\chi_1$

$$\begin{aligned} S_{\rm eff} &= \int d^3r\ \alpha_1\partial_\mu\phi_1\partial_\mu\phi_1 + \alpha_1\partial_\mu\chi_1\partial_\mu\chi_1 \\ &+\alpha_2\partial_\mu\overline{\Delta}_2\partial_\mu\Delta_2 \\ &+\begin{pmatrix}\phi_1 & \overline{\Delta}_2\end{pmatrix}\begin{pmatrix}\frac{b^2}{a_2} & b \\ b & a_2\end{pmatrix}\begin{pmatrix}\phi_1 \\ \Delta_2\end{pmatrix} + \ldots, \end{aligned}$$

Exceptional point is when this is non-diagonalisable

Action is U(1) symmetric !

Mass matrix

$$= \int d^3r\ \frac{1}{T^2}\begin{pmatrix}\phi_1 & \overline{\Delta}_2\end{pmatrix}\begin{pmatrix}\nu_1 & 0 \\ 0 & \nu_2\end{pmatrix}\left[-\Box\mathbb{I} + \begin{pmatrix}\frac{1}{\nu_1}\frac{b^2}{a_2} & \frac{1}{\nu_1}b \\ \frac{1}{\nu_2}b & \frac{1}{\nu_2}a_2\end{pmatrix}\right]\begin{pmatrix}\phi_1 \\ \Delta_2\end{pmatrix} + \ldots$$

Task 3

Mass matrix

$$\begin{pmatrix} \frac{1}{\nu_1}\frac{b^2}{a_2} & \frac{1}{\nu_1}b \\ \frac{1}{\nu_2}b & \frac{1}{\nu_2}a_2 \end{pmatrix}$$

Exceptional point of this mass matrix

$$1+\nu\left(-\delta+\frac{\epsilon}{\gamma}\frac{r_1}{r_2^2+\frac{1}{\gamma^2}}+i\epsilon\frac{r_1 r_2}{r_2^2+\frac{1}{\gamma^2}}\right)^2=0.$$

Extended London equation

$$\nabla^2\vec{B}=\frac{2e^2\alpha_2}{\kappa}\overline{\Delta_1^{\rm sol}}\Delta_1^{\rm sol}\left[1+\nu\left(-\delta+\frac{\epsilon}{\gamma}\frac{r_1}{r_2^2+\frac{1}{\gamma^2}}+i\epsilon\frac{r_1 r_2}{r_2^2+\frac{1}{\gamma^2}}\right)^2\right]\vec{B}.$$

Task 1

Free ⇔ Bath (γ) — GKSL

g Hermitian BCS ⇔ (μ) Non-Hermitian effective theory $i\gamma$ — HS

Two-component complex Ginzburg-Landau

⇩

Extended London equation

⇩

Breakdown of the Meissner effect

Task 2

Task 3

- beyond the critical temp while gapless
- at the exceptional point while gapful

Conclusion

- We have shown that the Meissner effect breakdown at the exceptional point.
- However, this was observed for a particular example.
- The exceptional temperature is sensitive to small change of ϵ. But why?
- Experimental realisation?

Thank you for listening

Room: Y208

taira904@iis.u-tokyo.ac.jp

Motivation

[Kato1980Book], [Heiss1990JPhys]

$$H(x_1, x_2)v_i = \lambda_i v_i \qquad \mathcal{P}(x_1, x_2, \lambda) \coloneqq \det(H(x_1, x_2) - \lambda \mathbb{I})$$

$$\mathcal{P}(x_1, x_2, \lambda) = 0, \quad \frac{\mathrm{d}\mathcal{P}(x_1, x_2, \lambda)}{d\lambda} = 0,$$

Diabolic point	Exceptional point
$\{x_1, x_2\} = \{x_1^{DP}, x_2^{DP}\}$	$x_1 = x_{EP}$
$\lambda_i = \lambda_j, \quad \boldsymbol{v_i \neq v_j}$	$\lambda_i = \lambda_j, \quad \boldsymbol{v_i = v_j}$
e.g. Dirac cone	e.g. (edges of) Fermi arc

Task 3

Mass matrix

$$\begin{pmatrix} \frac{1}{\nu_1}\frac{b^2}{a_2} & \frac{1}{\nu_1}b \\ \frac{1}{\nu_2}b & \frac{1}{\nu_2}a_2 \end{pmatrix}$$

$$a_1 \equiv r_1 - {}^1\!/_g + r_2\delta^2$$

$$a_2 \equiv r_2 - {}^1\!/_{i\gamma}$$

$$b \equiv -i\epsilon r_1 + \delta r_2 + i\epsilon\, {}^1\!/_g$$

Exceptional point of this mass matrix

$$\left[1+\nu\left(-\delta+\frac{\epsilon}{\gamma}\frac{r_1}{r_2^2+\frac{1}{\gamma^2}}+i\epsilon\frac{r_1 r_2}{r_2^2+\frac{1}{\gamma^2}}\right)\right]^2=0.$$

London equation

$$\nabla^2\vec{B}=\frac{2e^2\alpha_2}{\kappa}\overline{\Delta_1^{\rm sol}}\Delta_1^{\rm sol}\left[1+\nu\left(-\delta+\frac{\epsilon}{\gamma}\frac{r_1}{r_2^2+\frac{1}{\gamma^2}}+i\epsilon\frac{r_1 r_2}{r_2^2+\frac{1}{\gamma^2}}\right)\right]^2\vec{B}.$$

Task 3

Exceptional point is when this is non-diagonalisable

$$S_{\rm eff} = \int d^3r\ \alpha_1\partial_\mu\phi_1\partial_\mu\phi_1 + \alpha_1\partial_\mu\chi_1\partial_\mu\chi_1 + \alpha_2\partial_\mu\overline{\Delta}_2\partial_\mu\Delta_2 + \begin{pmatrix}\phi_1 & \overline{\Delta}_2\end{pmatrix}\begin{pmatrix}\frac{b^2}{a_2} & b \\ b & a_2\end{pmatrix}\begin{pmatrix}\phi_1 \\ \Delta_2\end{pmatrix} + \dots,$$

$\Delta_1 \equiv \phi_1 + i$

Action is

$$\begin{aligned} S_{\rm eff} &= \int d^3r\ \begin{pmatrix}\phi_1 & \overline{\Delta}_2\end{pmatrix}\left[\begin{pmatrix}-\alpha_1\Box & 0 \\ 0 & -\alpha_2\Box\end{pmatrix} + \begin{pmatrix}\frac{b^2}{a_2} & b \\ b & a_2\end{pmatrix}\right]\begin{pmatrix}\phi_1 \\ \Delta_2\end{pmatrix} + \dots \\ &= \int d^3r\ \frac{1}{T^2}\begin{pmatrix}\phi_1 & \overline{\Delta}_2\end{pmatrix}\begin{pmatrix}\nu_1 & 0 \\ 0 & \nu_2\end{pmatrix}\left[-\Box\mathbb{I} + \begin{pmatrix}\frac{1}{\nu_1}\frac{b^2}{a_2} & \frac{1}{\nu_1}b \\ \frac{1}{\nu_2}b & \frac{1}{\nu_2}a_2\end{pmatrix}\right]\begin{pmatrix}\phi_1 \\ \Delta_2\end{pmatrix} + \dots \\ &= \int d^3r\ \frac{1}{T^2}\begin{pmatrix}\phi_1 & \overline{\Delta}_2\end{pmatrix}\begin{pmatrix}\nu_1 & 0 \\ 0 & \nu_2\end{pmatrix}\left[-\Box\mathbb{I} + U\begin{pmatrix}0 & 0 \\ 0 & \lambda\end{pmatrix}U^\dagger\right]\begin{pmatrix}\phi_1 \\ \Delta_2\end{pmatrix} + \dots \\ &= \int d^3r\ \partial_\mu\overline{\Psi}_1\partial_\mu\Psi_1 + \partial_\mu\overline{\Psi}_2\partial_\mu\Psi_2 + \lambda\overline{\Psi}_2\Psi, \end{aligned}$$

Mass matrix

2 Lindblad master equation and open system

$$\frac{d\rho_S(t)}{dt} = -i\left[H_S, \rho_S\right] - \gamma \sum_n \left(L_n^\dagger L_n \rho_S + \rho_S L_n^\dagger L_n - 2L_n \rho_S L_n^\dagger\right) \tag{40}$$

$$\frac{d\rho_S(t)}{dt} = -i\left(H_{\text{eff}}\rho_S - \rho_S H_{\text{eff}}^\dagger\right) + 2\gamma \sum_n L_n \rho_S L_n^\dagger \tag{41}$$

$$H_{\text{eff}} = H_S - i\gamma \sum_n L_n^\dagger L_n \tag{42}$$

The operators $\{L^\dagger, L\}$ are said to describe the particle entering and existing the system. This is because these operators are required to satisfy the following conditions

$$\left[H_S, L\right] = -\omega L \tag{43}$$

$$\left[H_S, L^\dagger\right] = \omega L^\dagger \tag{44}$$

therefore if we consider a eigensystem $H_S\left|n\right\rangle = E_n\left|n\right\rangle$, we have

$$H_S L^\dagger\left|n\right\rangle = \left(L^\dagger H_S + \omega L^\dagger\right)\left|n\right\rangle = \left(E_n + \omega\right) L^\dagger\left|n\right\rangle \tag{45}$$

$$H_S L\left|n\right\rangle = \left(L H_S - \omega L\right)\left|n\right\rangle = \left(E_n - \omega\right) L\left|n\right\rangle \tag{46}$$

therefore $L^\dagger, L$ creates/destroy particles from the system.

3.1 modified parafermi statistics of rank 1

Let us consider the creation and annihilation operators of Cooper pairs.

$$A_p \equiv a_{p\downarrow}a_{p\uparrow} \quad , \qquad A_p^\dagger \equiv a_{p\uparrow}^\dagger a_{p\downarrow}^\dagger \tag{47}$$

$$B_p \equiv b_{p\downarrow}b_{p\uparrow} \quad , \qquad B_p^\dagger \equiv b_{p\uparrow}^\dagger b_{p\downarrow}^\dagger \tag{48}$$

These operators does not satisfy the standard Fermi-Dirac statistic. Instead, they satisfies modified parafermi statistic of rank 1 [1].

$$[a_{p\alpha}^\dagger a_{p\alpha}, A_q^\dagger] = \delta_{pq}A_p^\dagger \ \text{ for all } \ \alpha = \uparrow,\downarrow \tag{49}$$

$$[a_{p\alpha}^\dagger a_{p\alpha}, A_q] = -\delta_{pq}A_p \tag{50}$$

$$[A_p, A_q^\dagger] = \delta_{pq}\left(1 - \hat{n}_{p\downarrow}^{(a)} - \hat{n}_{p\uparrow}^{(a)}\right) \ , \quad \hat{n}_{p\alpha}^{(a)} \equiv a_{p\alpha}^\dagger a_{p\alpha} \tag{51}$$

$$[A_p, A_q] = [A_p^\dagger, A_q^\dagger] = 0 \tag{52}$$

$$[A_p, B_q] = [A_p^\dagger, B_q] = [A_p, B_q^\dagger] = 0 \tag{53}$$

$$[B_p, B_q^\dagger] = \delta_{pq}\left(1 - \hat{n}_{p\downarrow}^{(b)} - \hat{n}_{p\uparrow}^{(b)}\right) \ , \quad \hat{n}_{p\alpha}^{(b)} \equiv b_{p\alpha}^\dagger b_{p\alpha} \tag{54}$$

$$\left[\hat{n}_{p\alpha}^{(a)}, A_q^\dagger\right] = \delta_{pq}L_p^\dagger \ \text{ for all } \ \alpha = \uparrow,\downarrow \tag{55}$$

$$\left[\hat{n}_{p\alpha}^{(a)}, A_q\right] = -\delta_{pq}L_p \ \text{ for all } \ \alpha = \uparrow,\downarrow \tag{56}$$

GKSL

Let us consider a system Hamiltonian

$$H_S = \sum_{p\alpha} \epsilon_p^{(a)} a_{p\alpha}^\dagger a_{p\alpha} + \epsilon_p^{(b)} b_{p\alpha}^\dagger b_{p\alpha} \tag{75}$$

Then commutation relation is reduced to

$$[H_S, A_p] = -\epsilon_p^{(a)} A_p \quad , \qquad [H_S, A_p^\dagger] = \epsilon_p^{(a)} A_p \tag{76}$$

$$[H_S, B_p] = -\epsilon_p^{(b)} B_p \quad , \qquad [H_S, B_p^\dagger] = \epsilon_p^{(b)} B_p \tag{77}$$

$$H_S A_p^\dagger |n\rangle = \left(E_n + \epsilon_p^{(a)}\right) A_p^\dagger |n\rangle$$