# SPTO in Open Quantum Systems

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# Topological phases in open systems

What are the topological phases of open systems? Of mixed states?

In particular, which closed topological phases are robust against dissipation?

# Defining phase equivalence

- In closed systems...

  paths in the space of gapped Hamiltonians... or equivalently...

  fast, local Hamiltonian evolution $\psi \xrightarrow{H} \psi'$

  - ... approximated by low depth circuits of local gates (e.g. polylog(L) depth of constant size gates)

- In open systems... Coser & Pérez-García [1810.05092] propose using

  fast, local Lindbladian evolutions $\rho \xrightarrow{\mathcal{L}_1} \rho'$ and $\rho' \xrightarrow{\mathcal{L}_2} \rho$

  - motivation: local observables are analytic within phases.

# Defining phase equivalence for SPT phases

- In closed systems...

  paths in the space of symmetric gapped Hamiltonians

  fast, local, symmetric Hamiltonian evolution

  - ... approximated by low depth circuits of local symmetric gates

- In open systems...?

  Not obvious what symmetry condition to impose on Lindbladian evolutions.

  Two natural guesses...

  - weak symmetry: $\mathcal{U}_g \circ \mathcal{L} \circ \mathcal{U}_g^\dagger = \mathcal{L}$ .
  - strong symmetry: $\mathcal{L}^\dagger(U_g) = 0$ .

# Dynamical and static definitions of phases

## Key idea

- A "good" definition of phase equivalence is one for which order parameters are constant within phases...

dynamical definition = static definition

- For SPT orders in 1D... we consider string order parameters.

... more on these later

# Proposed SPT phase equivalence

## Main claim

*SPT order, as defined by string order parameters,*

*is preserved by a fast, local Lindbladian evolution $\mathcal{L}$*

*if and only if $\mathcal{L}$ satisfies a strong symmetry condition.*

## Interpretations

1. Simply a rule for determining whether or not SPT order is robust against certain couplings to the environment.
2. A "good" definition of SPT phase equivalence in open systems.
   - Two mixed states belong to the same SPT phase if they are related by...

   fast, local, strongly symmetric Lindbladian evolutions $\rho \xrightarrow{\mathcal{L}_1} \rho'$ and $\rho' \xrightarrow{\mathcal{L}_2} \rho$.

# Weak and Strong Symmetry Conditions

# Weakly symmetric noise destroys 1D SPT order

- C&PG constructs a fast, local Lindbladian evolution that does

$$|\psi\rangle \quad \longrightarrow \quad |\text{product}\rangle$$

  to any 1D state $|\psi\rangle$.

  This evolution destroys SPT order.

- Solution...? Maybe this Lindbladian is not symmetric.
- However it satisfies

$$\mathcal{U}_g \circ \mathcal{L} \circ \mathcal{U}_g^\dagger = \mathcal{L} \ . \qquad \text{(weak symmetry)}$$

- Is there a stronger symmetry condition on $\mathcal{L}$ that protects SPT order?

# Weak symmetry allows interference between trajectories

- Work with channels $\mathcal{E}$. Have in mind Lindbladian evolutions $\mathcal{E}_t = e^{t\mathcal{L}}$.
- $\mathcal{L}$ is weakly symmetric precisely when $\mathcal{E}_t$ satisfies, at all times $t$,

$$\mathcal{U}_g \circ \mathcal{E}_t \circ \mathcal{U}_g^\dagger = \mathcal{E}_t \ . \qquad \text{(weak symmetry)}$$

- Weak symmetry (WS) in terms of Kraus operators:

$$\sum (U_g K_i U_g^\dagger)\, \rho\, (U_g K_i U_g^\dagger)^\dagger = \sum K_i \rho K_i^\dagger \ , \quad \forall g \ .$$

$\Rightarrow$ For each $g$, there is a basis of Kraus operators $K_i^g$ where

$$U_g K_i^g U_g^\dagger = e^{i\theta_i(g)} K_i^g \ , \quad \forall i \ .$$

- Different phases $\theta_i \neq \theta_j$ give rise to interference between trajectories.

# Strong symmetry

- Strong symmetry (SS): trajectories transform with *the same* phase:

$$U_g K_i U_g^\dagger = e^{i\theta(g)} K_i \ , \quad \forall\, i, g \ .$$

- This condition is basis-independent, so the superscript $g$ can be dropped.
- Neglect the weak invariant $\theta(g)$. It vanishes for $\mathcal{E} = e^{t\mathcal{L}}$.

- Comment: WS and SS are the same for reversible channels $\rho \mapsto W\rho W^\dagger$.

# Strong symmetry of Lindbladians

- Using the relation between the $K_i$ and the jump operators $L_i$, we see that...

...families $\mathcal{E}_t = e^{t\mathcal{L}}$ of SS channels are generated by SS Lindbladians:

$$U_g H = H U_g \ , \qquad U_g L_i = L_i U_g \ , \qquad \forall\, i, g \ .$$

- The SS condition on $\mathcal{L}$ has appeared previously. [ Buča , Prosen 12 ] [ Albert , Jiang 13 ]

# Strong symmetry in the Heisenberg picture

- The Heisenberg picture is convenient for studying the evolution of observables:

$$\mathrm{Tr}[\mathcal{E}(\rho)\,\mathcal{O}] = \mathrm{Tr}\big[\rho\,\mathcal{E}^\dagger(\mathcal{O})\big] \ .$$

- Strong symmetry of $\mathcal{E}$ means that $\mathcal{E}^\dagger$ fixes the symmetry operators:

$$U_g K_i U_g^\dagger = K_i \ , \quad \forall\, i, g \qquad \Longleftrightarrow \qquad \mathcal{E}^\dagger(U_g) = U_g \ , \quad \forall\, g \ .$$

- ... in other words, charge is conserved in the system alone.

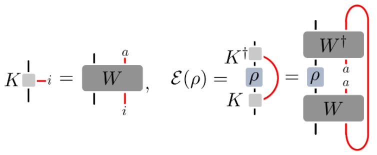
- In terms of the Lindbladian, strong symmetry means...

$$\mathcal{L}^\dagger(U_g) = 0 \ , \quad \forall\, g \ .$$

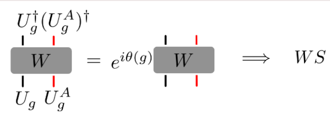
- The SPT-destroying Lindbladian of C&PG fails this condition.

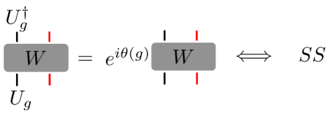
# Strong symmetry in terms of purifications

- Purification: unitary $W$ on $\mathcal{H} \otimes A$ such that $K_i = \langle e_i | W | a \rangle$.




- Claim: If $\exists$ a $W$ symmetric with respect to some $U_g \otimes U_g^A$, the channel $\mathcal{E}$ is WS.




- Claim: The channel is SS if and only if there exists a $W$ with $U_g^A = \mathbb{1}$.

- Strong symmetry means the system and bath couple by symmetric terms:

$$W = e^{-itH/\hbar}\,, \quad H = \sum_i H_i^S \otimes H_i^E \quad \overset{\text{SS}}{\Longrightarrow} \quad U_g H_i^S = H_i^S U_g \,,\ \forall\, i, g\,.$$

# Review: String Order for SPT Phases

# Symmetry fractionalization in one dimension

- SPT phases are characterized by patterns of symmetry fractionalization on the edge.


- Boundary action $V_g$ may be projective

$$V_g V_h = \omega(g,h) V_{gh} \ , \qquad \omega : G \times G \to U(1)$$

[ Chen, Gu, Wen 10 ] [ Schuch, Perez-Garcia, Cirac 11 ] [ Else, Nayak 14 ]

[ Pollmann, Turner, Berg, Oshikawa 10 ]

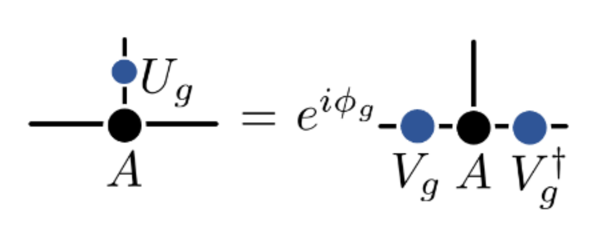
- $\omega$ satisfies a cocycle condition and is defined up to coboundaries
$\implies$ patterns of symmetry fractionalization are classified by group cohomology

$$[\omega] \in H^2(G, U(1))$$

# Fractionalization in MPS

- Symmetry means $U_g^N|\psi\rangle = |\psi\rangle$. Represent each side as an MPS.
- Tensors $A^i$ and $(U_g)^{ij}A_j$ define the same state $\Rightarrow$ related by a gauge trans.:

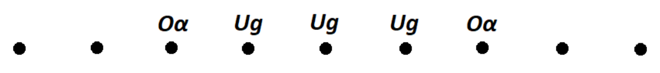
$$U_g^{ij}A_j = e^{i\phi_g}V_gA_iV_g^\dagger \ .$$



# String operators

- For simplicity... assume the symmetry $G$ is a finite abelian group.

- String operators: long strings of symmetry operators capped by end operators:
  [ den Nijs, Rommelse 89 ] [ Perez-Garcia, Wolf, Sans, Verstraete, Cirac 08 ] [ Pollmann, Turner 12 ]




- End operators $O^l_\alpha$ and $O^r_\alpha$ are charged under $G$ according to the irreps $\alpha$ and $\alpha^*$

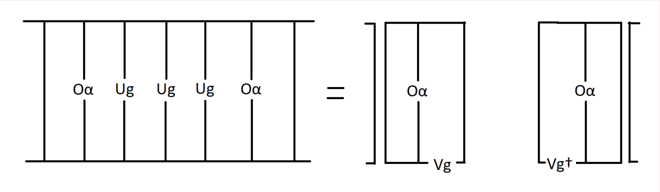
- Expectation values on MPS display a selection rule, aka "pattern of zeros"

$$\langle s(g, O_\alpha)\rangle = 0 \quad \text{unless} \quad \chi_\alpha(h) = \frac{\omega(h,g)}{\omega(g,h)} \text{ for all } h \; .$$

- The pattern determines the ratios $\omega/\omega$ and therefore the class $[\omega]$.

# Patterns of zeros of MPS states

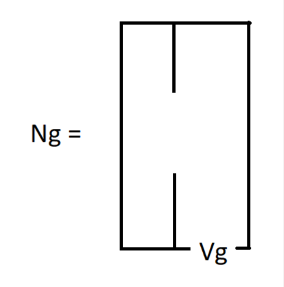
- To evaluate the string order parameter $\langle s(g,\alpha)\rangle$, sandwich the string operator between $\langle\psi|$ and $|\psi\rangle$, then apply some relations (injectivity) of the MPS tensor:




- Each end evaluates to $\mathrm{Tr}[N_g O_\alpha]$.
- $O_\alpha$ transforms as $\alpha$, while $N_g$ transforms as $\omega/\omega$.
- Vanishes unless these representations are equal:

$$\langle s(g,O_\alpha)\rangle = 0 \quad \text{unless} \quad \chi_\alpha(h) = \frac{\omega(h,g)}{\omega(g,h)} \text{ for all } h\,.$$

- If they are equal, *generically* $\mathrm{Tr}[N_g O_\alpha] \neq 0$.



# Reconstructing the SPT invariant

- Represent the pattern as an array with columns $g$, rows $\alpha$.
- For each $g$, there is a unique $\alpha$ with $\langle s(g, O_\alpha)\rangle \neq 0$.

$$\langle s(g, O_\alpha)\rangle_{\text{trivial}} = \begin{pmatrix} \star & \star & \star & \star \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \langle s(g, O_\alpha)\rangle_{\text{Haldane}} = \begin{pmatrix} \star & 0 & 0 & 0 \\ 0 & \star & 0 & 0 \\ 0 & 0 & \star & 0 \\ 0 & 0 & 0 & \star \end{pmatrix}$$

- Knowledge of all values $\langle s(g, O_\alpha)\rangle$ determines the ratios $\omega/\omega$.

- The ratios $\omega/\omega$ completely determine the cohomology class $[\omega]$.
  - Follows from Schur's lemma.

- Comment: If $G$ is non-abelian, other nonlocal order parameters may be needed to fully reconstruct $[\omega]$. [ Pollmann, Turner 12 ]

# Strong Symmetry and String Order

(three slides of technical argument)

# Strongly symmetric uncorrelated noise on string operators

- Consider uncorrelated noise

$$\mathcal{E} = \mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_L \ .$$

- $\mathcal{E}$ is strongly symmetric if and only if the $\mathcal{E}_s$ are.

- The labels $(g, \alpha)$ of the string operator are preserved

$$\mathcal{E}^\dagger(s(g, O_\alpha)) = s(g, O'_\alpha)$$

since the bulk and end parts of the string transform as

$$\mathcal{E}_s^\dagger(U_g) = U_g \ , \qquad U_g^\dagger \mathcal{E}_s^\dagger(O_\alpha) U_g = \chi_\alpha(g) \mathcal{E}_s^\dagger(O_\alpha) \ .$$

- Conversely, if a channel preserves all string operators, it must be SS.

- Next... what about the expectation values of these string operators?

# Strong symmetry of $\mathcal{L}$ is necessary and sufficient

*Fix any SPT phase, as defined by its string order.*

*A semigroup preserves the phase at all finite times*
*if and only if*
*it is generated by a strongly symmetric Lindbladian.*

## 'If' direction

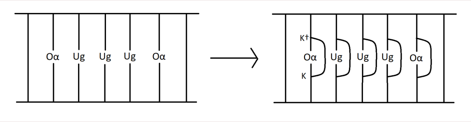
- Does SS protect the pattern of zeros in the expectation values $\langle s(g, O_\alpha) \rangle$?
- Yes. String operators $s(g, O_\alpha)$ and $\mathcal{E}^\dagger(s(g, O_\alpha))$ have the same labels, so they have the same pattern *for generic choices of end operators*...

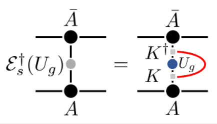
  ...i.e. if $O_\alpha$ is not annihilated and is orthogonal to neither $N_g$ nor $\mathcal{E}(N_g)$.
- Note: at $t \to \infty$, $\mathcal{E}_t$ may annihilate some end operators and spoil the pattern.

# Transfer matrix argument

- Conversely... suppose string order is preserved and show strong symmetry.

- String order vanishes unless the following transfer matrix has $\lambda_{\text{max}} = 1$.




- $\lambda_{\text{max}} = 1$ implies the insertion is a symmetry. [ Bridgeman, Chubb 17 ]

$$\implies \quad \mathcal{E}^\dagger(U_g) = U_{\sigma(g)} \ ,$$

- The family $\mathcal{E}_t$ defines a continuous path $\sigma_t$ from 1 to $\sigma$.
- For finite $G$, this implies that $\sigma = 1$, which is the strong symmetry condition:

$$\mathcal{E}_t^\dagger(U_g) = U_g \ , \forall\, t, g \qquad \implies \qquad \mathcal{L} \text{ is SS}$$

- Beyond finite $G$... $\sigma$ is an inner automorphism, and so $\mathcal{L}$ can also involve rotation by a generator of a continuous symmetry under which $\omega$ is invariant.

# Interpretation and Related Results

# Recent related results

- Ma & Wang [2209.02723] consider "Average SPT order" (ASPTO).
  - Systems evolve with subgroup $H$ "exact symmetry" (strong symmetry) and $G/H$ "average symmetry" (weak symmetry).
  - Nevertheless, some SPT order is robust.

- Lee, You, Xu [2210.16323] and Zhang, Qi, Bi [2210.17485] detect ASPTO with strange correlators.

- How can non-SS evolutions preserve SPTO?

- Resolution: these channels destroy *string order* but are nevertheless preserve SPTO in some other sense.