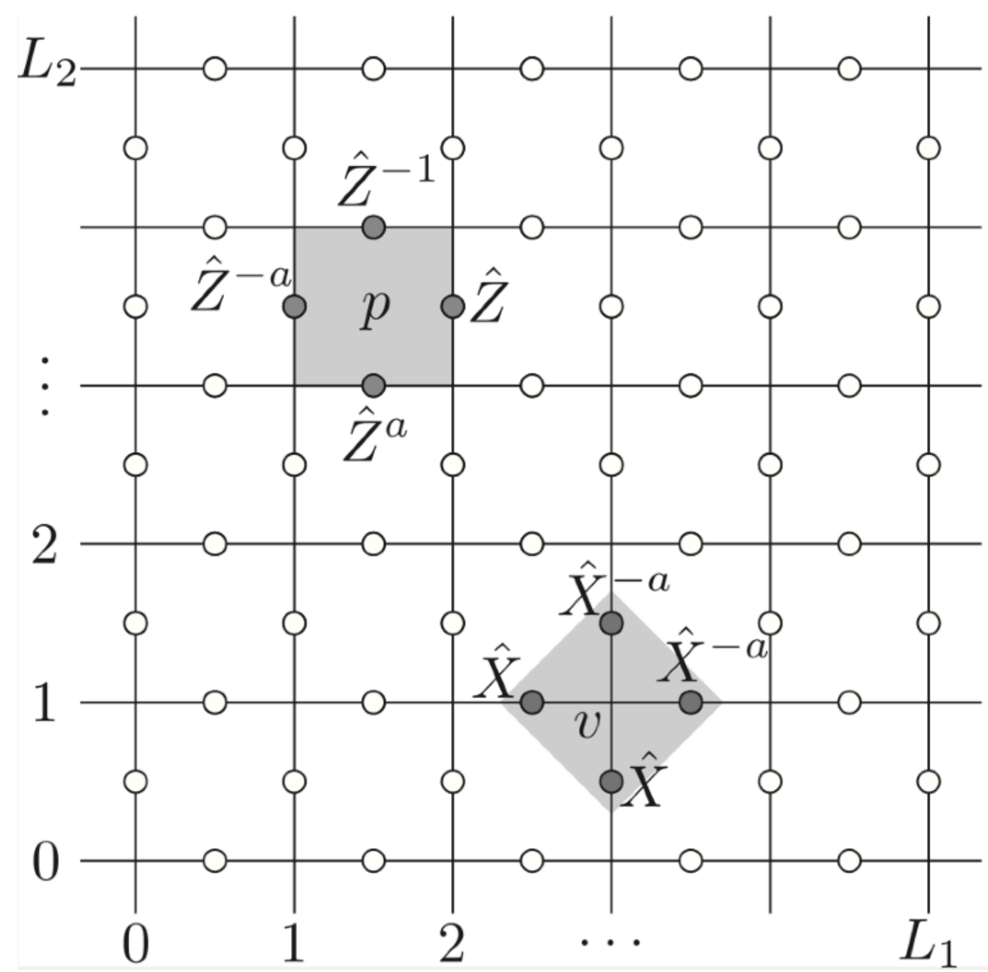
Ground state degeneracy on torus in topologically ordered phases

Haruki Watanabe (U Tokyo)
Collaborators: Yohei Fuji (U Tokyo), Meng Cheng (Yale)

Ref: HW, M. Cheng, Y. Fuji, arXiv:2211.00299

この絵を黒板に書く→
オンラインの人はメモ？



Introduction

Defining properties of phases (textbook)

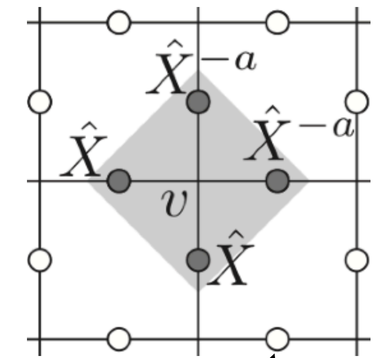
Symmetry-protected topological/trivial phases

1. Unique GS with excitation gap.
2. Stable against symmetry-preserving perturbations.
3. Can be connected to product states by local unitaries if symmetries are broken.
   (Exceptions: 1D Kitaev chain, integer Quantum Hall systems, …)
4. Trivial excitations in the bulk.
   Anomalous states on symmetry-preserving boundaries.

Topologically-ordered phases

1. Degenerate GS with excitation gap.
   Degeneracy does not originate from spontaneous breaking of symmetries.
2. Stable against any local perturbations.
3. Cannot be connected to product state by any local unitaries.
   Topological entanglement entropy.
4. Fractionalized (anyonic) excitations in the bulk. Anyons must appear in pairs.

Features of our model



- Our model contains a parameter $N \geq 2$ and $a = 1,2,\cdots,N$.
- Here we consider the case where N is a prime and a is chosen properly. (primitive root modulo N)
- Ground state degeneracy (GSD) N_{deg}:
  - $N_{\text{deg}} = N^2$ when both L_1 and L_2 are multiples of $N-1$.
  - $N_{\text{deg}} = 1$ when L_1 or L_2 is not a multiple of $N-1$.
- e.g. $N = 11$ and $a = 2$.
  To observe $N_{\text{deg}} > 1$, both L_1 and L_2 need to be multiples of 10.
  Minimum system size is 10×10; total Hilbert space dimension = 11^{200} .
  → Nearly impossible to realize in any numerical study.
- Topological entanglement entropy (TEE): $S_{\text{top}} = -\log N$.
- Anyons: N^2 species characterized by the electric charge $q_e = 1,2,\cdots,N$ and the magnetic charge $q_m = 1,2,\cdots,N$.

Defining properties of phases (updated)

Symmetry-protected topological/trivial phases

1. Unique GS with excitation gap
2. Stable against symmetry-preserving perturbations.
3. Can be connected to product states by local unitaries if symmetries are broken.
4. Trivial excitations in the bulk.
   Anomalous states on symmetry-preserving boundaries.

Topologically-ordered phases

1. Degenerate GS with excitation gap
   Degeneracy does not originate from spontaneous breaking of symmetries.
2. Stable against any local perturbations.
3. Cannot be connected to product state by any local unitaries.
   Topological entanglement entropy.
4. Fractionalized (anyonic) excitations in the bulk. Anyons must appear in pairs.

Defining properties of phases (updated)

Symmetry-protected topological/trivial phases

1. Unique GS with excitation gap **for any sequence of L_1 and L_2.**
2. Stable against symmetry-preserving perturbations.
3. Can be connected to product states by local unitaries if symmetries are broken.
4. Trivial excitations in the bulk.
   Anomalous states on symmetry-preserving boundaries.

Topologically-ordered phases

1. Degenerate GS with excitation gap **for a sequence of L_1 and L_2.**
   Degeneracy does not originate from spontaneous breaking of symmetries.
2. Stable against any local perturbations.
3. Cannot be connected to product state by any local unitaries.
   Topological entanglement entropy.
4. Fractionalized (anyonic) excitations in the bulk. ~~Anyons must appear in pairs.~~

Definition & basic properties of our model

N level spins

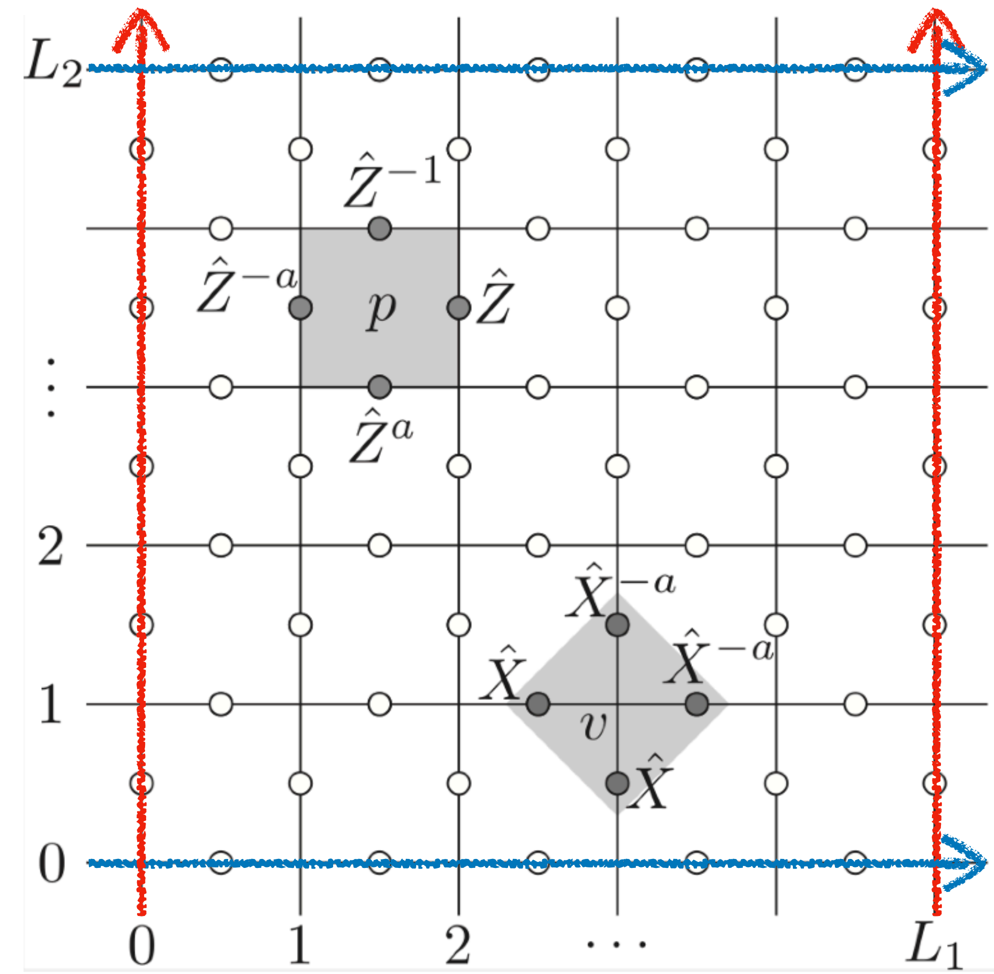
- Generalization of Pauli matrices to $N \geq 2$ level spins

$$X = \begin{pmatrix} & & & & 1 \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{N-1} \end{pmatrix}$$

- $\mathbb{Z}_N \times \mathbb{Z}_N$ group with phase $\omega = e^{\frac{2\pi i}{N}}$

$$ZX = \omega XZ = \begin{pmatrix} & & & & 1 \\ \omega & & & & \\ & \omega^2 & & & \\ & & \ddots & & \\ & & & \omega^{N-1} & \end{pmatrix}$$

$$Z^N = X^N = 1$$



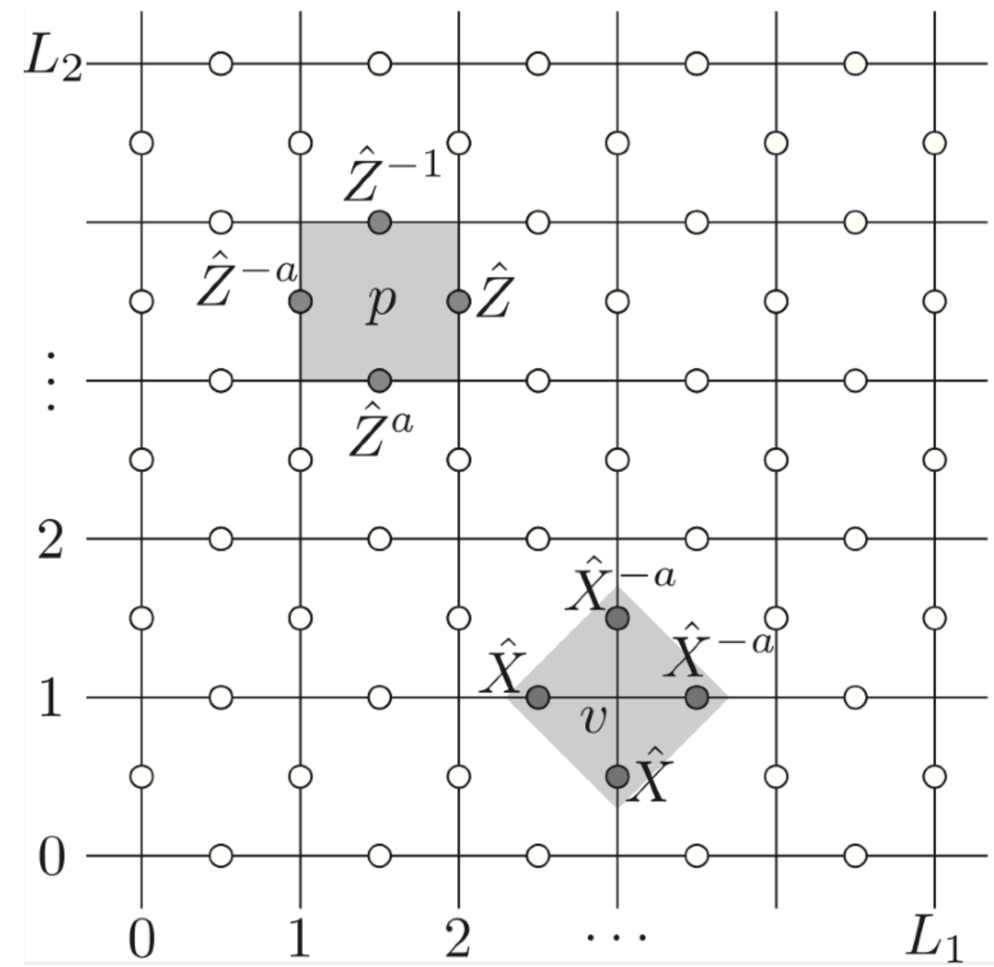
- A N level spin is placed on every bond. They satisfy $\hat{Z}_{\boldsymbol{r}}\hat{X}_{\boldsymbol{r}'} = \omega^{\delta_{r,r'}}\hat{X}_{\boldsymbol{r}'}\hat{Z}_{\boldsymbol{r}}$ and $\hat{Z}_{\boldsymbol{r}}^N = \hat{X}_{\boldsymbol{r}}^N = 1$.
- Periodic boundary condition (PBC) with system size L_1 and L_2 → System is put on torus.
- Total Hilbert space dimension is $N^{2L_1L_2}$.

$\mathbb{Z}_N$ toric code

- Vertex and plaquette terms (all commute regardless of a)

$$\hat{A}_{(m_1,m_2)} = \hat{X}^{-a}_{(m_1+\frac{1}{2},m_2)}\hat{X}^{-a}_{(m_1,m_2+\frac{1}{2})}\hat{X}_{(m_1-\frac{1}{2},m_2)}\hat{X}_{(m_1,m_2-\frac{1}{2})}$$

$$\hat{B}_{(m_1+\frac{1}{2},m_2+\frac{1}{2})} = \hat{Z}_{(m_1+1,m_2+\frac{1}{2})}\hat{Z}^{-1}_{(m_1+\frac{1}{2},m_2+1)}\hat{Z}^{-a}_{(m_1,m_2+\frac{1}{2})}\hat{Z}^{a}_{(m_1+\frac{1}{2},m_2)}$$



- $a = 1,2,\cdots,N$ is a very important parameter.
- Hamiltonian is sum of stabilizers:

$$\hat{H} = -\sum_{v\in\mathscr{V}}\frac{1}{2}(\hat{A}_v + \text{h.c.}) - \sum_{p\in\mathscr{P}}\frac{1}{2}(\hat{B}_p + \text{h.c.})$$

$$\mathscr{V} = \{(m_1, m_2)\,|\,m_i = 0,1,\cdots,L_i - 1\}$$

$$\mathscr{P} = \{(m_1+\tfrac{1}{2}, m_2+\tfrac{1}{2})\,|\,m_i = 0,1,\cdots,L_i - 1\}$$

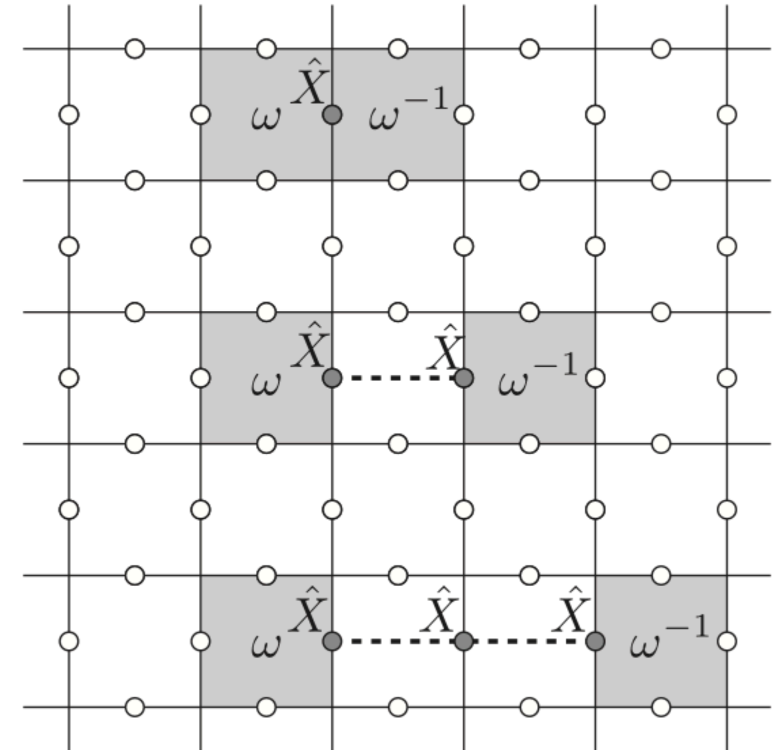
- Translation symmetry: $\hat{T}_i\hat{X}_{\boldsymbol{r}}\hat{T}_i^\dagger = \hat{X}_{\boldsymbol{r}+\boldsymbol{e}_i}$ and $\hat{T}_i\hat{Z}_{\boldsymbol{r}}\hat{T}_i^\dagger = \hat{Z}_{\boldsymbol{r}+\boldsymbol{e}_i}$.

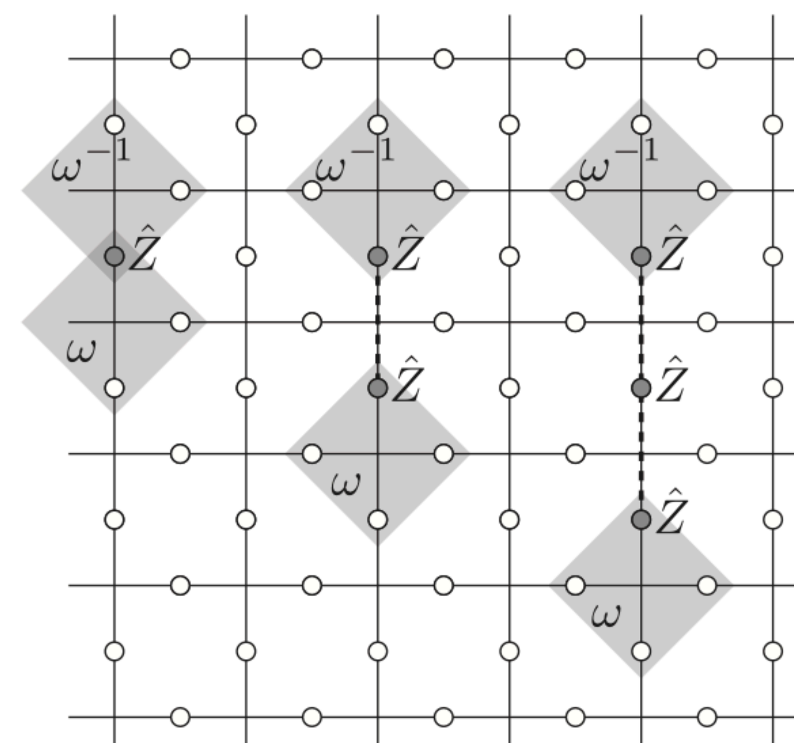
A ground state

- Since $\hat{A}_v^N = \hat{B}_p^N = 1$, the eigenvalues of $\hat{A}_v$ and $\hat{B}_p$ are N-fold: $1, \omega, \cdots, \omega^{N-1}$.
- Any state with eigenvalues +1 for all vertices and plaquettes is a ground state.
- A ground state can be constructed by
  (i) starting from the ferromagnetic state $\hat{Z}_{\boldsymbol{r}} | \phi_0 \rangle = | \phi_0 \rangle \quad (\forall \boldsymbol{r} \in \Lambda)$
  (ii) applying the projector $\hat{P} = \frac{1}{N^{L_1 L_2}} \prod_{v \in \mathcal{V}} \sum_{\ell=0}^{N-1} \hat{A}_v^{\ell}$ with the following properties.
  Then $| \Phi_0 \rangle \propto \hat{P} | \phi_0 \rangle$ has eigenvalues +1 for all vertices and plaquettes.
  - $\hat{P}^2 = \hat{P}$
  - $\hat{A}_v \hat{P} = \hat{P} \hat{A}_v = \hat{P}$
  - $\hat{B}_p \hat{P} = \hat{P} \hat{B}_p$
  - $\hat{T}_i \hat{P} = \hat{P} \hat{T}_i$

Simple case 1: $a = 1$

- This case is basically the same as the original toric code.
- Global constraints $\prod_{v \in \mathscr{V}} \hat{A}_v = \prod_{p \in \mathscr{P}} \hat{B}_p = 1.$
- X loops: $\hat{X}^{(1)} = \prod_{\ell=0}^{L_1-1} \hat{X}_{(\ell, L_2-\frac{1}{2})}, \hat{X}^{(2)} = \prod_{\ell=0}^{L_2-1} \hat{X}_{(L_1-\frac{1}{2}, \ell)}.$
- Z loops: $\hat{Z}^{(1)} = \prod_{\ell=0}^{L_1-1} \hat{Z}_{(\ell+\frac{1}{2}, 0)}, \hat{Z}^{(2)} = \prod_{\ell=0}^{L_2-1} \hat{Z}_{(0, \ell+\frac{1}{2})}.$
    - $\hat{Z}^{(1)}\hat{X}^{(2)} = \omega \hat{X}^{(2)}\hat{Z}^{(1)}.$
    - $\hat{Z}^{(2)}\hat{X}^{(1)} = \omega \hat{X}^{(1)}\hat{Z}^{(2)}.$
    - $[\hat{Z}^{(1)}, \hat{Z}^{(2)}] = [\hat{X}^{(1)}, \hat{X}^{(2)}] = 0.$
    - $[\hat{Z}^{(1)}, \hat{X}^{(1)}] = [\hat{X}^{(2)}, \hat{Z}^{(2)}] = 0.$
- GSD: $N_{\text{deg}} = N^2$. TEE: $S_{\text{top}} = -\log N$. N^2 species of anyons.





Open string operators

Control the eigenvalues of

- $\hat{A}_v \ (v \in \mathscr{V}, v \neq v_0)$
- $\hat{B}_p \ (p \in \mathscr{P}, p \neq p_0)$

$$\hat{Z}_{\boldsymbol{r}} \hat{X}_{\boldsymbol{r}'} = \omega^{\delta_{\boldsymbol{r},\boldsymbol{r}'}} \hat{X}_{\boldsymbol{r}'} \hat{Z}_{\boldsymbol{r}}$$

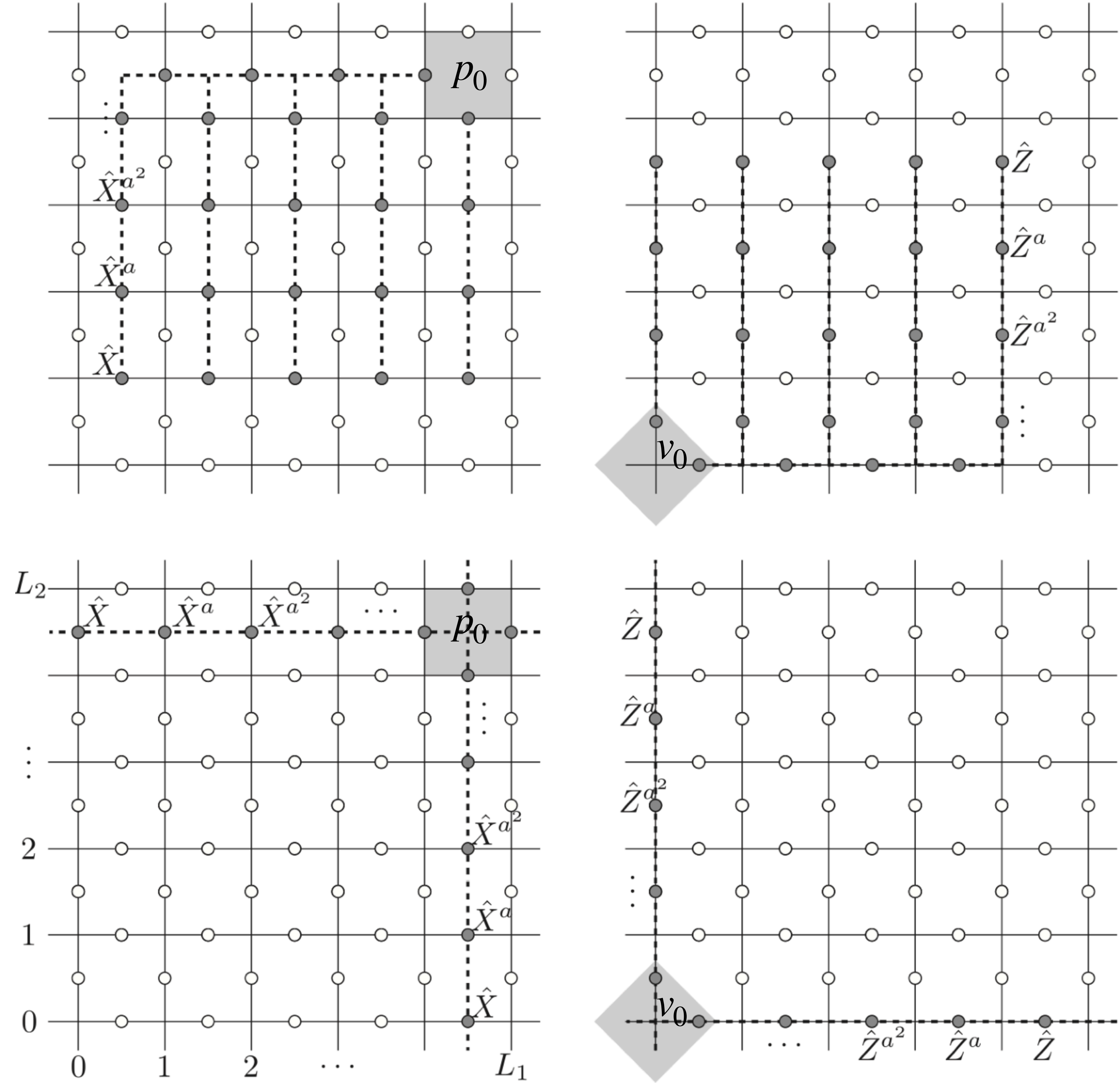
$\hat{A}_{v_0}$ and $\hat{B}_{p_0}$ are fixed by global constraints.

Closed string operators $\hat{X}^{(2)}$ and $\hat{Z}^{(2)}$

Control the eigenvalues of

- $\hat{X}^{(1)}$ and $\hat{Z}^{(1)}$

$$\hat{Z}^{(2)} \hat{X}^{(1)} = \omega \, \hat{X}^{(1)} \hat{Z}^{(2)}$$
$$\hat{Z}^{(1)} \hat{X}^{(2)} = \omega \, \hat{X}^{(2)} \hat{Z}^{(1)}$$

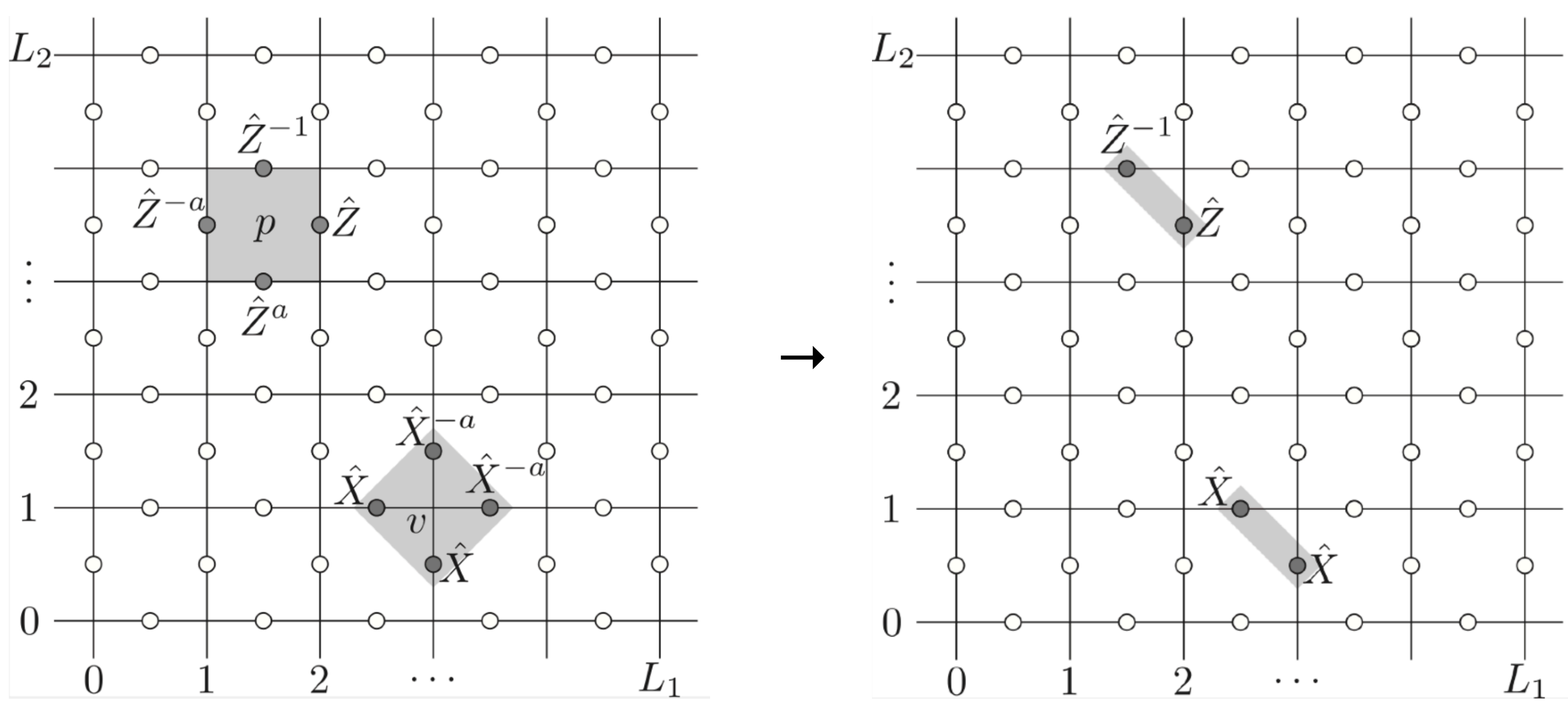


Simple case 2: $a = N$

- $\hat{H} = \sum_{\boldsymbol{r} \in \Lambda} \hat{h}_{\boldsymbol{r}}$ with

$$\hat{h}_{(m_1,m_2)} = \frac{1}{2}\Big(\hat{X}_{(m_1-\frac{1}{2},m_2)}\hat{X}_{(m_1,m_2-\frac{1}{2})} + \text{h.c.}\Big) - \frac{1}{2}\Big(\hat{Z}_{(m_1,m_2-\frac{1}{2})}\hat{Z}^{-1}_{(m_1-\frac{1}{2},m_2)} + \text{h.c.}\Big)$$

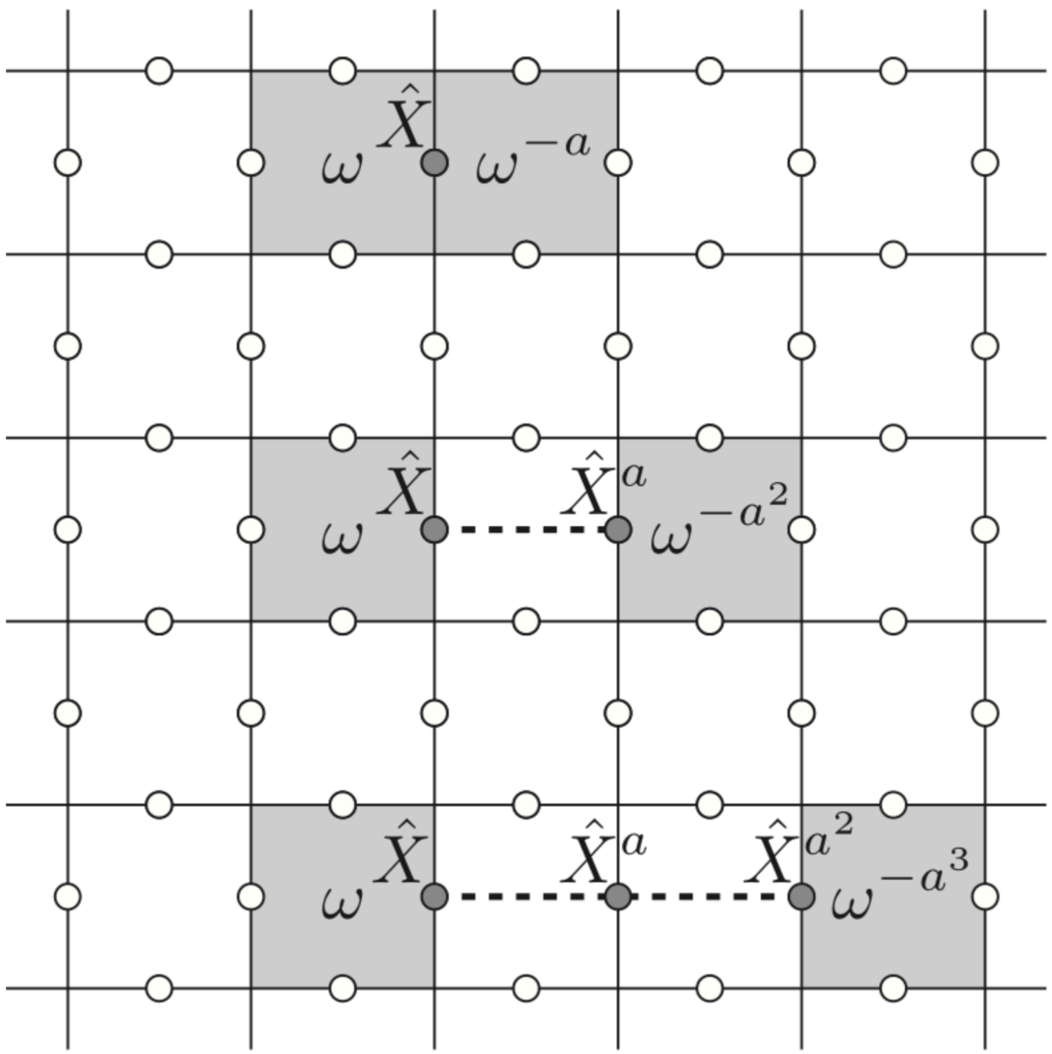
- Completely decoupled → product state.
- GSD: $N_{\text{deg}} = 1$. TEE: $S_{\text{top}} = 0$. No anyons.

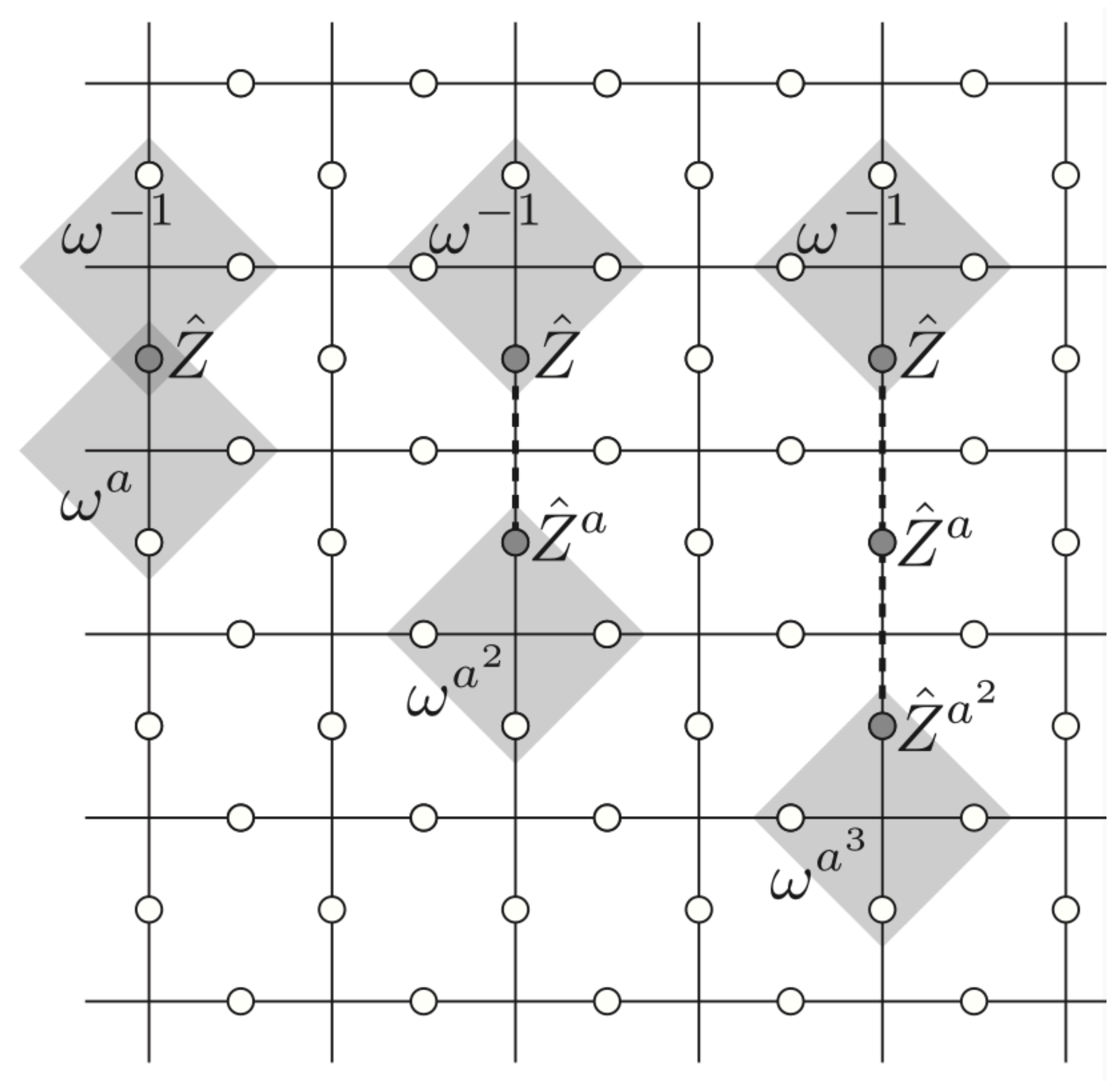


More general cases

Excited states

- Applying open strings of $\hat{X}$ and $\hat{Z}$ creates a pair of electric and magnetic particles.
- Translation permutes anyons (i.e. changes the charges of e and m particles).





$$\hat{X}^{(1)}_{(m_1-\frac{1}{2},m_2+\frac{1}{2}),(m_1'+\frac{1}{2},m_2+\frac{1}{2})} = \prod_{\ell=0}^{m_1'-m_1} \hat{X}^{a^\ell}_{(m_1+\ell,m_2+\frac{1}{2})}$$

$$\hat{Z}^{(2)}_{(m_1,m_2),(m_1,m_2'+1)} = \prod_{\ell=0}^{m_2'-m_2} \hat{Z}^{a^{m_2'-m_2-\ell}}_{(m_1,m_2+\ell+\frac{1}{2})}$$

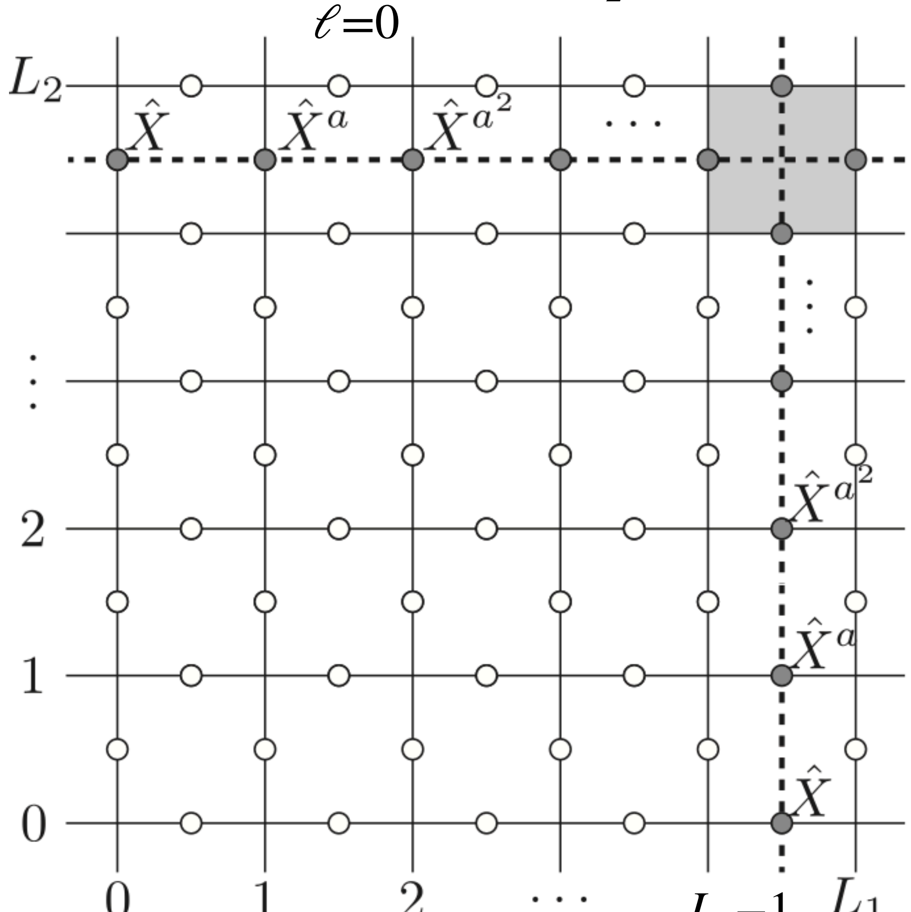
Global constraints

- Then GSD is given by $N_{\rm deg} = N_C^2$ (proof is given later),
  - where N_C is the number of global constraints: $\prod_{v \in \mathscr{V}} \hat{A}_v^{\ell_v} = 1\ (0 \le \ell_v \le N-1)$).
  - There will be an equal number of constraints: $\prod_{p \in \mathscr{P}} \hat{B}_p^{\ell_p} = 1\ (0 \le \ell_p \le N-1)$.
- Trial global constraints:
  - $$\prod_{m_1=0}^{L_1-1} \prod_{m_2=0}^{L_2-1} \hat{A}_{(m_1,m_2)}^{a^{m_1+m_2}} = \prod_{m_1=0}^{L_1-1} \hat{X}_{(m_1,-\frac{1}{2})}^{-a^{m_1}(a^{L_2}-1)} \prod_{m_2=0}^{L_2-1} \hat{X}_{(-\frac{1}{2},m_2)}^{-a^{m_2}(a^{L_1}-1)}.$$
  - $$\prod_{m_1=0}^{L_1-1} \prod_{m_2=0}^{L_2-1} \hat{B}_{(m_1+\frac{1}{2},m_2+\frac{1}{2})}^{a^{(L_1-1-m_1)+(L_2-1-m_2)}} = \prod_{m_1=0}^{L_1-1} \hat{Z}_{(m_1+\frac{1}{2},0)}^{a^{(L_1-1-m_1)}(a^{L_2}-1)} \prod_{m_2=0}^{L_2-1} \hat{Z}_{(0,m_2+\frac{1}{2})}^{-a^{(L_2-1-m_2)}(a^{L_1}-1)}.$$
- Consider their n-th power. We need $(a^{L_1}-1)n = (a^{L_2}-1)n = 0 \mod N$.

Loops

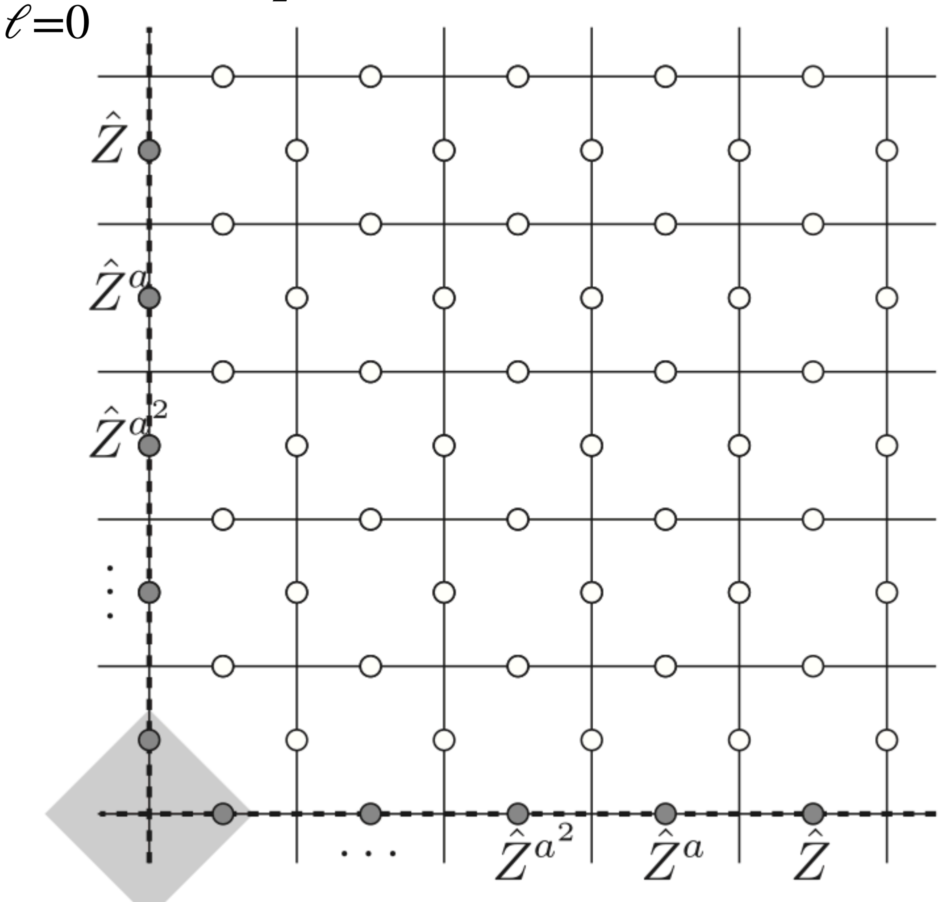
- $\hat{X}^{(i)}\hat{B}_{p_0} = \omega^{a^{L_i}-1}\,\hat{B}_{p_0}\hat{X}^{(i)}$ and $\hat{Z}^{(i)}\hat{A}_{v_0} = \omega^{1-a^{L_i}}\,\hat{A}_{v_0}\hat{Z}^{(i)}$.
- Closed loops are not really closed. Create an anyon without forming a pair.

$$\hat{X}^{(1)} = \prod_{\ell=0}^{L_1-1} \hat{X}^{a^\ell}_{(\ell, L_2-\frac{1}{2})}$$



$$\hat{X}^{(2)} = \prod_{\ell=0}^{L_2-1} \hat{X}^{a^\ell}_{(L_1-\frac{1}{2}, \ell)}$$

$$\hat{Z}^{(2)} = \prod_{\ell=0}^{L_2-1} \hat{Z}^{a^{L_2-1-\ell}}_{(0, \ell+\frac{1}{2})}$$



$$\hat{Z}^{(1)} = \prod_{\ell=0}^{L_1-1} \hat{Z}^{a^{L_1-1-\ell}}_{(\ell+\frac{1}{2}, 0)}$$

Math definitions

- Multiplicative order of a modulo n
  - Suppose n and a are coprime.
  - $M_n(a)$ is the smallest positive integer ℓ such that $a^\ell = 1 \pmod{n}$.
  - $M_n(a) = 1$ if and only if $a = 1 \pmod{n}$.
  - When $n \geq 3$, $M_n(a) = 2$ if $a = n - 1 \pmod{n}$.
- Euler's totient function $\varphi(n)$
  - The number of positive integers smaller than n that are relatively prime to n.
    e.g., $\varphi(n) = n - 1$ when n is a prime.
- Primitive root modulo n
  - a is a primitive root modulo n if $M_n(a) = \varphi(n)$.
  - The primitive roots exist if and only if n is either 2, 4, p^k, or $2p^k$ (p is an odd prime & k is a positive integer).

Case 3: N is a prime

- When N is a prime, $(a^{L_i}-1)n = 0$ (mod N) ⇔ $a^{L_i} = 1$ (mod N) ⇔ $L_i = 0$ (mod $M_N(a)$).
- If $L_i \neq 0$ (mod $M_N(a)$), $a^{L_i}-1$ is coprime to N. Eigenvalues of $\hat{A}_{v_0}$ and $\hat{B}_{p_0}$ can be freely controlled by using $\hat{X}^{(i)}\hat{B}_{p_0} = \omega^{a^{L_i-1}}\hat{B}_{p_0}\hat{X}^{(i)}$ and $\hat{X}^{(i)}\hat{A}_{p_0} = \omega^{1-a^{L_i}}\hat{A}_{p_0}\hat{X}^{(i)}$.
  This argument can be extended to all $v \in \mathscr{V}$ and $p \in \mathscr{P}$ → Absence of global constraints.
- GSD N_{deg} depends on L_1 and L_2:
  - $N_{\text{deg}} = N^2$ when both L_1 and L_2 are multiples of $M_N(a)$.
  - $N_{\text{deg}} = 1$ when L_1 or L_2 is not a multiple of $M_N(a)$.
- a is a primitive root if the smallest integer ℓ such that $a^{\ell} = 1$ (mod N) is $N-1$.
  c.f. Fermat's little theorem: $a^{N-1} = 1$ (mod N)
- TEE: $S_{\text{top}} = -\log N$. N^2 species of anyons.

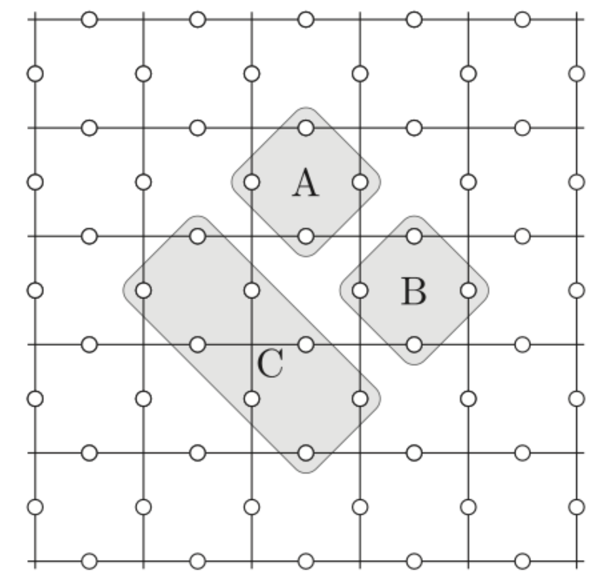
Example: $N = 11$

a	a^{ℓ} (mod N)
1	1
2	2, 4, 8, 5, 10, 9, 7, 3, 6, 1
3	3, 9, 5, 4, 1
4	4, 5, 9, 3, 1
5	5, 3, 4, 9, 1
6	6, 3, 7, 9, 10, 5, 8, 4, 2, 1
7	7, 5, 2, 3, 10, 4, 6, 9, 8, 1
8	8, 9, 6, 4, 10, 3, 2, 5, 7, 1
9	9, 4, 3, 5, 1
10	10, 1

Primitive roots are $a = 2,6,7,8$.

Calculation of TEE

- Kitaev-Preskill prescription
  - $S_{\text{topo}} = (S_A + S_B + S_C) - (S_{AB} + S_{BC} + S_{CA}) + S_{ABC}$
  - $S_R = -\operatorname{tr}[\hat{\rho}_R \log \hat{\rho}_R]$



- Useful formula: $S_R = n_R \log N - \log |G_R|$
  - n_R is the number of N-level spins in R.
  - G_R is the subgroup of G supported in R.
  - G is the multiplicative group generated by all $\hat{A}_v$'s ($v \in \mathscr{V}$), $\hat{B}_p$'s ($p \in \mathscr{P}$), and possible closed string operators for which $|\Phi_0\rangle$ has the eigenvalue +1.
- When N and a are coprime: $S_R = (n_R - m_R)\log N$
  - m_R is the number of generators of G supported in R.

General case

- Prime factorization $N = \prod_{j=1}^{n} p_j^{r_j} = p_1^{r_1} p_2^{r_2} \cdots p_n^{r_n}$.
- Radical of N: $\mathrm{rad}(N) = \prod_{j=1}^{n} p_j = p_1 p_2 \cdots p_n$.
- $N_a =$ the largest divisor of N that is coprime to a.
  - $N_a \neq 1$ if a is not a multiple of $\mathrm{rad}(N)$ → Topologically-ordered phases
  - $N_a = 1$ if a is a multiple of $\mathrm{rad}(N)$ → SPT phases
- GSD $N_{\mathrm{deg}} = d_a^2$ with $d_a = \mathrm{gcd}(a^{L_1} - 1, a^{L_2} - 1, N_a)$.
- N_a^2 anyon species characterized by the electric charge $q_e = 1,2,\cdots, N_a$ and the magnetic charge $q_m = 1,2,\cdots, N_a$.

Justification

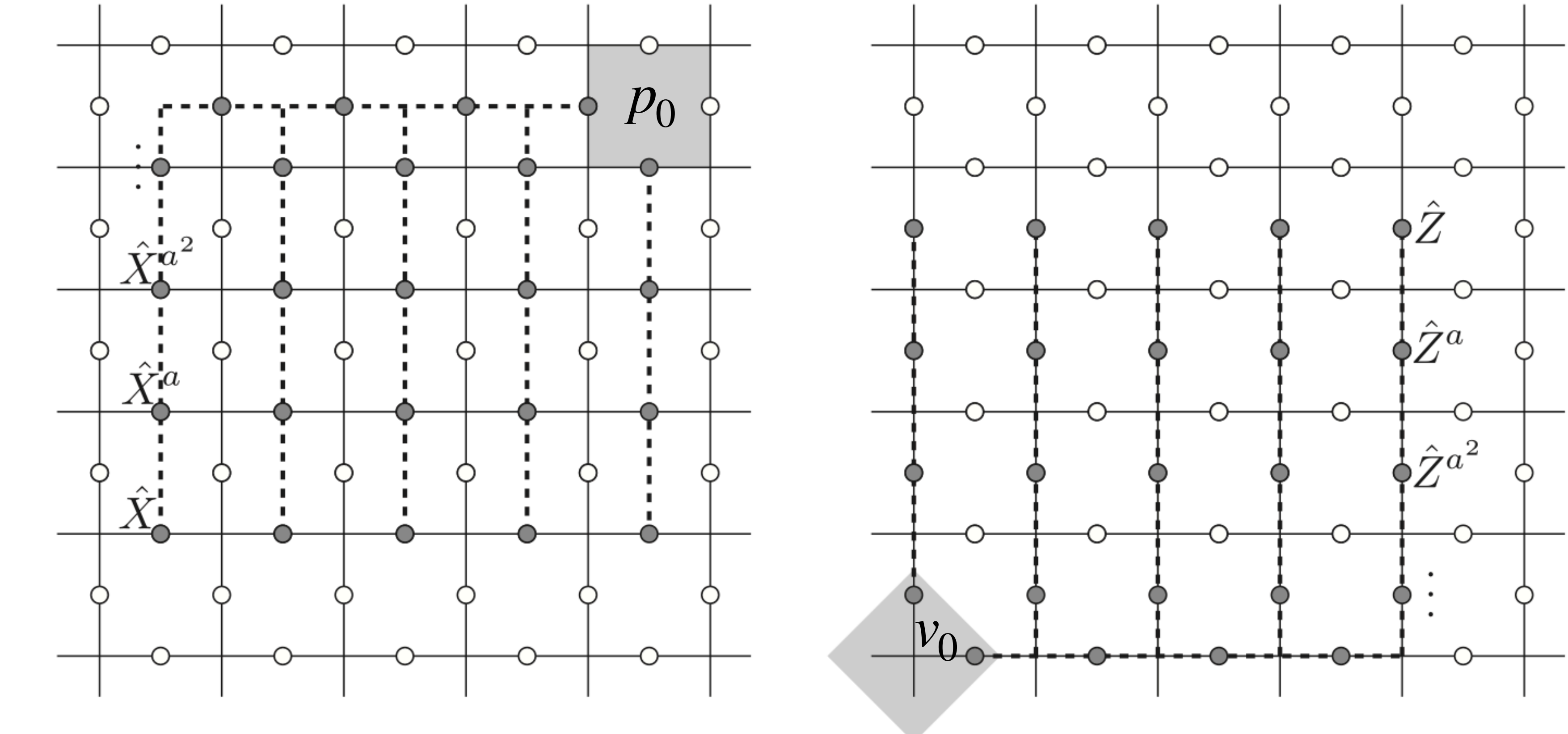
- Trial global constraints:
  - $$\prod_{m_1=0}^{L_1-1}\prod_{m_2=0}^{L_2-1}\hat{A}_{(m_1,m_2)}^{a^{m_1+m_2}} = \prod_{m_1=0}^{L_1-1}\hat{X}_{(m_1,-\frac{1}{2})}^{-a^{m_1}(a^{L_2}-1)}\prod_{m_2=0}^{L_2-1}\hat{X}_{(-\frac{1}{2},m_2)}^{-a^{m_2}(a^{L_1}-1)}.$$
  - $$\prod_{m_1=0}^{L_1-1}\prod_{m_2=0}^{L_2-1}\hat{B}_{(m_1+\frac{1}{2},m_2+\frac{1}{2})}^{a^{(L_1-1-m_1)+(L_2-1-m_2)}} = \prod_{m_1=0}^{L_1-1}\hat{Z}_{(m_1+\frac{1}{2},0)}^{a^{(L_1-1-m_1)}(a^{L_2}-1)}\prod_{m_2=0}^{L_2-1}\hat{Z}_{(0,m_2+\frac{1}{2})}^{-a^{(L_2-1-m_2)}(a^{L_1}-1)}.$$
- Consider their n-th power. We need $(a^{L_1}-1)n = (a^{L_2}-1)n = 0 \mod N$.
  Solution: $n = \dfrac{N}{d_a}$ with $d_a = \gcd(a^{L_1}-1, a^{L_2}-1, N_a)$.
- Similarly, $[\hat{X}^{(i)}]^{n_{i,a}}$ commutes with $\hat{B}_{p_0}$ if $(a^{L_i}-1)n_{i,a} = 0$ (mod N).
  Solution: $n_{i,a} = \dfrac{N}{\gcd(a^{L_i}-1, N_a)}$.

Open string operators

Control the eigenvalues of

- $\hat{A}_v$ ($v \in \mathscr{V}, v \neq v_0$)
- $\hat{B}_p$ ($p \in \mathscr{P}, p \neq p_0$)

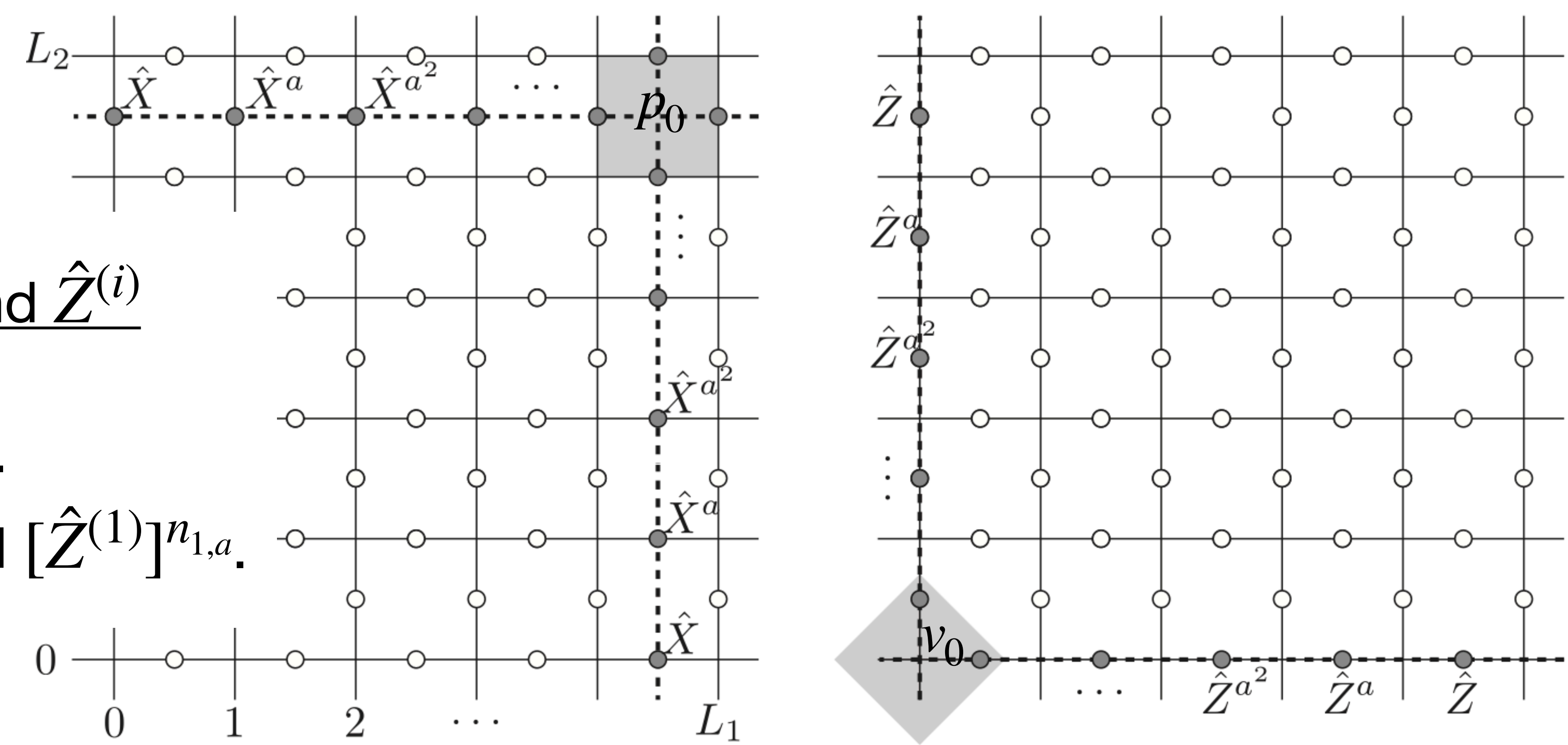
$[\hat{A}_{v_0}]^{\frac{N}{d_a}}$ and $[\hat{B}_{p_0}]^{\frac{N}{d_a}}$ are fixed by global constraints.



Closed string operators $\hat{X}^{(i)}$ and $\hat{Z}^{(i)}$

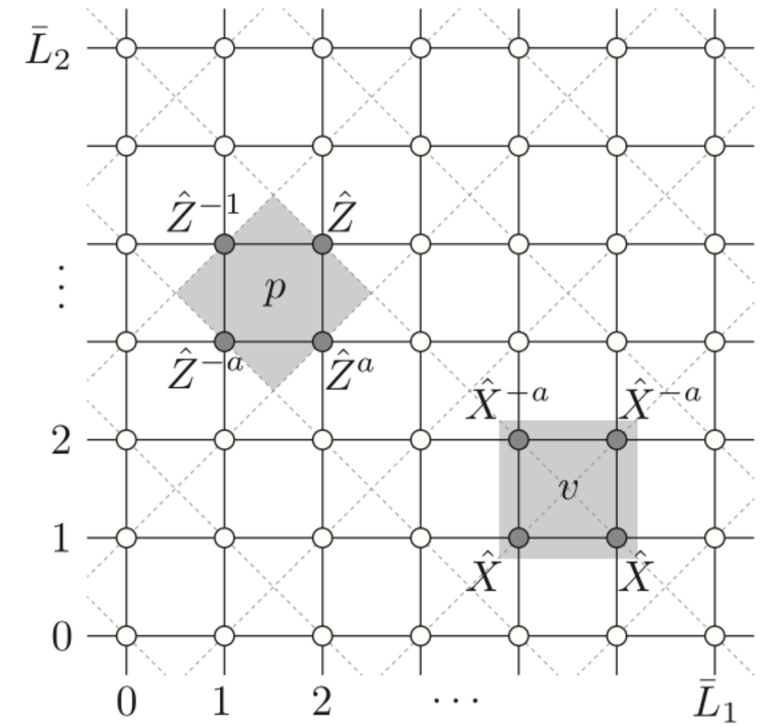
Control the eigenvalues of

- Residual part of $\hat{A}_{v_0}$ and $\hat{B}_{p_0}$.
- Closed strings: $[\hat{X}^{(1)}]^{n_{1,a}}$ and $[\hat{Z}^{(1)}]^{n_{1,a}}$.

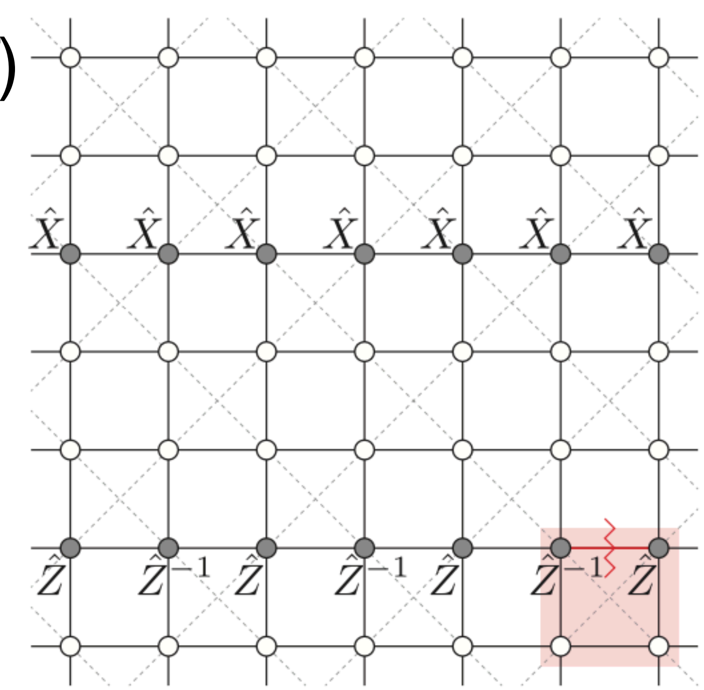


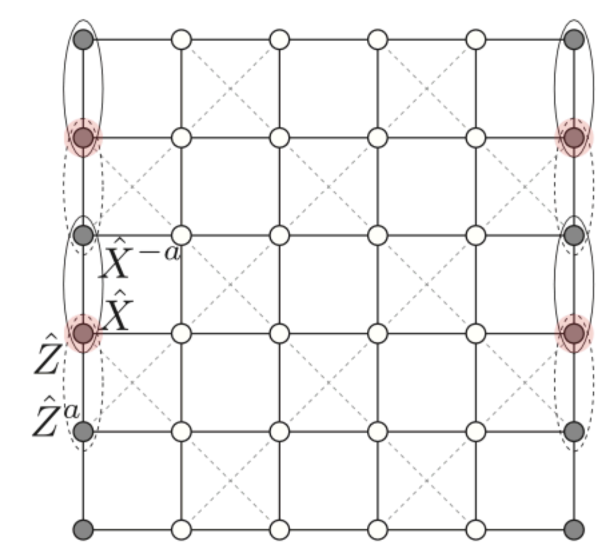
Case 4: $a^2 = N$

- Rotate the model by 45 degree.
- Subsystem symmetries for each row
  - $$\hat{X}_{\bar{m}_2} = \prod_{\bar{m}_1=0}^{\bar{L}_1-1} \hat{X}_{(\bar{m}_1,\bar{m}_2)}$$
  - $$\hat{Z}_{\bar{m}_2} = \prod_{\bar{m}_1=0}^{\bar{L}_1-1} \hat{Z}_{(\bar{m}_1,\bar{m}_2)}^{(-1)^{\bar{m}_1}}$$



- Subsystem-Protected Topological (SSPT)
  - Charge pumping
  - Zero energy edge states





Summary 1

- We introduced a family of $\mathbb{Z}_N$ toric code with an integer parameter a.
- $N_a =$ the largest divisor of N that is coprime to a.
  - $N_a \neq 1$ if a is not a multiple of $\mathrm{rad}(N)$ → Topologically-ordered phases
  - $N_a = 1$ if a is a multiple of $\mathrm{rad}(N)$ → (S)SPT phases
- GSD $N_{\mathrm{deg}} = d_a^2$ with $d_a = \mathrm{gcd}(a^{L_1} - 1, a^{L_2} - 1, N_a)$.
- N_a^2 anyon species characterized by the electric charge $q_e = 1,2,\cdots,N_a$ and the magnetic charge $q_m = 1,2,\cdots,N_a$.

Ref: HW, M. Cheng, Y. Fuji, arXiv:2211.00299

Summary 2

Symmetry-protected topological/trivial phases

1. Unique GS with excitation gap **for any sequence of L_1 and L_2.**
2. Stable against symmetry-preserving perturbations.
3. Can be connected to product states by local unitaries if symmetries are broken.
4. Trivial excitations in the bulk.
   Anomalous states on symmetry-preserving boundaries.

Topologically-ordered phases

1. Degenerate GS with excitation gap **for a sequence of L_1 and L_2.**
   Degeneracy does not originate from spontaneous breaking of symmetries.
2. Stable against any local perturbations.
3. Cannot be connected to product state by any local unitaries.
   Topological entanglement entropy.
4. Fractionalized (anyonic) excitations in the bulk. ~~Anyons must appear in pairs.~~