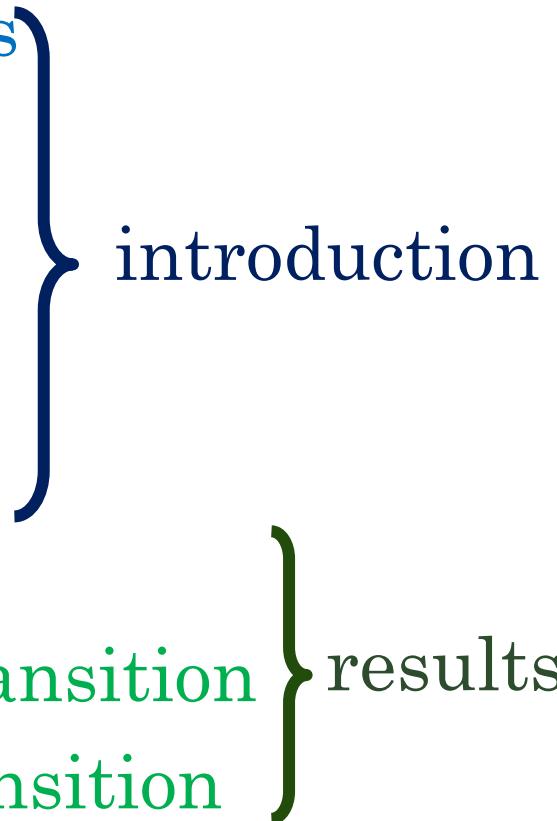


Complexity transitions of boson sampling dynamics in non-unitary regime

Ken Mochizuki (RIKEN)

Ken Mochizuki and Ryusuke Hamazaki, arXiv: 2207.12624 .

Outline

- post-selected open quantum systems
 - PT symmetry breaking
 - photonic experiments
 - boson sampling problem
 - motivation
 - model and PT symmetry breaking
 - short-time dynamical complexity transition
 - long-time dynamical complexity transition
- 

Post selected open quantum systems

unitary evolution of the total system

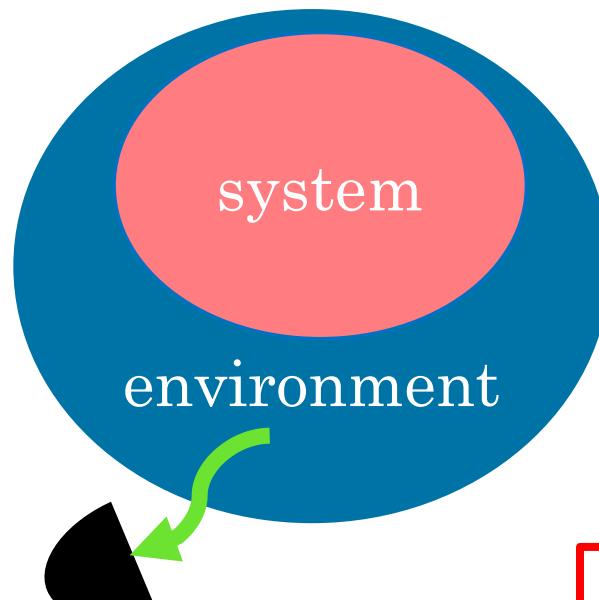
$$|\psi_t^{\text{in}}\rangle = |\psi_s^{\text{in}}\rangle \otimes |\psi_e^{\text{in}}\rangle \rightarrow V |\psi_t^{\text{in}}\rangle, \quad V^\dagger = V^{-1}$$

t : total
s : system
e : environment

non-unitary evolution of the partial system

$$|\psi_t^{\text{out}}\rangle \propto |\phi_e\rangle \langle \phi_e| V |\psi_t^{\text{in}}\rangle \propto |\psi_s^{\text{out}}\rangle \otimes |\phi_e\rangle \quad U^\dagger \neq U^{-1}$$

$$|\psi_s^{\text{out}}\rangle \propto U |\psi_s^{\text{in}}\rangle, \quad U = \langle \phi_e| V |\psi_e^{\text{in}}\rangle$$



post selection:
measurement on the environment
 \Rightarrow choosing a specific outcome $|\phi_e\rangle$

$$U = \exp(-iH\tau) \quad \leftrightarrow \quad H \neq H^\dagger$$



various intriguing phenomena
absent in isolated systems!

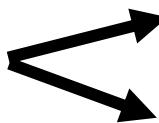
PT symmetry breaking

Parity-Time symmetry (PT symmetry)

$$H \neq H^\dagger$$

PT symmetry of a Hamiltonian

$$(\mathcal{PT})H(\mathcal{PT})^{-1} = H$$



PT symmetry of eigenstates

preserved (oscillation) :

$$\mathcal{PT}|\phi_l\rangle = |\phi_l\rangle \rightarrow \varepsilon_l : \text{real}$$

broken (attenuation/divergence) :

$$\mathcal{PT}|\phi_l\rangle \neq |\phi_l\rangle \rightarrow \varepsilon_l : \text{complex}$$

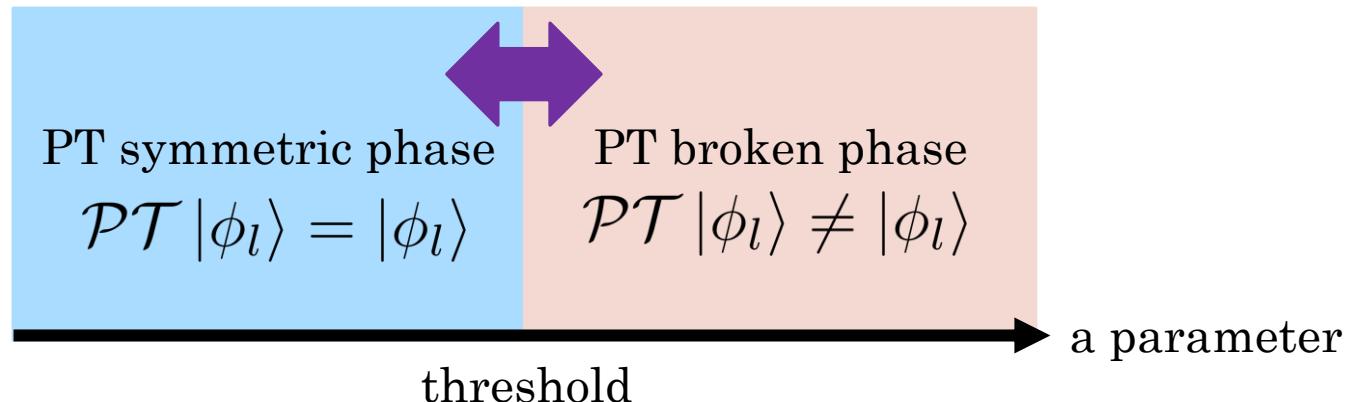
C. M. Bender and S. Boettcher,
Phys. Rev. Lett., **80**, 5243 (1998).

$$H |\phi_l\rangle = \varepsilon_l |\phi_l\rangle$$

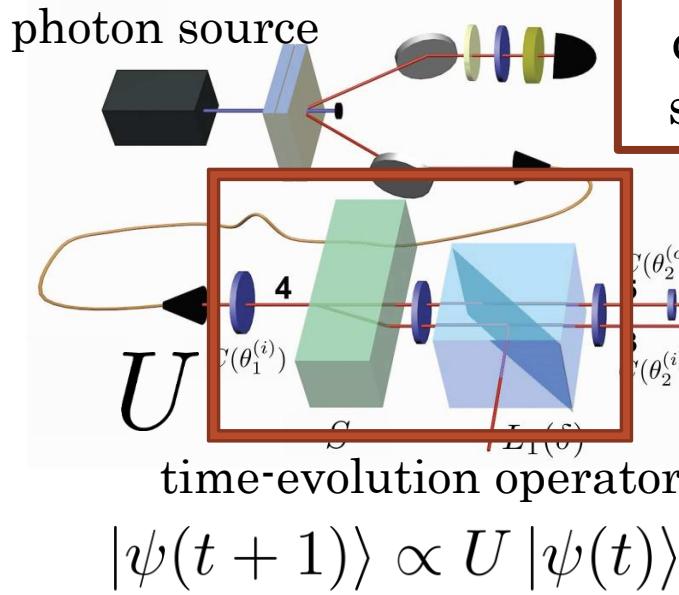
typical behavior in PT symmetric systems:

unique transition in open systems!

drastic change of dynamics!



A quantum optical experiment (quantum walk)



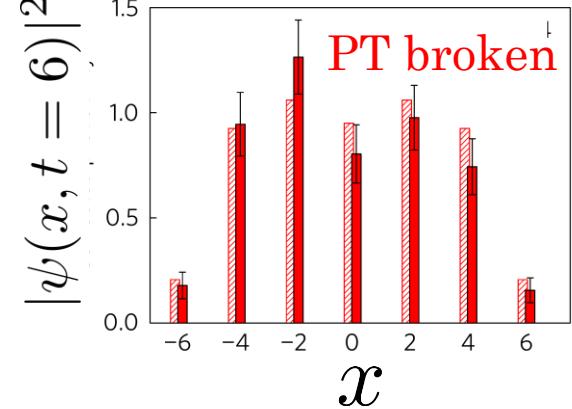
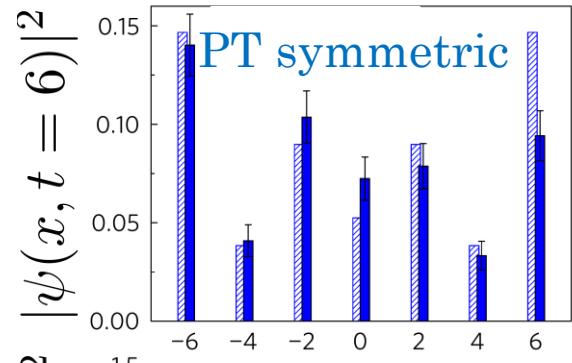
direction of photon propagation \Rightarrow effective time separation of paths \Rightarrow hoppings between lattices

photon loss effect & post selection $|\psi_e^{in}\rangle, |\phi_e\rangle$:

$\rightarrow U^\dagger \neq U^{-1} \leftrightarrow H \neq H^\dagger$ vacuum

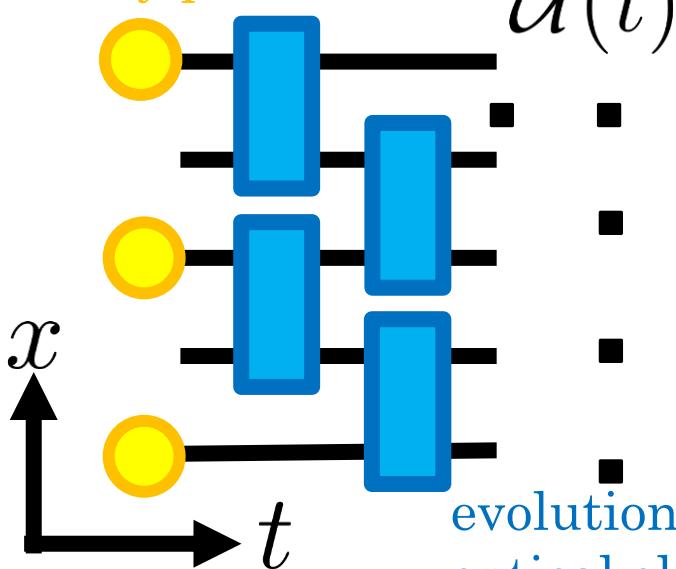
conventional: single photon dynamics
many photons \Rightarrow boson sampling problem

Xiao, Zhan, Bian, Wang, Zhang, Wang, Li, **Mochizuki**, Kim, Kawakami, Obuse, Sanders, Xue, Nature Physics **13**, 1117 (2017).



Boson sampling problem

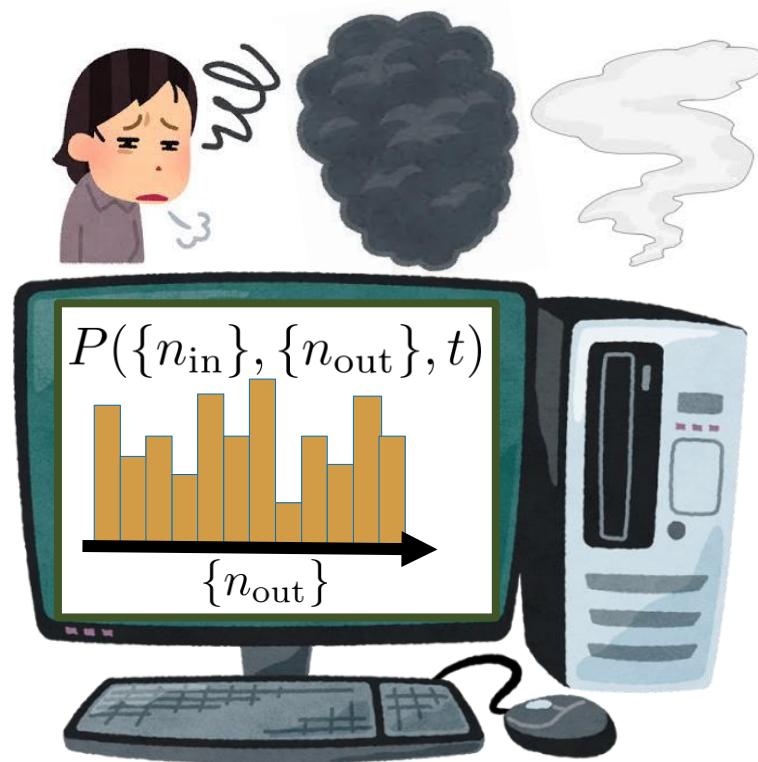
injection of
many photons



$$\mathcal{U}^T(t) = U^t \quad \hat{b}_x^\dagger \rightarrow \sum_y \mathcal{U}_{xy}(t) \hat{b}_y^\dagger$$

detection of
output photons

$$P(\{n_{\text{in}}\}, \{n_{\text{out}}\}, t)??$$



Photon probabilities can be **hard** to compute by classical computers
⇒ computational complexity / quantum supremacy

Computational complexity/Quantum supremacy

probability distribution of many photons :

can be hard to compute

$$P(\{n_{\text{in}}\}, \{n_{\text{out}}\}, t) = |\text{Per}[\mathcal{U}_{\text{in,out}}(t)]|^2 / N(t) \prod_x n_x^{\text{in}}! n_x^{\text{out}}!$$

$$\text{Per}[\mathcal{U}(t)] = \sum_{\omega} \prod_x \mathcal{U}_{x\omega(x)}$$

$$\{n_{\text{in/out}}\} = (n_1^{\text{in/out}}, n_2^{\text{in/out}}, \dots), \sum_x n_x^{\text{in/out}} = n$$

properties of states, $\mathcal{U}(t)$

hardness/easiness for
computing $\text{Per}[\mathcal{U}_{\text{in,out}}(t)]$
sampling $P(\{n_{\text{in}}\}, \{n_{\text{out}}\}, t)$

examples :

$\mathcal{U}(t)$: random matrix \Rightarrow hard

$\mathcal{U}(t)$: sparse/positive \Rightarrow easy

input : coherent/thermal state \Rightarrow easy
easy \Leftrightarrow classical

S. Aaronson, A. Arkhipov, STOC'11, 333.

S. R. Keshari, *et al*, PRL. **114**, 060501 (2015).

W. Roga, M. Takeoka, Sci. Rep. **10**, 14739 (2020).

* hard/easy \Leftrightarrow exponential/polynomial computational time of n

characterization of dynamics/phase by computational complexity

Objective

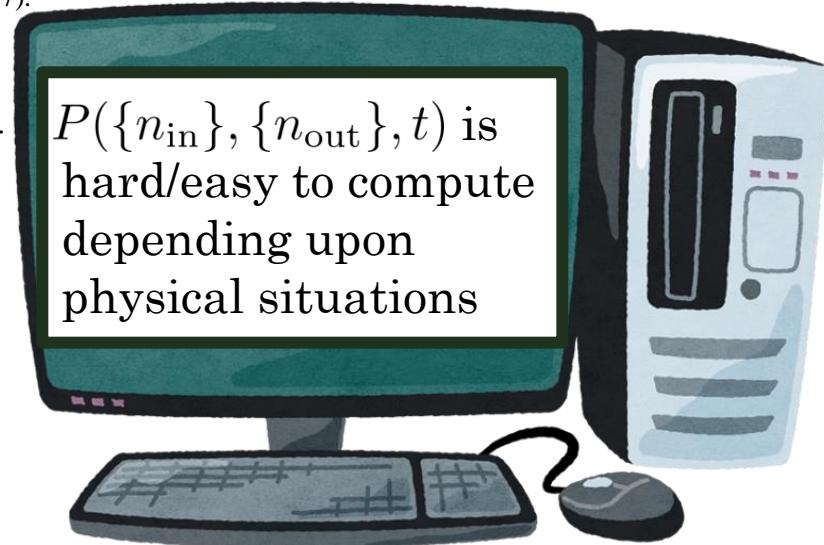
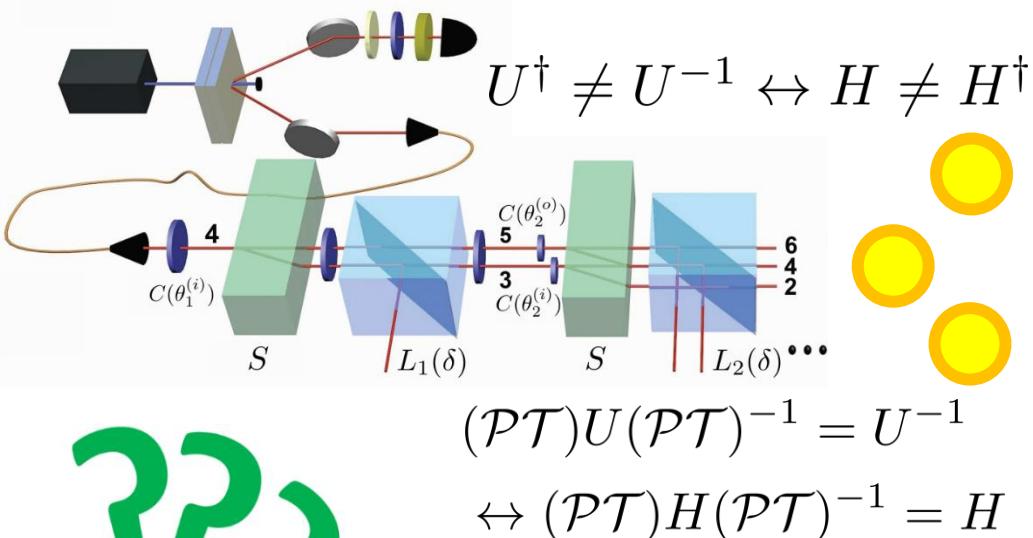
versatile platform of
boson sampling problem

relation between them?

unique phenomenon in
open quantum systems

Open photonic systems can exhibit PT symmetry breaking transition

L. Xiao, X. Zhan, Z. H. Bian, K. K. Wang, X. Zhang, X. P. Wang, J. Li, **Ken Mochizuki**, D. Kim,
N. Kawakami, W. Yi, H. Obuse, B. C. Sanders, and P. Xue, Nature Physics **13**, 1117 (2017).



computational complexity in PT symmetric open systems?

Outline

- post-selected quantum systems
- PT symmetry breaking
- photonic experiments
- boson sampling problem
- motivation

(0) model and PT symmetry breaking ○

(1) short-time dynamical complexity transition

(2) long-time dynamical complexity transition

Model

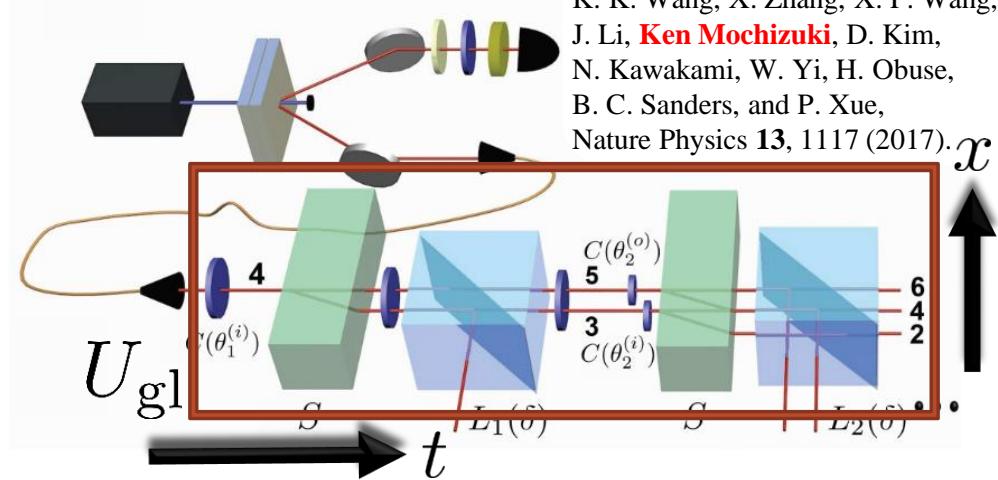
$$|\psi(t+1)\rangle = U_{\text{gl}} |\psi(t)\rangle = e^{-iH_{\text{gl}}} |\psi(t)\rangle$$

$$|\psi(t)\rangle = \sum_{x=1}^X \begin{pmatrix} \psi_h(x, t) \\ \psi_v(x, t) \end{pmatrix} |x\rangle \quad U_{\text{gl}} = C(\theta_1/2) S G(+\gamma) C(\theta_2) G(-\gamma) S C(\theta_1/2)$$

$$P(\{n_{\text{in}}\}, \{n_{\text{out}}\}, t) \quad i = (x, h/v)$$

$$= |\text{Per}[\mathcal{U}_{\text{in,out}}(t)]|^2 / N(t) \prod_i n_i^{\text{in}}! n_i^{\text{out}}!$$

L. Xiao, X. Zhan, Z. H. Bian,
 K. K. Wang, X. Zhang, X. P. Wang,
 J. Li, **Ken Mochizuki**, D. Kim,
 N. Kawakami, W. Yi, H. Obuse,
 B. C. Sanders, and P. Xue,
Nature Physics **13**, 1117 (2017).



Parity-Time (PT) symmetry :

$$(\mathcal{PT})H_{\text{gl}}(\mathcal{PT})^{-1} = H_{\text{gl}}, \quad \mathcal{P} = \sum_x | -x \rangle \langle x | \otimes \sigma_1, \quad \mathcal{T} = \sum_x | x \rangle \langle x | \otimes \sigma_2 \mathcal{K}$$

wave plates : changes of polarizations

$$C(\theta) = \sum_{x=1}^X |x\rangle \langle x| \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

beam displacers : position shift of photons

$$S = \sum_{x=1}^X \begin{pmatrix} |x-1\rangle \langle x| & 0 \\ 0 & |x+1\rangle \langle x| \end{pmatrix}$$

partially polarizing beam splitters : polarization dependent photon losses
 ⇒ nonunitary dynamics $U_{\text{gl}}^\dagger \neq U_{\text{gl}}^{-1}$

$$G(\gamma) = \sum_{x=1}^X |x\rangle \langle x| \begin{pmatrix} e^{+\gamma} & 0 \\ 0 & e^{-\gamma} \end{pmatrix}$$

$\mathcal{P} : +\gamma \rightarrow -\gamma, \mathcal{T} : -\gamma \rightarrow +\gamma$

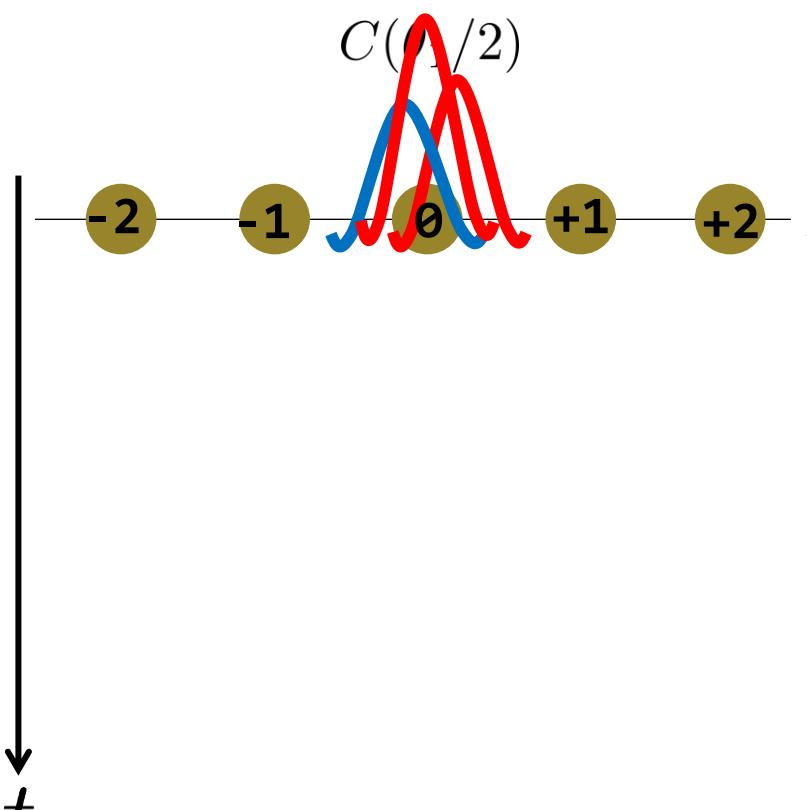
Single-particle dynamics

$$|\psi(t+1)\rangle = U_{\text{gl}} |\psi(t)\rangle$$

$$U_{\text{gl}} = C(\theta_1/2) S G(+\gamma) C(\theta_2) G(-\gamma) S C(\theta_1/2)$$

initial state

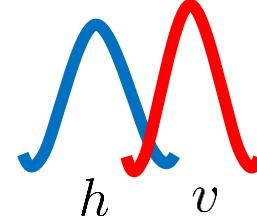
$$C(\theta_1/2)$$



$$C(\theta) = \sum_{x=1}^X |x\rangle \langle x| \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

$$S = \sum_{x=1}^X \begin{pmatrix} |x-1\rangle \langle x| & 0 \\ 0 & |x+1\rangle \langle x| \end{pmatrix}$$

$$G(\gamma) = \sum_{x=1}^X |x\rangle \langle x| \begin{pmatrix} e^{+\gamma} & 0 \\ 0 & e^{-\gamma} \end{pmatrix}$$

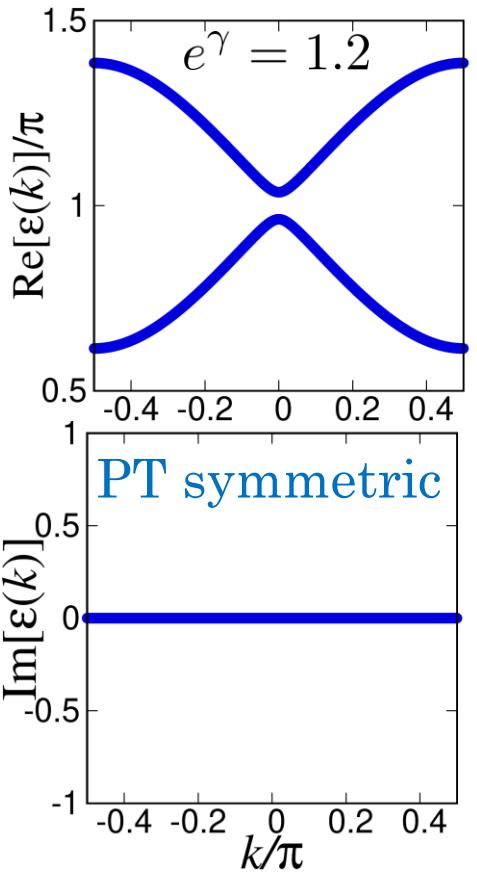


$$|\psi(t)\rangle = \sum_{x=1}^X \begin{pmatrix} \psi_h(x, t) \\ \psi_v(x, t) \end{pmatrix} |x\rangle$$

PT symmetry breaking

Ken Mochizuki, D. Kim, and H. Obuse, Physical Review A **93**, 062116 (2016).

$$\tilde{H}_{\text{gl}}(k) |\phi_{\pm}(k)\rangle = \varepsilon_{\pm}(k) |\phi_{\pm}(k)\rangle$$



$$\tilde{H}_{\text{gl}}(k) = i \log[\tilde{U}_{\text{gl}}(k)]$$

$$U_{\text{gl}} = \sum_k |k\rangle \langle k| \otimes \tilde{U}_{\text{gl}}(k)$$

$$\tilde{H}_{\text{gl}}^\dagger(k) \neq \tilde{H}_{\text{gl}}(k)$$

$$(\text{PT}) \tilde{H}_{\text{gl}}(k) (\text{PT})^{-1} = \tilde{H}_{\text{gl}}(k)$$

$$\text{PT} = \sigma_3 \mathcal{K}$$

$$e^{-i\varepsilon_{\pm}(k)} = d(k) \pm \sqrt{d^2(k) - 1}$$

$$d(k) = \text{tr}[\tilde{U}(k)]/2$$

PT symmetry breaking
crucial effects upon
computational complexity!

threshold :

$\text{PT} |\phi_{\pm}(k)\rangle = |\phi_{\pm}(k)\rangle$ $\text{PT} |\phi_{\pm}(k)\rangle \neq |\phi_{\pm}(k)\rangle$

γ

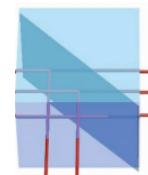
$$d(k=0) = -1 \rightarrow e^{\gamma \text{PT}} \simeq 1.22$$

$$\theta_1 = 0.65\pi, \theta_2 = 0.25\pi$$

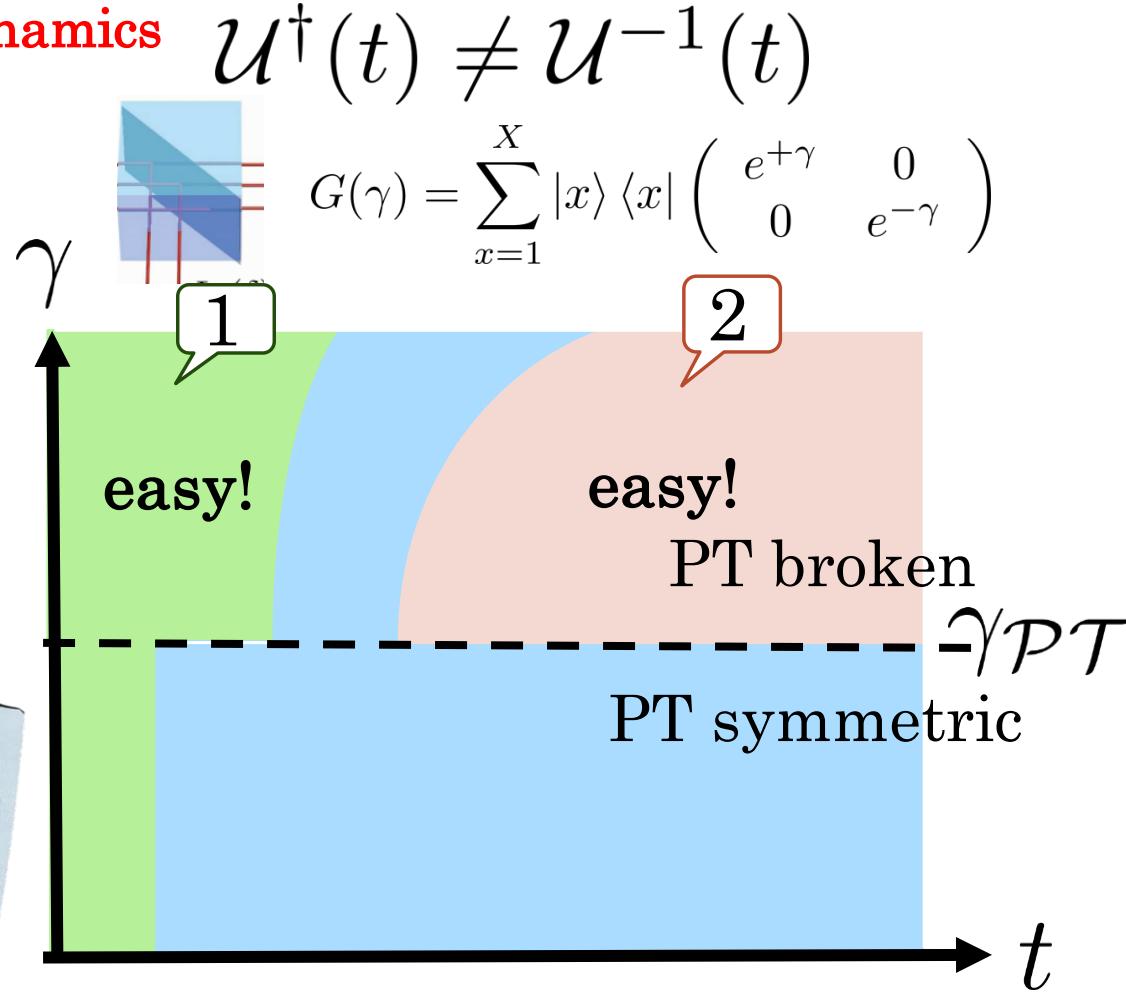
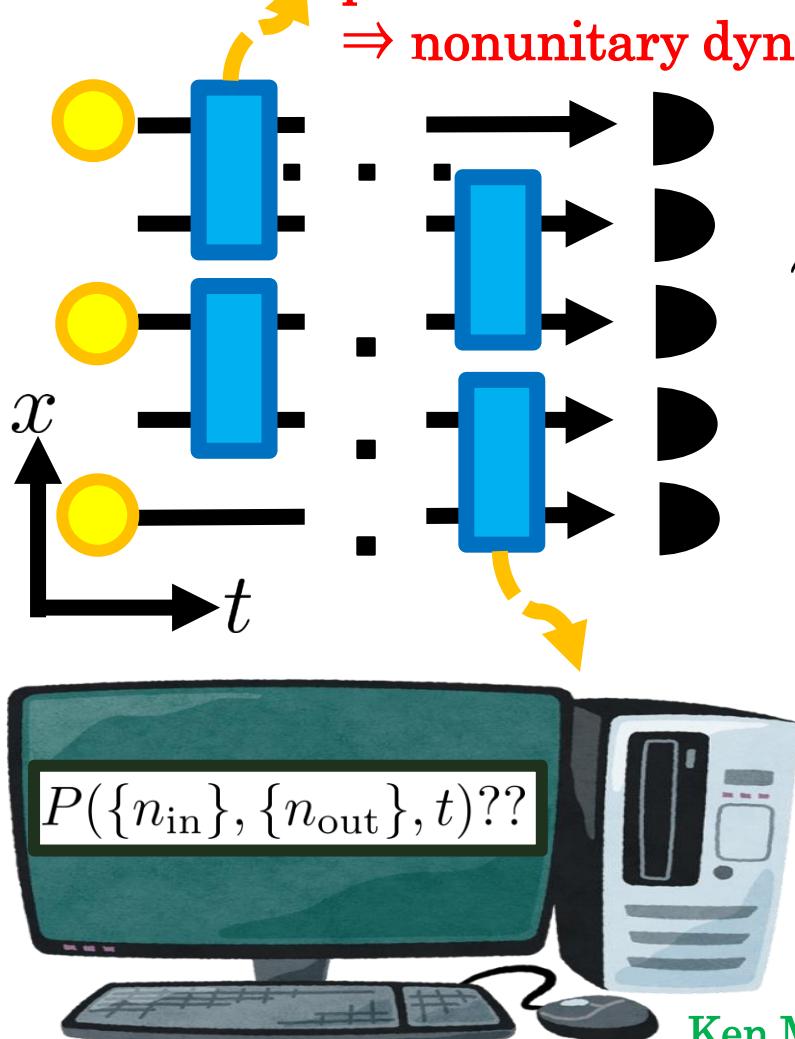
Phase diagram

photon loss effect & postselection
⇒ nonunitary dynamics

$$\mathcal{U}^\dagger(t) \neq \mathcal{U}^{-1}(t)$$



$$G(\gamma) = \sum_{x=1}^X |x\rangle \langle x| \begin{pmatrix} e^{+\gamma} & 0 \\ 0 & e^{-\gamma} \end{pmatrix}$$



Outline

- post-selected quantum systems
- PT symmetry breaking
- photonic experiments
- boson sampling problem
- motivation

(0) model and PT symmetry breaking

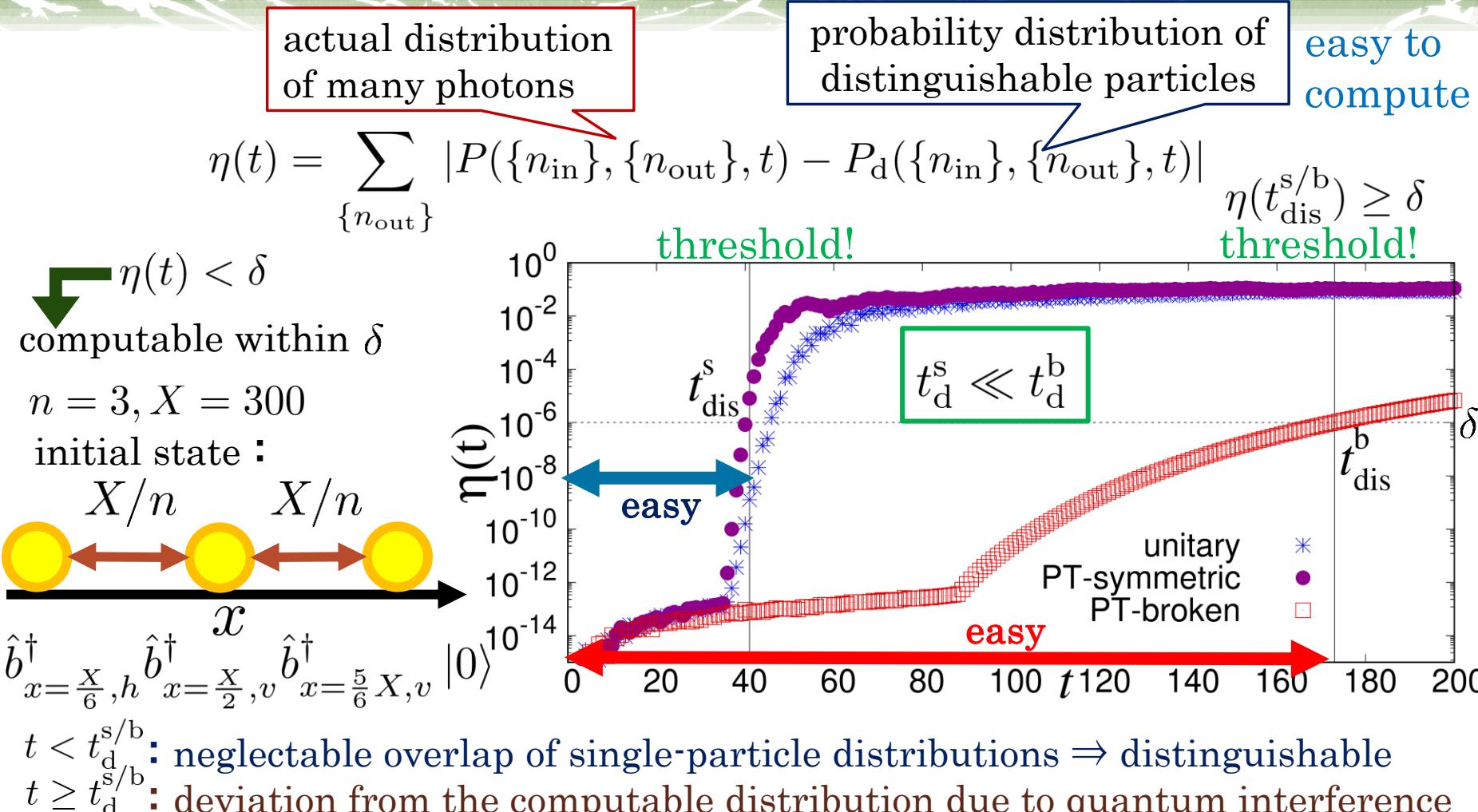
(1) short-time dynamical complexity transition



(2) long-time dynamical complexity transition

Short-time dynamical complexity transition (1)

[Ken Mochizuki](#) and Ryusuke Hamazaki, arXiv: 2207.12624.

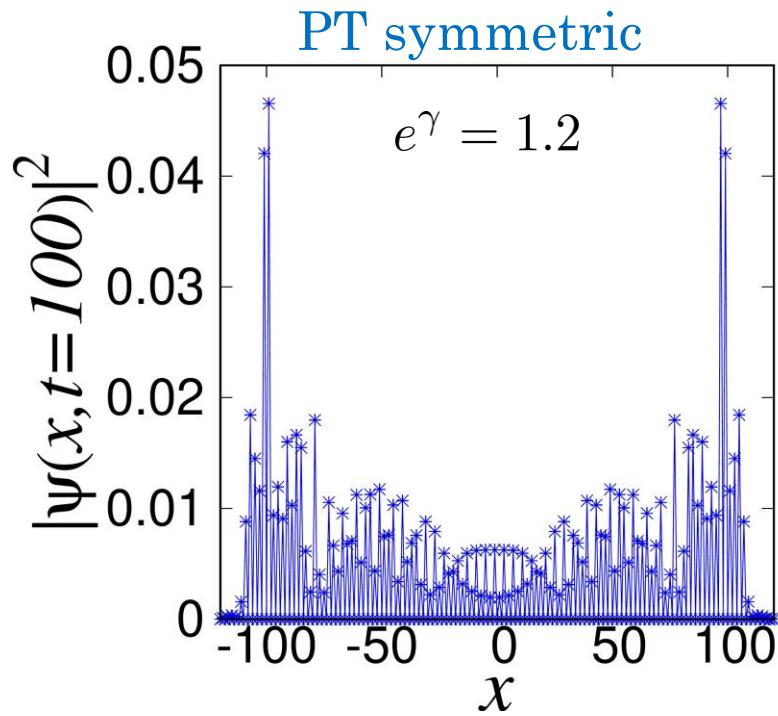


PT symmetry breaking enlarges the computable region

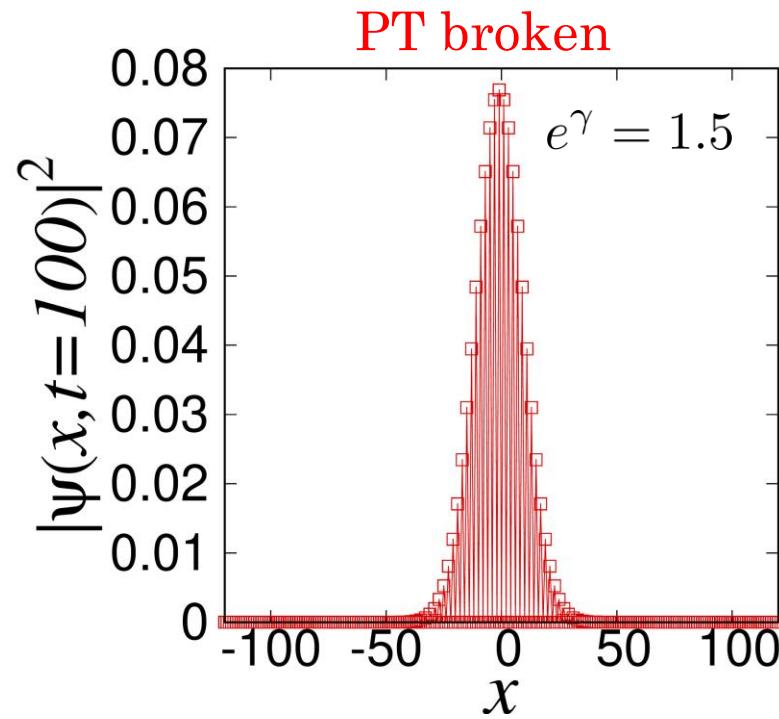
Origin of the enlargement for the computable region

Ken Mochizuki and Ryusuke Hamazaki, arXiv: 2207.12624.

single-particle dynamics with $|\psi(t=0)\rangle = |x=0\rangle \otimes (|h\rangle + i|v\rangle)/\sqrt{2}$



ballistic $\sqrt{\langle x^2(t) \rangle - \langle x(t) \rangle^2} \propto t$



diffusive $\sqrt{\langle x^2(t) \rangle - \langle x(t) \rangle^2} \propto \sqrt{t}$

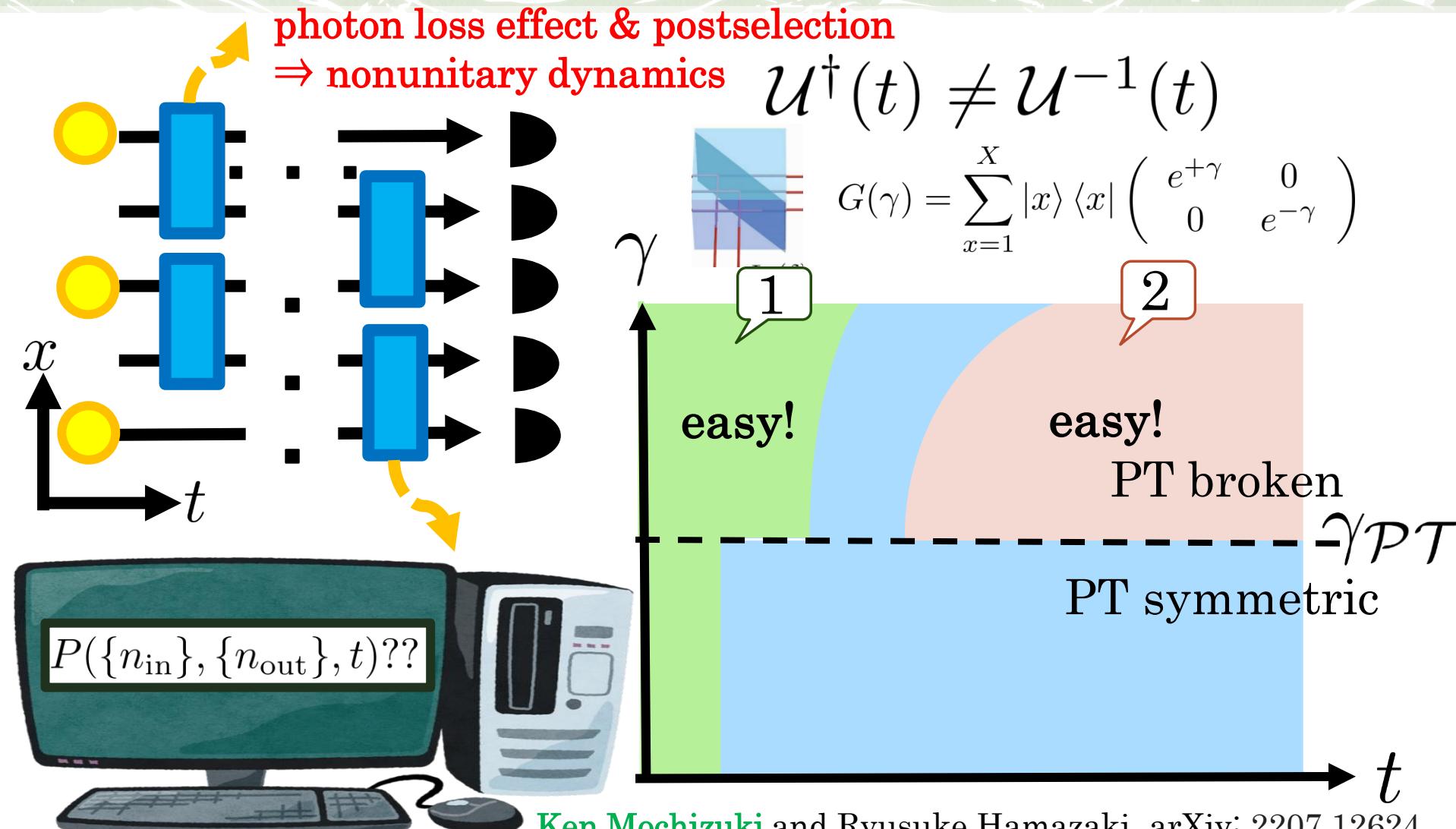
PT symmetry breaking leads to diffusive dynamics
⇒ prolongation of transition times $t_d^s \ll t_d^b$

Diffusive dynamics can
be derived analytically.

Outline

- post-selected quantum systems
 - PT symmetry breaking
 - photonic experiments
 - boson sampling problem
 - motivation
- (0) model and PT symmetry breaking
- (1) short-time dynamical complexity transition
- (2) long-time dynamical complexity transition

Phase diagram



Long-time dynamical complexity transition (2)

[Ken Mochizuki](#) and Ryusuke Hamazaki, arXiv: 2207.12624.

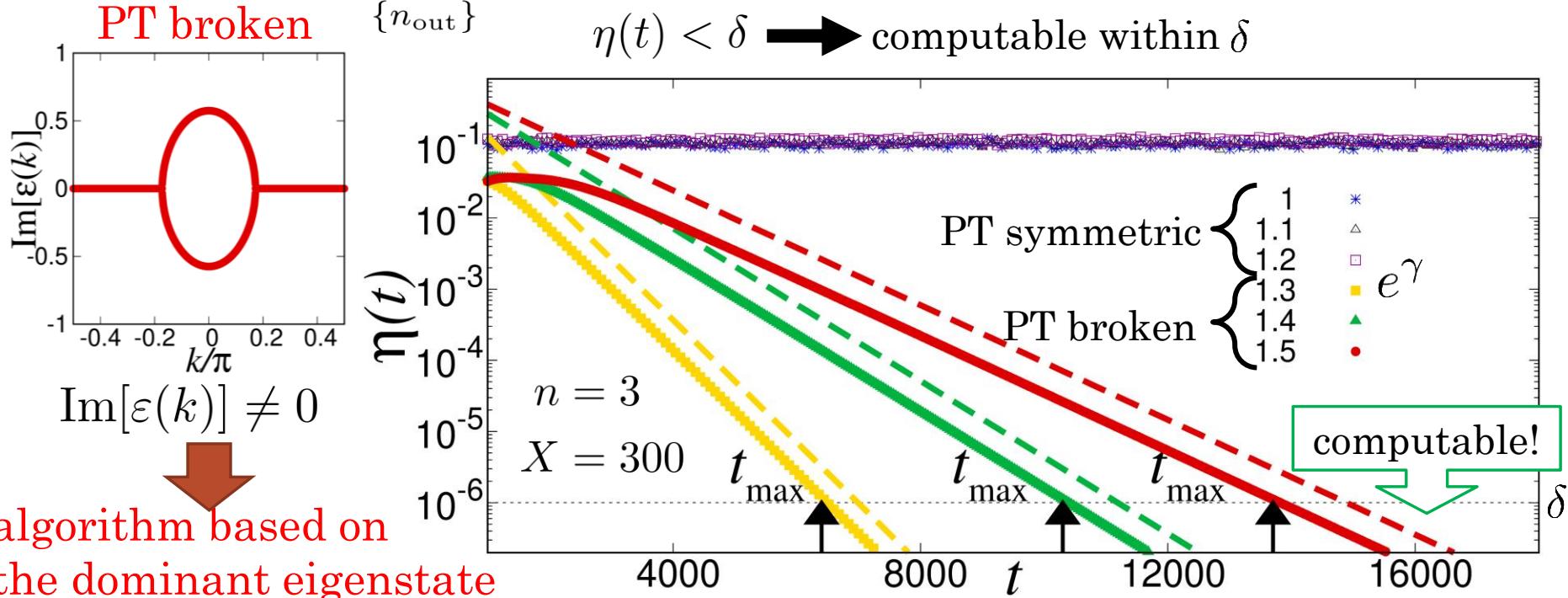
$$P_d(\{n_{\text{in}}\}, \{n_{\text{out}}\}, t) = \frac{\sum_{\omega} \prod_{p=1}^n |\mathcal{U}_{\text{in}_p \text{out}_{\omega(p)}}(t)|^2}{N_d(t) \prod_{j=1}^Y n_j^{\text{in}}! n_j^{\text{out}}!}$$

probability distribution of
distinguishable particles
easy to compute

$$\eta(t) = \sum_{\{n_{\text{out}}\}} |P(\{n_{\text{in}}\}, \{n_{\text{out}}\}, t) - P_d(\{n_{\text{in}}\}, \{n_{\text{out}}\}, t)|$$

PT broken

$\eta(t) < \delta \rightarrow$ computable within δ



PT symmetry breaking \Rightarrow The distribution becomes computable again

Analysis based on the dominant state

[Ken Mochizuki](#) and Ryusuke Hamazaki, arXiv: 2207.12624.

$$P(\{n_{\text{in}}\}, \{n_{\text{out}}\}, t) = |\text{Per}[W(t)]|^2 / N(t) \prod_j n_j^{\text{in}}! n_j^{\text{out}}! \quad \text{approximation based on the dominant eigenstate}$$

ubiquitous in non-unitary dynamics!

$$H_{\text{gl}} |\phi_l^R\rangle = \varepsilon_l |\phi_l^R\rangle, \langle \phi_l^L | H_{\text{gl}} = \varepsilon_l \langle \phi_l^L |$$

$$\mathcal{U}^T(t) = e^{-iH_{\text{gl}}t} \simeq e^{-i\varepsilon_m t} |\phi_m^R\rangle \langle \phi_m^L|, \text{Im}(\varepsilon_m) = \max_l \text{Im}(\varepsilon_l)$$

$$W_{pq}(t) = [\mathcal{U}(t)]_{\text{in}_p \text{out}_q} \simeq e^{-i\varepsilon_m t} \langle \text{out}_q | \phi_m^R \rangle^* \langle \phi_m^L | \text{in}_p \rangle^*$$

 $\text{Per}[W(t)] \simeq e^{-i\varepsilon_m n t} n! \prod_{p=1}^n \langle \text{out}_p | \phi_m^R \rangle^* \langle \phi_m^L | \text{in}_p \rangle^*$

in_p: p-th input state
out_q: q-th output state

$$\begin{aligned} P(\{n_{\text{in}}\}, \{n_{\text{out}}\}, t) &\simeq P_m(\{n_{\text{out}}\}) \simeq \underline{P_d(\{n_{\text{in}}\}, \{n_{\text{out}}\}, t)} \text{ computable} \\ &= \frac{\prod_{p=1}^n |\langle \text{out}_p | \phi_m^R \rangle|^2}{N_m \prod_{j=1}^Y n_j^{\text{out}}!}, \quad N_m = \frac{\langle \phi_m^R | \phi_m^R \rangle^n}{n!} \end{aligned}$$

In the PT broken phase, the true distribution approaches a distribution computable through the dominant eigenstate

Summary

[Ken Mochizuki](#) and Ryusuke Hamazaki, arXiv: 2207.12624.

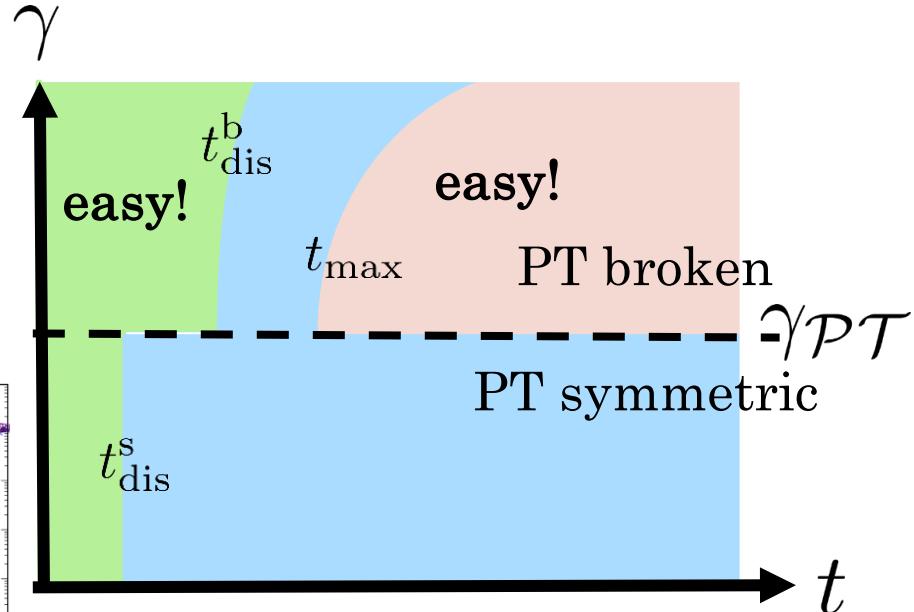
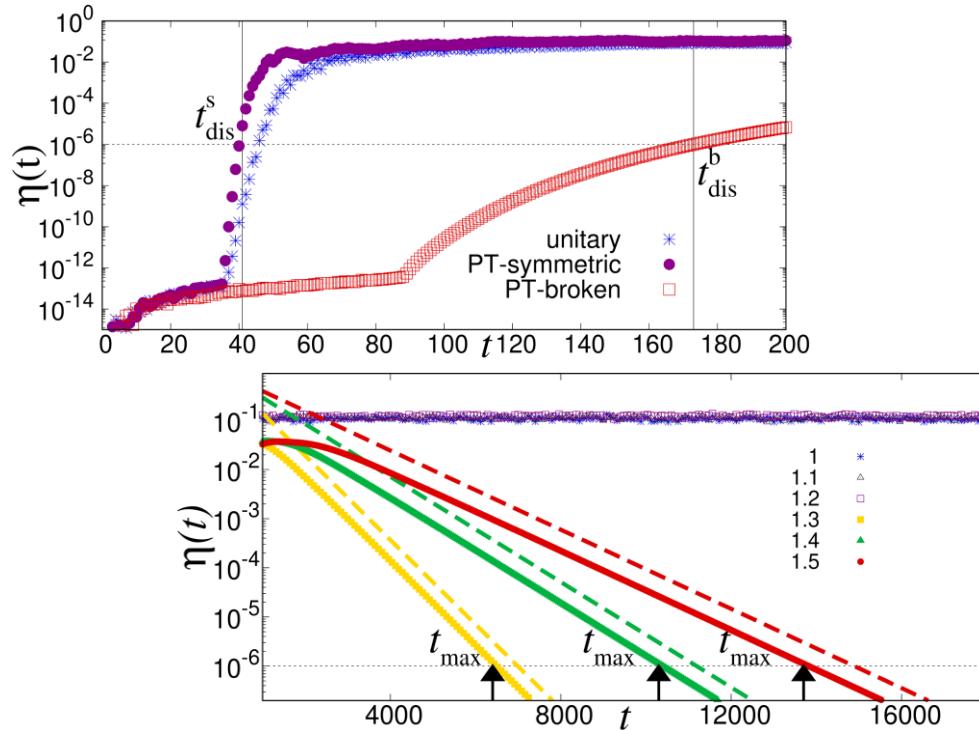
extending single-photonic non-Hermitian system into many-photon system

→ computational complexity of boson sampling problem

PT symmetry breaking

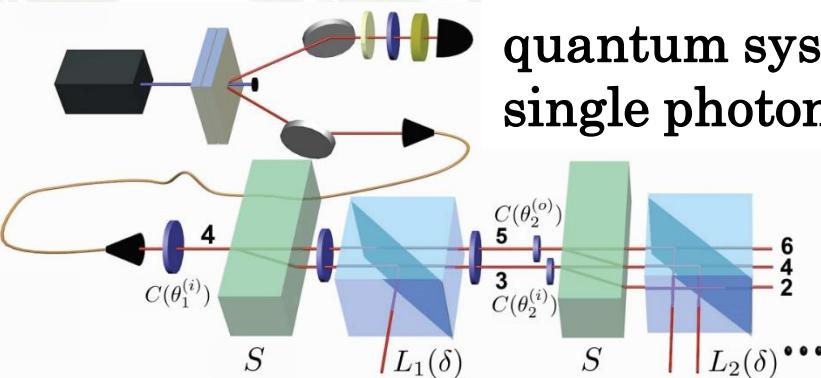
enhancement of classical nature

-
- prolongation of the threshold for the short-time complexity transition
 - long-time complexity transition unique to open systems



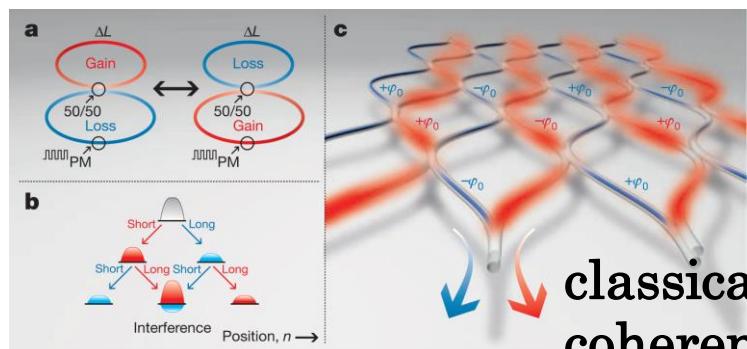
easy \Leftrightarrow classically computable

Optical experiments in both quantum and classical systems



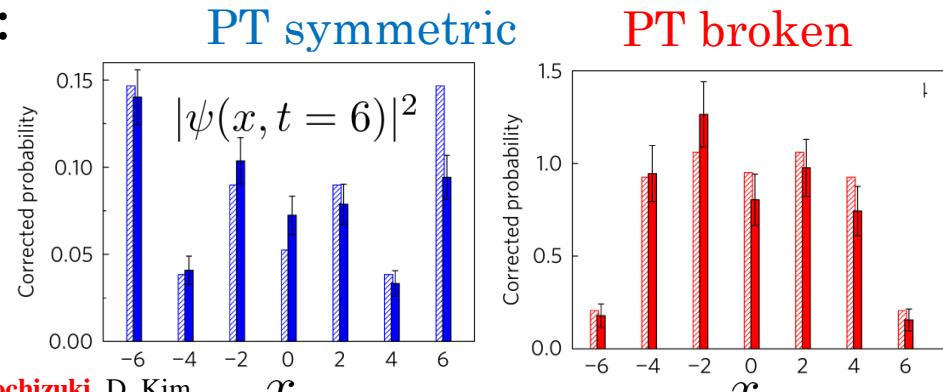
quantum system:
single photons

L. Xiao, X. Zhan, Z. H. Bian, K. K. Wang, X. Zhang, X. P. Wang, J. Li, **Ken Mochizuki**, D. Kim, N. Kawakami, W. Yi, H. Obuse, B. C. Sanders, and P. Xue, Nature Physics **13**, 1117 (2017).



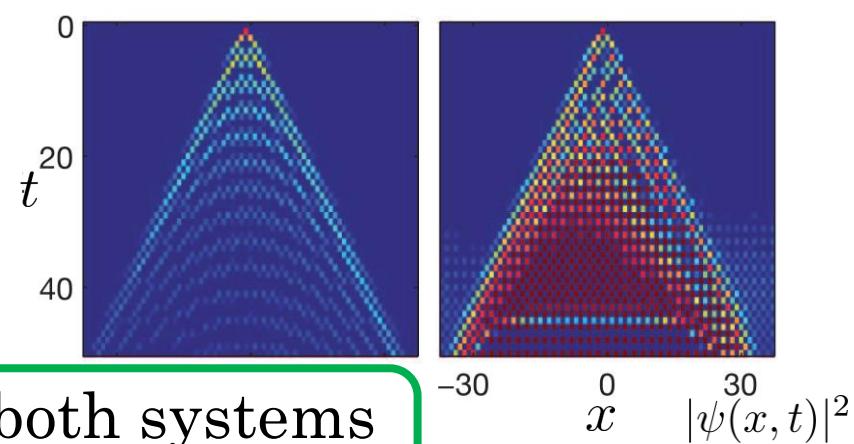
classical system:
coherent light

A. Regensburger, C. Bersch, M. A. Miri, G. Onishchukov, D. N. Christodoulides, and U. Peschel, Nature **488**, 167 (2012)



probability distribution of single photons
 \Leftrightarrow intensity configuration of coherent light

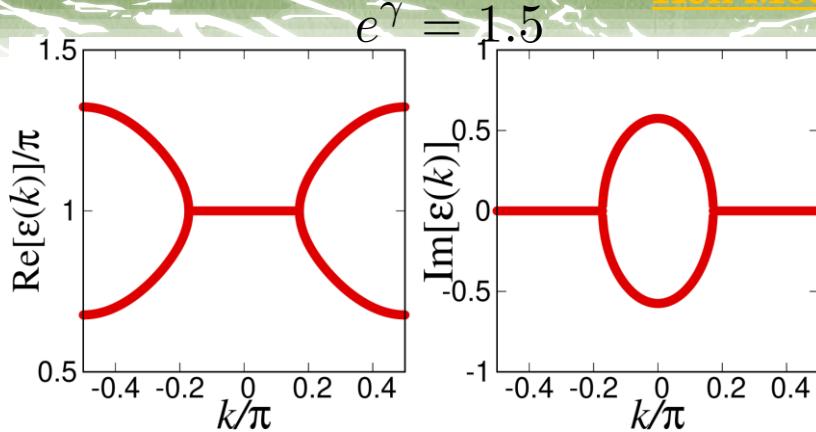
*Formulations
are different



Results are similar in both systems
 \Rightarrow Is there a distinct feature?

Derivation of the diffusive dynamics

[Ken Mochizuki](#) and Ryusuke Hamazaki, arXiv: 2207.12624.



$$U_{\text{gl}}^t(k) = \sum_s e^{-i\varepsilon_s(k)t} |\phi_s^R(k)\rangle \langle \phi_s^L(k)|$$

$$\varepsilon_\pm(k) \simeq \varepsilon_\pm(k=0) \pm i \frac{D}{2} k^2, D > 0$$

$|\psi(t=0)\rangle = |x=0\rangle |\sigma_0\rangle$ quadratic dispersion

$$N(t) = \int_{-\pi}^{\pi} dk \langle \sigma_0 | [U_{\text{gl}}^\dagger(k)]^t U_{\text{gl}}^t(k) | \sigma_0 \rangle$$

$$\langle x^r(t) \rangle = \frac{\sum_x x^r |\psi(x,t)|^2}{\sum_x |\psi(x,t)|^2} = \frac{(-i)^r}{N(t)} \int_{-\pi}^{\pi} dk \langle \sigma_0 | [U_{\text{gl}}^\dagger(k)]^t \frac{d^r}{dk^r} U_{\text{gl}}^t(k) | \sigma_0 \rangle$$

extraction of leading terms

$$\sim \frac{(-i)^r \int_{-\infty}^{+\infty} dk e^{-\frac{Dk^2}{2}t} \frac{d^r}{dk^r} e^{-\frac{Dk^2}{2}t}}{\int_{-\infty}^{+\infty} dk e^{-Dk^2t}} = \frac{i^r \int_{-\infty}^{+\infty} dk e^{-Dk^2t} \left(\frac{Dt}{2}\right)^{\frac{r}{2}} H_r \left(\sqrt{\frac{Dt}{2}}k\right)}{\int_{-\infty}^{+\infty} dk e^{-Dk^2t}}$$

extension of the integral range

$$\langle \exp[\xi x(t)] \rangle \simeq \frac{\int_{-\infty}^{+\infty} dk \exp\left(-Dk^2t + \frac{Dt}{2}\xi^2 + i\xi Dt k\right)}{\int_{-\infty}^{+\infty} dk \exp(-Dk^2t)} = \exp\left[\frac{\xi^2}{2} \left(\sqrt{\frac{Dt}{2}}\right)^2\right]$$

$$H_r \left(\sqrt{\frac{Dt}{2}}k\right) : \text{Hermite Polynomial} \quad \sum_{r=0}^{\infty} H_r(y) z^r / r! = \exp(2yz - z^2) : \text{generating function}$$

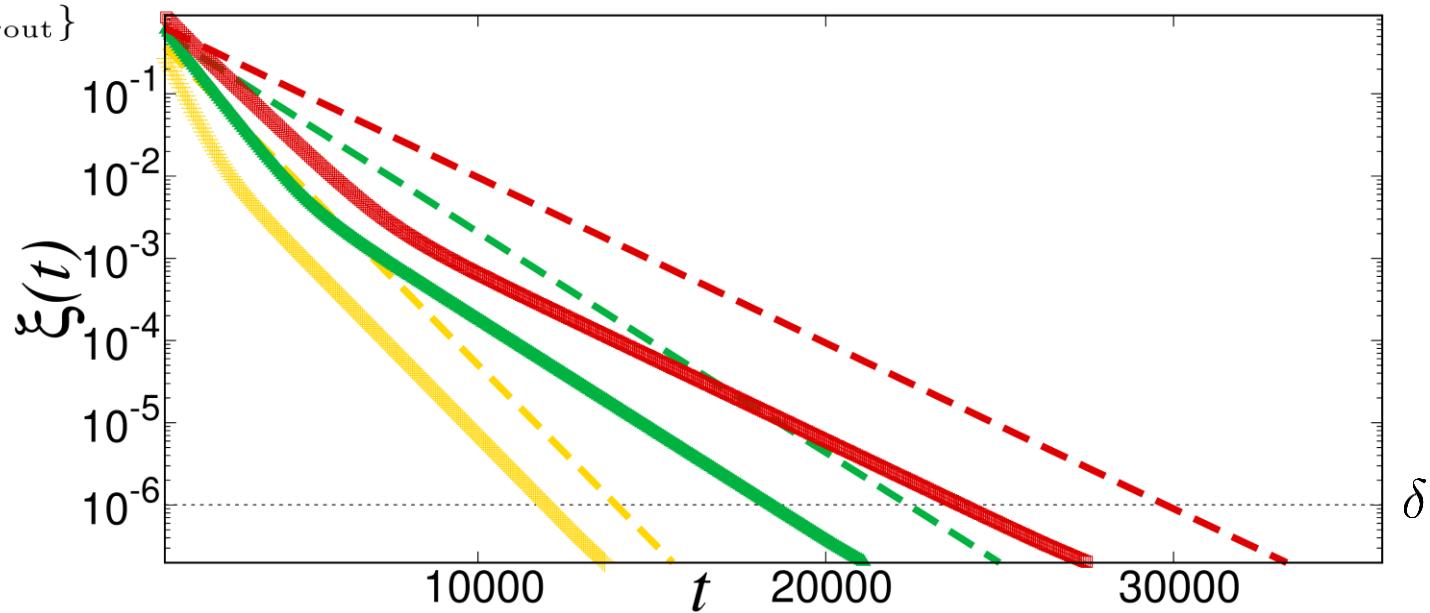
Moment generating function corresponds to that of Gaussian distribution

Numerical confirmation

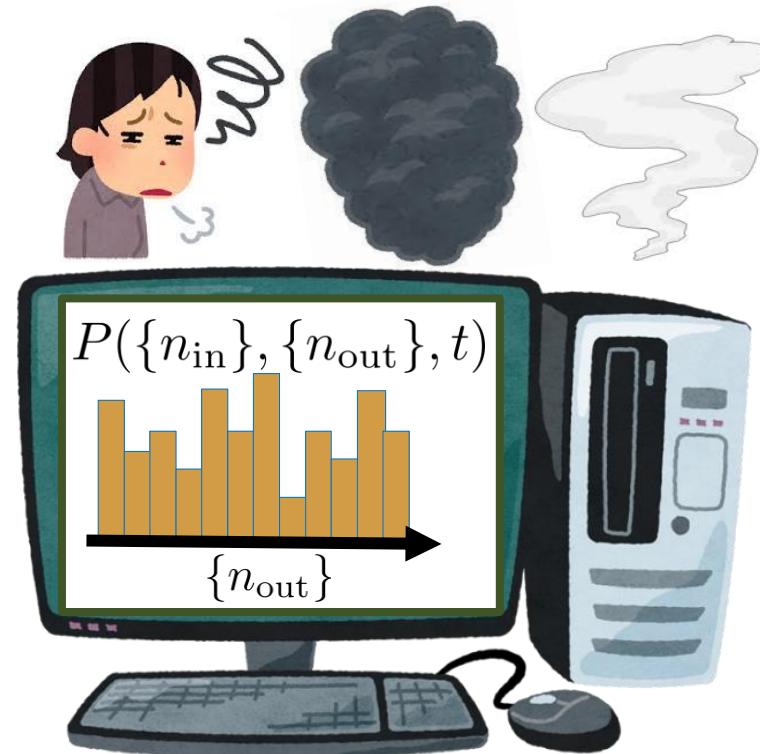
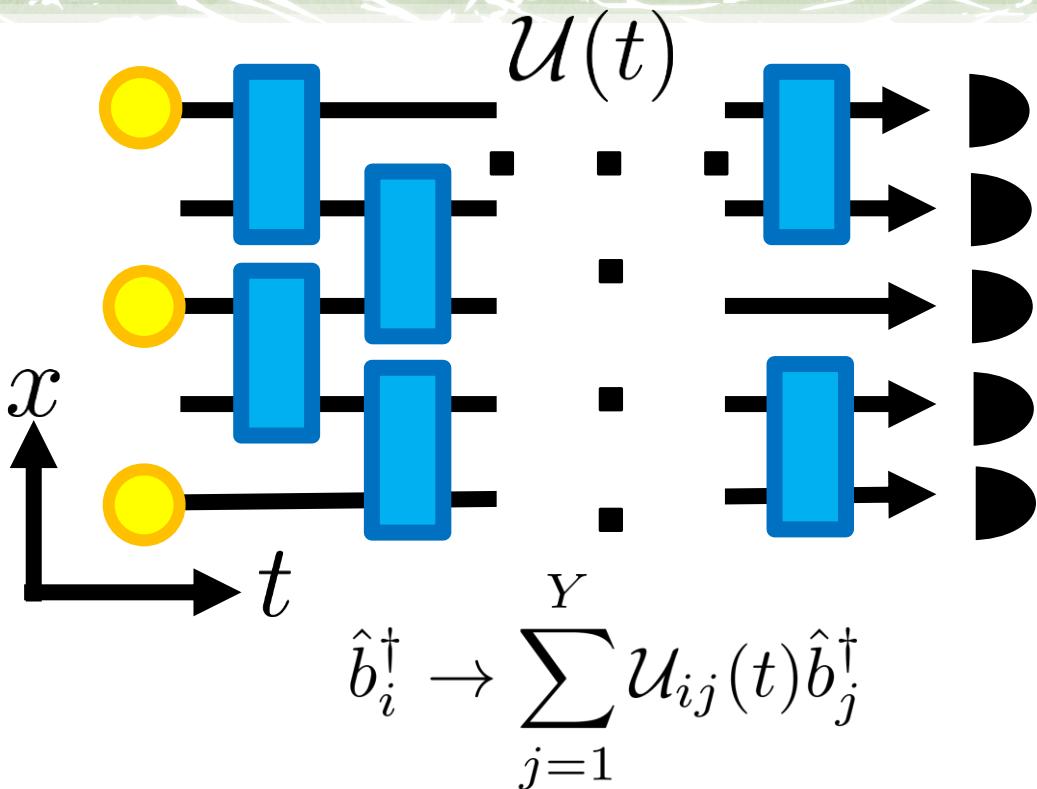
$$P_m(\{n_{\text{out}}\}) = \frac{\prod_{p=1}^n |\langle \text{out}_p | \phi_m^R \rangle|^2}{N_m \prod_{j=1}^Y n_j^{\text{out}}!}, \quad N_m = \frac{\langle \phi_m^R | \phi_m^R \rangle^n}{n!}$$

$$\lim_{t \rightarrow \infty} P(\{n^{\text{in}}\}, \{n^{\text{out}}\}, t) = \underline{P_m(\{n^{\text{out}}\})} = \lim_{t \rightarrow \infty} \underline{P_d(\{n^{\text{in}}\}, \{n^{\text{out}}\}, t)}$$

$$\xi(t) = \sum_{\{n_{\text{out}}\}} |P(\{n_{\text{in}}\}, \{n_{\text{out}}\}, t) - P_m(\{n_{\text{out}}\})| \quad \text{computable}$$



Outlook



Relation between the computational complexity
and the **entanglement entropy**??
Behavior in more **general non-unitary dynamics**??
(time-dependent, without postselection, • • •)

