Unconventional Symmetries in Many-Body Physics

Sanjay Moudgalya

Walter Burke Institute of Theoretical Physics
California Institute of Technology

SM, Olexei I. Motrunich $\left\{ \begin{array}{l} \text{arXiv: 2108.10824 [PRX 12, 011050 (2022)]} \\ \text{arXiv: 2209.03370, arXiv: 2209.03377} \end{array} \right.$

14th November 2022

Symmetries in Quantum Many-Body Physics

- Lattice system with a finite-dimensional tensor product Hilbert space $\mathcal{H} = \bigotimes_{j=1}^{L} \mathbb{C}^d$ with a Hamiltonian H
- Symmetries/conserved quantities $\{Q_\alpha\}$ are defined as operators that commute with the Hamiltonian, $[H, Q_\alpha] = 0$.
- Conventionally, additional structure is imposed on $\{Q_\alpha\}$.
- Internal symmetries: On-site unitary representations of a (Lie) group G

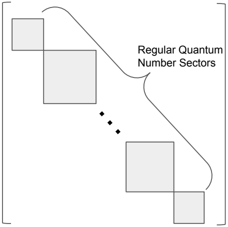
$$Q_\alpha = \hat{u}(g) \otimes \hat{u}(g) \otimes \cdots \otimes \hat{u}(g), \text{ e.g. } \hat{u}(g) = \begin{cases} e^{i\alpha Z} & \text{if } G = U(1) \\ e^{i\vec{\alpha}\cdot\vec{\sigma}} & \text{if } G = SU(2) \end{cases}$$

- Continuous symmetries: Conserved quantities are typically sums of local operators, e.g. total charge, number of domain walls, etc.
- Lattice symmetries: Unitary operators that implement translation, rotation, reflection, etc.

Symmetries in Quantum Many-Body Physics

- Symmetric Hamiltonians can be block-diagonalized into symmetry quantum number sectors.

- Sectors are uniquely labelled by eigenvalues under (a maximally commuting subset of) the $\{Q_\alpha\}$.



- Various generalizations of conventional symmetries are under active exploration: Categorical symmetries, MPO symmetries, etc.[1]

- This talk: Different (?) generalization motivated by recent work on the dynamics of certain quantum systems.

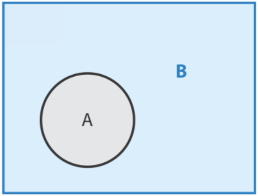
[1] J.McGreevy (2022)

Quantum Many-Body Dynamics and Symmetries

Ergodicity in Isolated Quantum Systems

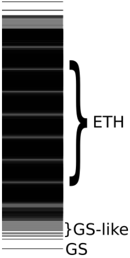
- A quantum Hamiltonian is said to be ergodic if *any* initial state $|\psi(0)\rangle$ evolves into a "thermal" state $|\psi(t)\rangle = e^{-iHt}\,|\psi(0)\rangle$

- Reduced density matrix of a thermal state is the Gibbs density matrix of the subsystem



$$\rho = |\psi\rangle\,\langle\psi|\,, \quad \rho_A = \mathrm{Tr}_B\,(\rho)\,, \quad \rho_A \sim e^{-\beta H|_A}$$

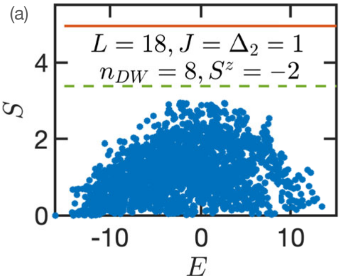
- Eigenstate Thermalization Hypothesis (ETH): Eigenstates $|E_n\rangle$ in the middle of the spectrum are thermal[2]
  - Entanglement entropy obeys a **volume law** $S \sim \log D \sim L$
  - Eigenstate properties are a "smooth" function of energy

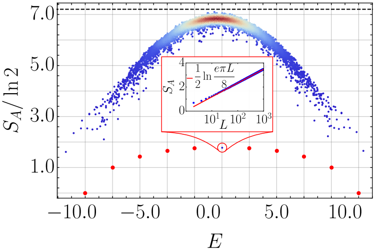


[2] J.M.Deutsch (1991), M. Srednicki (1994)

Ergodicity in Symmetric Isolated Quantum Systems

- With symmetries: $\rho_A \sim e^{-\beta(H-\mu N)|_A}$ ETH should hold for eigenstates *within* each symmetry sector
- Recent analytical and experimental progress has identified two new types of "weak" violations[3]
  - Hilbert Space Fragmentation
  - Quantum Many-Body Scars
- Violations can be seen in several diagnostics, e.g., entanglement entropy of the eigenstates.[4]





[2]M.Serbyn, D.A.Abanin, Z.Papic (2020); **SM**, B.A.Bernevig, N.Regnault (2021)

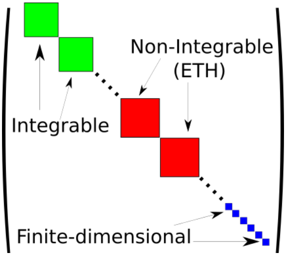
[4]Z.C.Yang, F.Liu, A.V.Gorshkov, T.Iadecola (2020); M.Schecter, T.Iadecola (2019)

Hilbert Space Fragmentation

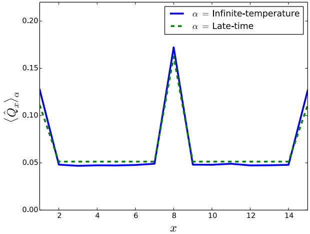
- Dynamics under certain local Hamiltonians splits the Hilbert space into *exponentially many* dynamically disconnected subspaces $\{\mathcal{K}(H, |R_\alpha\rangle)\}$, $|R_\alpha\rangle$ being product states

$$\mathcal{H} = \bigoplus_{j=1}^{K\sim\exp(L)} \mathcal{K}\left(H, |R_\alpha\rangle\right)$$

$$\mathcal{K}\left(H, |R\rangle\right) = \mathrm{span}_t \left\{ e^{-iHt} |R\rangle \right\}$$



- Different subspaces are *not distinguished* by obvious symmetry quantum numbers, can show vastly different properties![5]
- Initial product states never thermalize w.r.t. the full Hilbert space due to "hidden" blocks *after resolving known symmetries*



[5] **SM**, A.Prem, R.Nandkishore, N.Regnault, B.A.Bernevig (2019)

Hilbert Space Fragmentation

- Fragmentation *generically* occurs in one dimensional systems conserving dipole moment ($\sum_j jS_j^z$ with OBC)[6,7]
- Example: spin-1 dipole conserving Hamiltonian that implements the following rules ($H = \sum_j (S_{j-1}^-(S_j^+)^2 S_{j+1}^- + h.c.)$)

$$|+-0\rangle \leftrightarrow |0+-\rangle\,, \;\; |0-+\rangle \leftrightarrow |-+0\rangle$$
$$|+-+\rangle \leftrightarrow |0+0\rangle\,, \;\; |-+-\rangle \leftrightarrow |0-0\rangle$$

- Exponentially many one-dimensional subspaces ("frozen" eigenstates)

$$|++--\cdots++--\rangle\,, \;\; |0++0++\cdots0++\rangle$$

- Subspaces with non-local conserved quantities, e.g. a product state $|0\cdots0+0\cdots0\rangle$ can only evolve to states with "string-order" $|0\cdots0+0\cdots0-0\cdots0+\cdots0\rangle$

---

[6] P.Sala, T.Rakovszky, R.Verresen, M.Knap, F.Pollmann (2019)

[7] V.Khemani, M.Hermele, R.Nandkishore (2019)

Quantum Many-Body Scars

- Non-integrable models with quasiparticle towers of eigenstates deep in the spectrum have been discovered[8]
- AKLT spin chain:[9] $\mathcal{P} = \sum_j (-1)^j (S_j^+)^2$, states with N quasiparticles dispersing with $k = \pi$ are **exact eigenstates** for finite system sizes L!

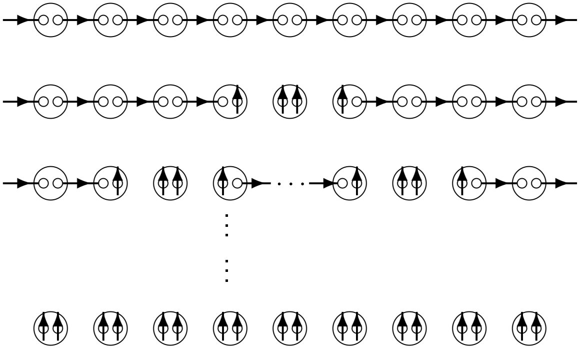
$$|G\rangle$$

$$|S_2\rangle = \mathcal{P}\,|G\rangle$$

$$|S_4\rangle = \mathcal{P}^2\,|G\rangle$$

$$\vdots$$

$$|S_L\rangle = \mathcal{P}^{\frac{L}{2}}\,|G\rangle = |F\rangle$$

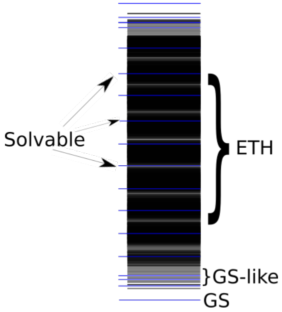


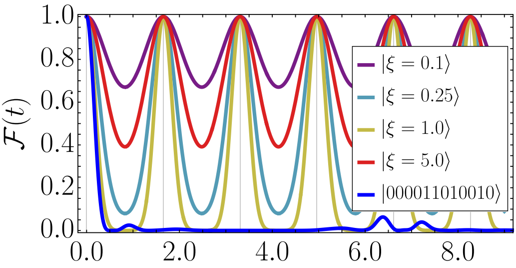
[8] **SM**, B.A.Bernevig, N.Regnault (2021)

[9] D.P.Arovas (1989); **SM**, S.Rachel, B.A.Bernevig, N.Regnault (2017)

Quantum Many-Body Scars

- States have entanglement entropy $S \sim \log L \implies$ Violation of Strong ETH!
- Equally spaced tower: leads to exact revivals from simple initial states[10]
- Alternate view: Existence of small dynamically disconnected subspace[11]



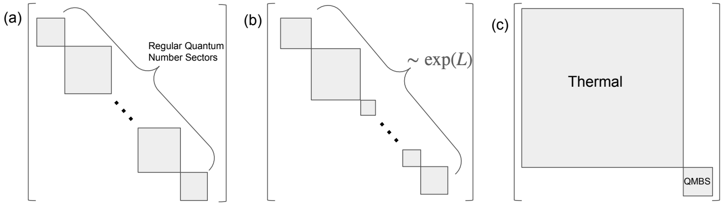


[10] T.Iadecola, M.Schecter (2019)

[11] M.Serbyn, D.A.Abanin, Z.Papic (2020)

Dynamically Disconnected Subspaces

- Weak ergodicity breaking = existence of unexpected "dynamically disconnected subspaces"



- While on-site or other conventional symmetries do not explain these blocks, allowing *arbitrary* operators to be conserved quantities is problematic

$$[H, |E_n\rangle\langle E_n|] = 0 \quad \implies \text{exponentially many conserved quantities?!}$$

What is an appropriate definition of a conserved quantity?[12]

[12] Similar problems exist in defining integrability in finite-dimensional systems: E.A.Yuzbashyan, B.S.Shastry (2013)

Symmetries and Commutant Algebras

Commutant algebras

- Key observation: Same block structure appears for entire classes of Hamiltonians $\{\sum_j J_j h_{j,j+1}\}$
- Natural to look for operators that commute with this entire family.

$$[\widehat{O}, \sum_j J_j h_{j,j+1}] = 0 \;\; \forall \{J_j\}.$$

- Commutant Algebra $\mathcal{C}$: algebra of operators $\widehat{O}$ (not necessarily local) such that $[h_{j,j+1}, \widehat{O}] = 0 \;\; \forall j$

$$\widehat{O}_1 \in \mathcal{C}, \;\; \widehat{O}_2 \in \mathcal{C} \quad \implies \quad \begin{cases} \alpha_1 \widehat{O}_1 + \alpha_2 \widehat{O}_2 \in \mathcal{C} \;\; \text{for any} \;\; \alpha_1, \alpha_2 \in \mathbb{C} \\ \widehat{O}_1 \widehat{O}_2, \, \widehat{O}_2 \widehat{O}_1 \in \mathcal{C} \end{cases}$$

- $\mathcal{C}$ commutes with the full "bond algebra" $\mathcal{A}$ generated by $\{h_{j,j+1}\}$ $(\mathcal{A} = \langle\!\langle \{h_{j,j+1}\} \rangle\!\rangle)$.

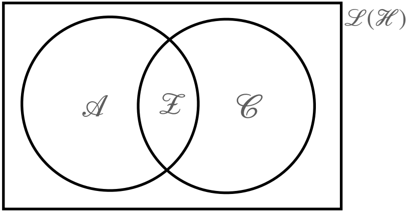
Commutant Algebras

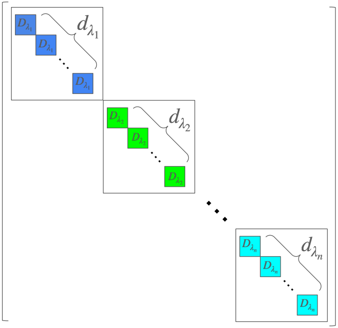
- $\mathcal{A}$ and $\mathcal{C}$ are unital $\dagger$-closed (von Neumann) algebras, centralizers of each other (Double Commutant Theorem)
- Representation theory: Can unitarily transform into a basis in which $\widehat{h}_{\mathcal{A}} \in \mathcal{A}$ and $\widehat{h}_{\mathcal{C}} \in \mathcal{C}$ have the matrix representations

$$W^\dagger \widehat{h}_{\mathcal{A}} W = \bigoplus_{\lambda} (M_{D_\lambda} \otimes \mathbb{1}_{d_\lambda})$$

$$W^\dagger \widehat{h}_{\mathcal{C}} W = \bigoplus_{\lambda} (\mathbb{1}_{D_\lambda} \otimes N_{d_\lambda})$$

- $\{D_\lambda\}$ and $\{d_\lambda\}$: dimensions of irreducible representations of $\mathcal{A}$ and $\mathcal{C}$.





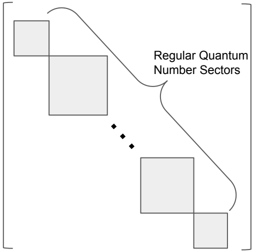
Dynamically Disconnected Subspaces

- Equivalently: Basis in which *all* elements of $\mathcal{A}$ are maximally block diagonal
- Hamiltonian $H = \sum_j J_j h_{j,j+1} \in \mathcal{A}$, block diagonal form defines quantum number sectors/dynamically disconnected "Krylov subspaces"
- For each λ: d_λ number of degenerate D_λ-dimensional blocks, total number of blocks: $K = \sum_\lambda d_\lambda$
- K can be bounded using $\dim(\mathcal{C}) = \sum_\lambda d_\lambda^2$, the number of linearly independent operators in $\mathcal{C}$, given by

$$\frac{1}{2}\log(\dim(\mathcal{C})) \leq \log K \leq \log(\dim(\mathcal{C}))$$

$\log(\dim(\mathcal{C}))$	**Example**
$\sim \mathcal{O}(1)$	Discrete Global Symmetry
$\sim \log L$	Continuous Global Symmetry
$\sim L$	Fragmentation

Conventional Symmetries



Simple Examples: Abelian $\mathcal{C}$

$$\begin{bmatrix} D_{\lambda_1} & & & \\ & D_{\lambda_2} & & \\ & & \ddots & \\ & & & D_{\lambda_n} \end{bmatrix}$$

- Abelian $\mathcal{C} \implies d_\lambda = 1, \ \ K = \dim(\mathcal{C})$
- Generic Hamiltonians $\sum_j J_j h_{j,j+1}$ with no symmetries, solve for $[h_{j,j+1}, \widehat{O}] = 0$

$$\mathcal{C} = \{\mathbb{1}\}, \ K = \dim(\mathcal{C}) = 1$$

- Ising models $H = \sum_{j=1}^{L} [J_j X_j X_{j+1} + h_j Z_j]$, solve for $[X_j X_{j+1}, \widehat{O}] = 0$ and $[Z_j, \widehat{O}] = 0$

$$\mathcal{C} = \mathrm{span}\{\mathbb{1}, \prod_j Z_j\}, \quad K = \dim(\mathcal{C}) = 2.$$

- Spin-$\frac{1}{2}$ XX models $H = \sum_{j=1}^{L} [J_j (X_j X_{j+1} + Y_j Y_{j+1}) + h_j Z_j]$, solve for $[X_j X_{j+1} + Y_j Y_{j+1}, \widehat{O}] = 0$ and $[Z_j, \widehat{O}] = 0$

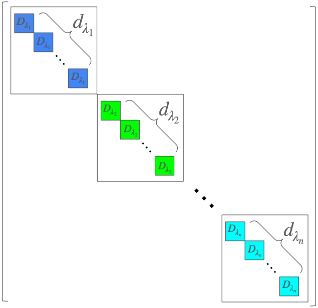
$$\mathcal{C} = \langle\langle Z_{\mathrm{tot}} \rangle\rangle = \mathrm{span}\{\mathbb{1}, Z_{\mathrm{tot}}, (Z_{\mathrm{tot}})^2, \cdots, (Z_{\mathrm{tot}})^L\}, \ \ Z_{\mathrm{tot}} = \sum_j Z_j$$
$$K = \dim(\mathcal{C}) = L + 1.$$

Simple Examples: Non-Abelian $\mathcal{C}$

- Non-Abelian $\mathcal{C} \implies$ some $d_\lambda > 1 \implies$ degeneracies
- Example: spin-$\frac{1}{2}$ Heisenberg model
  $H = \sum_j J_j \; \vec{S}_j \cdot \vec{S}_{j+1}$, $\mathcal{A} = \langle\!\langle \{\vec{S}_j \cdot \vec{S}_{j+1}\} \rangle\!\rangle$

$$\begin{aligned}\mathcal{C} &= \langle\!\langle S^x_{\rm tot}, S^y_{\rm tot}, S^z_{\rm tot} \rangle\!\rangle \\ &= \text{span}_{\alpha,\beta,\gamma}\{(S^x_{\rm tot})^\alpha (S^y_{\rm tot})^\beta (S^z_{\rm tot})^\gamma\}\end{aligned}$$

- Block-diagonal form (Schur-Weyl duality):
  $0 \leq \lambda \leq L/2$: S^2 eigenvalues
  $d_\lambda = 2\lambda + 1$: irreps of $\mathfrak{su}(2)$
  D_λ: irreps of S_L

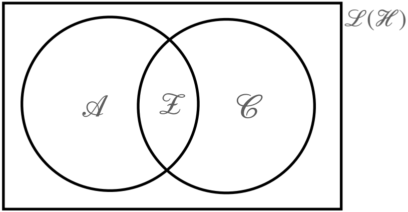


- Example: Stabilizer codes – $\mathcal{A}$ is the group algebra of the stabilizer group, $\mathcal{C}$ consists of $\mathcal{A}$ and the non-trivial logical operators.

[12] **SM**, O.I.Motrunich (2021)

New View on Symmetries

- Symmetries well defined for families of Hamiltonians, pair of algebras $\mathcal{A}$ and $\mathcal{C}$ associated with any symmetry.
- $\mathcal{A}$ is generated by a set of local operators, $\mathcal{C}$ is its centralizer.

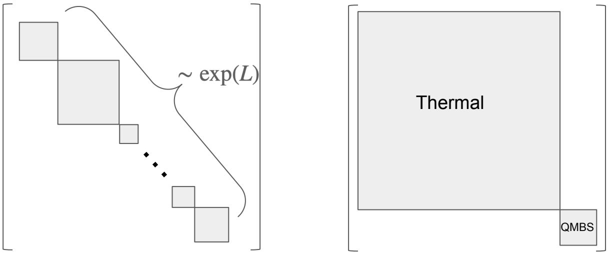


- Symmetries of several standard Hamiltonians can be understood this way, including free-fermion models, Hubbard models[13,14]
- Conventional commutants $\mathcal{C}$: Full commutant generated by "conventional" conserved quantities, $\dim(\mathcal{C})$ scales sub-exponentially with system size.
- In general: Start with any set of non-commuting local operators, generate their algebra $\mathcal{A}$, then determine commutant $\mathcal{C}$ – gives rise to novel unconventional symmetries!

[13] **SM**, O.I.Motrunich (2022)

[14] Some of them have mildly non-standard symmetries

Unconventional Symmetries



“Classical” Fragmentation

- Fragmentation occurs when $\dim(\mathcal{C}) \sim \exp(L)$
- Consider the $t - J_z$ Hamiltonian: hopping with two species of particles $|\uparrow 0\rangle \leftrightarrow |0 \uparrow\rangle$, $|\downarrow 0\rangle \leftrightarrow |0 \downarrow\rangle$

$$H_{t-J_z} = \sum_j (-t_{j,j+1} \sum_{\sigma \in \{\uparrow,\downarrow\}} \left(\tilde{c}_{i,\sigma} \tilde{c}^\dagger_{j,\sigma} + h.c. \right) + J^z_{j,j+1} S^z_i S^z_j)$$

$$\tilde{c}_{j,\sigma} = c_{j,\sigma} \left(1 - c^\dagger_{j,-\sigma} c_{j,-\sigma} \right)$$

- Has two $U(1)$ symmetries $N^\uparrow = \sum_j N^\uparrow_j$ and $N^\downarrow = \sum_j N^\downarrow_j$
- Full pattern of spins ($\uparrow$ or $\downarrow$) preserved in one dimension with OBC, number of Krylov subspaces $K = \sum_{j=0}^{L} 2^j = 2^{L+1} - 1$

$$|0 \uparrow\downarrow 0 \downarrow\uparrow 0\rangle \not\leftrightarrow |0 \uparrow\uparrow 0 \downarrow\downarrow 0\rangle$$

- Fragmentation in the product state basis $\implies$ essentially classical[15]

[15] D.Dhar, M.Barma (1993); G.I.Menon, M.Barma, D.Dhar (1997)

"Classical" Fragmentation

- Local operators $N_j^{\uparrow}$ and $N_j^{\downarrow}$ satisfy the relations

$$[h_{j,j+1}, N_j^{\alpha} + N_{j+1}^{\alpha}] = 0, \quad [h_{j,j+1}, N_j^{\alpha} N_{j+1}^{\beta}] = 0, \quad \alpha, \beta \in \{\uparrow, \downarrow\}$$

- The full commutant algebra $\mathcal{C}$ can be explicitly constructed, $\dim(\mathcal{C}) = 2^{L+1} - 1 \sim \exp(L)$

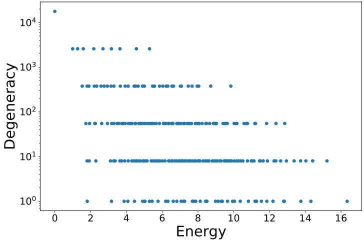
$$N^{\sigma_1 \sigma_2 \cdots \sigma_k} = \sum_{j_1 < j_2 < \cdots < j_k} N_{j_1}^{\sigma_1} N_{j_2}^{\sigma_2} \cdots N_{j_k}^{\sigma_k}, \quad \sigma_j \in \{\uparrow, \downarrow\}$$

- Most of these are functionally independent from the conventional conserved quantities $N^{\uparrow}$ and $N^{\downarrow}$ $\implies$ new dynamically disconnected subspaces
- Classical fragmentation: All conserved quantities diagonal, Hamiltonian block-diagonal in product state basis
- Similar construction works for dipole-conserving models, exact results in some cases (e.g. $\dim(\mathcal{C}) \sim (1 + \sqrt{2})^L$ for range-3 spin-1 model)

"Quantum" Fragmentation

- Disordered $SU(3)$-symmetric spin-1 biquadratic model, eigenstate degeneracies grow exponentially with $L \implies$ hidden symmetries

$$H = \sum_{j=1}^{L} J_j(\vec{S}_j \cdot \vec{S}_{j+1})^2$$



- $\mathcal{A} = \langle\!\langle (\vec{S}_j \cdot \vec{S}_{j+1})^2 \rangle\!\rangle = TL_L(q = \frac{3+\sqrt{5}}{2})$, commutant $\mathcal{C}$ can be explicitly constructed,[16] $\dim(\mathcal{C}) \sim \exp(L)$

$$[(\vec{S}_j \cdot \vec{S}_{j+1})^2, (M^{\alpha}_{\beta})_j + (M^{\alpha}_{\beta})_{j+1}] = 0, \quad [(\vec{S}_j \cdot \vec{S}_{j+1})^2, (M^{\alpha}_{\beta})_j (M^{\gamma}_{\delta})_{j+1}] = 0,$$

$$M^{\alpha_1\alpha_2\cdots\alpha_k}_{\beta_1\beta_2\cdots\beta_k} = \sum_{j_1<j_2<\cdots<j_k} (M^{\alpha_1}_{\beta_1})_{j_1} (M^{\alpha_2}_{\beta_2})_{j_2} \cdots (M^{\alpha_k}_{\beta_k})_{j_k}.$$

- Quantum fragmentation: Block-diagonal structure of the Hamiltonian understood in the spin-1 singlet basis, not product state basis

[16] N.Read, H.Saleur (2007)

Quantum Many Body Scars

- Aim: Given QMBS eigenstates $\{|S_n\rangle\}$, find a locally-generated algebra $\mathcal{A}_{scar}$ that $\mathcal{C}_{scar} = \langle\!\langle \{|S_n\rangle \langle S_n|\} \rangle\!\rangle$
- Example: Spin-$\frac{1}{2}$ ferromagnetic multiplet $\{|S_n\rangle = (S^-_{\text{tot}})^n |F\rangle\}$, $|F\rangle = |\uparrow \cdots \uparrow\rangle$ – Start with $SU(2)$ symmetry and systematically break it[17]

$$
\begin{aligned}
&\mathcal{A}_{sym} = \langle\!\langle \{\vec{S}_j \cdot \vec{S}_{j+1}\} \rangle\!\rangle && \mathcal{C}_{sym} = \langle\!\langle S^x_{\text{tot}}, S^y_{\text{tot}}, S^z_{\text{tot}} \rangle\!\rangle \\
&\mathcal{A}_{dyn} = \langle\!\langle \{\vec{S}_j \cdot \vec{S}_{j+1}\}, S^z_{\text{tot}} \rangle\!\rangle && \mathcal{C}_{dyn} = \langle\!\langle \vec{S}^2, S^z_{\text{tot}} \rangle\!\rangle \\
&\mathcal{A}_{scar} = \langle\!\langle \{\vec{S}_j \cdot \vec{S}_{j+1}\}, S^z_{\text{tot}}, \{D^{\alpha}_{j-1,j,j+1}\} \rangle\!\rangle && \mathcal{C}_{scar} = \langle\!\langle \{|S_n\rangle\langle S_n|\} \rangle\!\rangle
\end{aligned}
$$

- $\mathcal{A}_{\text{scar}}$ can be explicitly constructed for several known examples of QMBS[18]
- Generators of $\mathcal{A}_{\text{scar}}$ are building blocks for constructing quantum scarred Hamiltonians $\implies$ Lots of local perturbations that exactly preserve the QMBS!

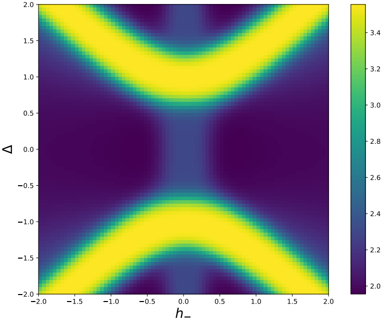
[17] D.K.Mark, O.I.Motrunich (2020); N.O'Dea, F.J.Burnell, A.Chandran, V.Khemani (2020)

[18] **SM**, O.I.Motrunich (2022)

Constraints on Realizable Symmetries

- Locality of generators of $\mathcal{A}$ restricts realizable commutants $\mathcal{C}$
- No-go result: No locally generated $\mathcal{A}$ with $\mathcal{C} = \langle\langle \sum_j S_j^z, \sum_j j S_j^z \rangle\rangle$[19]
- Can systematically search for symmetries realizable using, e.g., spin-1/2 n.n. S^z_{tot}-conserving terms[20]

$$\mathcal{A} = \langle\langle \{X_j X_{j+1} + Y_j Y_{j+1} + \Delta Z_j Z_{j+1} + h_-(Z_j - Z_{j+1})\} \rangle\rangle$$



- Detects presence of an unconventional $SU(2)_q$ symmetry for $(\Delta, h_-) = (\frac{q+q^{-1}}{2}, \frac{q-q^{-1}}{2})$
- Also leads to discovery of non-integrable models with Strong Zero Modes[21]

[19] P.Sala, T.Rakovszky, R.Verresen, M.Knap, F.Pollmann (2019); V.Khemani, M.Hermele, R.Nandkishore (2020)

[21] **SM**, O.I.Motrunich (in preparation)

[21] P.Fendley (2016); D.V.Else, P.Fendley, J.Kemp, C.Nayak (2017)

Summary & Outlook

- Symmetry $\iff$ Pair of $(\mathcal{A}, \mathcal{C})$
  $\mathcal{A}$: Local algebra $\mathcal{C}$: Commutant algebra
- Conventional symmetries: $\mathcal{C}$ generated by conventional conserved quantities, $\dim(\mathcal{C}) \sim \mathcal{O}(1)$ or $\dim(\mathcal{C}) \sim \mathrm{poly}(L)$
- Concrete definitions:
  - Fragmentation: $\dim(\mathcal{C}) \sim \exp(L)$
  - QMBS: Simultaneous eigenstates of multiple non-commuting local operators*
- Double Commutant Theorem: Building blocks for *all* symmetric local Hamiltonians
- Interesting $\mathcal{C}$? Connections to categorical/MPO symmetries?
- Approximate Commutants? PXP Model?
- Implications for equilibrium physics? Non-interacting models?