Zipper entanglement renormalization for free fermions

arXiv: 2206.11761

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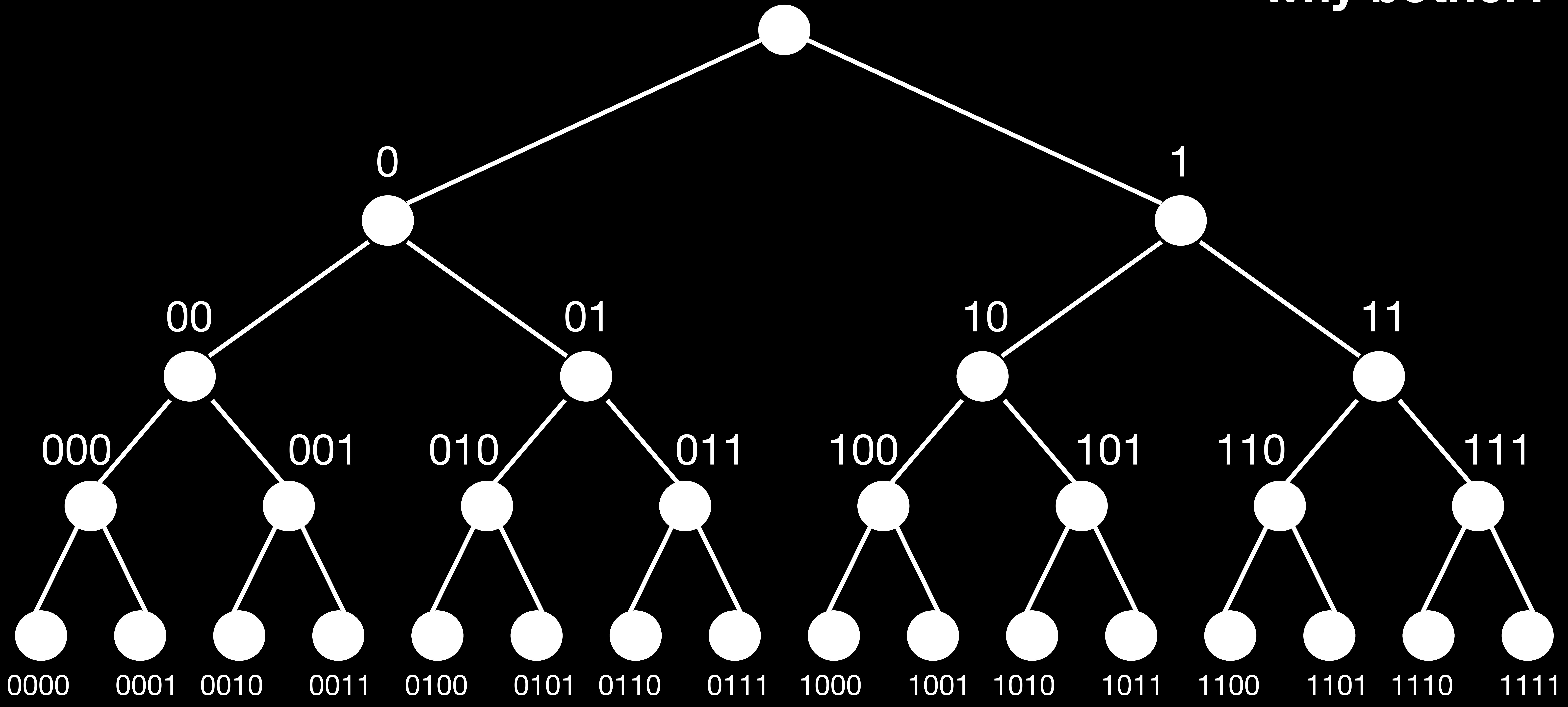
Outline

- Entanglement, renormalization, & entanglement renormalization
- “Zipper”
  - What is it?
  - What could it do?
  - How is it constructed?
- Going beyond 1D!…

Quantum entanglement —

why bother?

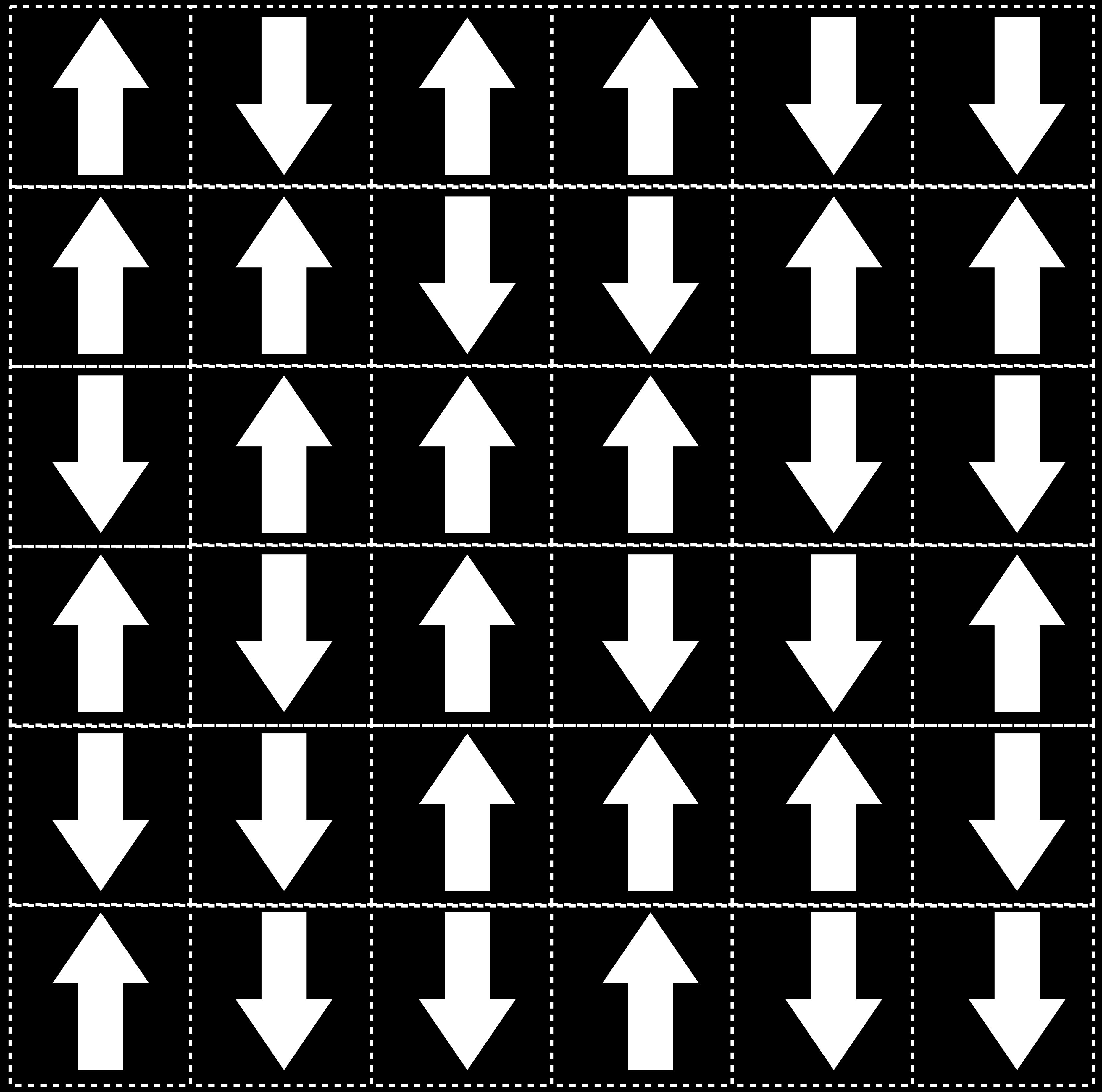


Exponentials are (computationally) hard

but the classical world suffers from the same problem!

\# of configurations :

$$2^{36} = 68{,}719{,}476{,}736$$



Yet!

NO difficulty at all drawing *one* configuration!

Now, into the quantum world…

in which exponentials are much more overwhelming/ powerful

E.g., a $|\text{quantum state}\rangle$ with 8 quantum bits =

$$
\begin{aligned}
&|00000000\rangle + |00000001\rangle + |00000010\rangle + |00000011\rangle + |00000100\rangle + |00000101\rangle + |00000110\rangle + |00000111\rangle + |00001000\rangle + |00001001\rangle + |00001010\rangle + |00001011\rangle \\
&+ |00001100\rangle + |00001101\rangle + |00001110\rangle + |00001111\rangle + |00010000\rangle + |00010001\rangle + |00010010\rangle + |00010011\rangle + |00010100\rangle + |00010101\rangle + |00010110\rangle + |00010111\rangle \\
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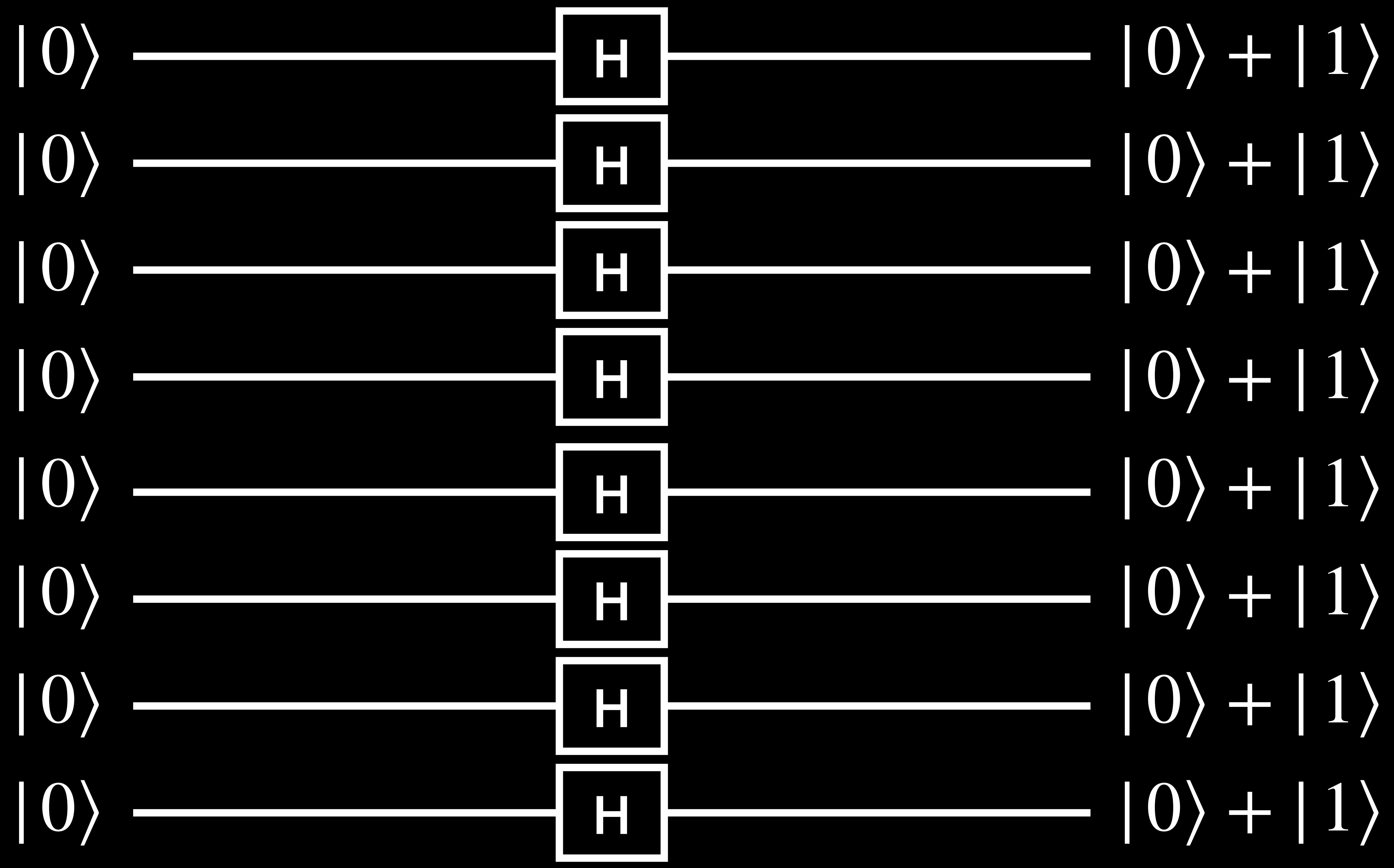
&+ |11111100\rangle + |11111101\rangle + |11111110\rangle + |11111111\rangle
\end{aligned}
$$

Now, into the quantum world…

in which exponentials are much more overwhelming/ powerful

E.g., a $|\text{quantum state}\rangle$ with 8 quantum bits $= \bigotimes_{i=1}^{8} \left(|0\rangle_i + |1\rangle_i \right)$



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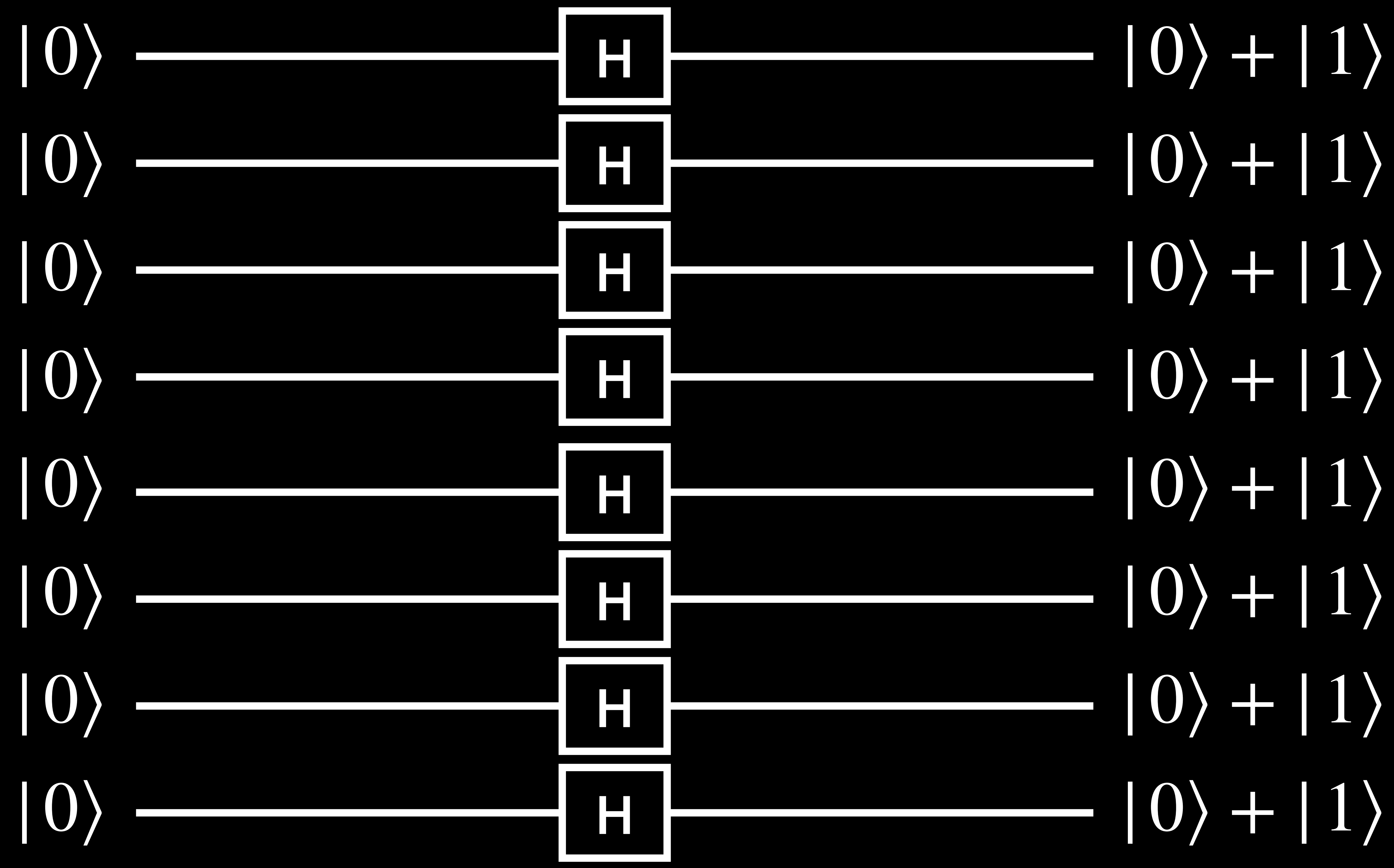

$= | \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rangle$

NO difficulty at all describing this state!

Now, into the quantum world…

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$|00000000\rangle + |00000001\rangle \quad + |00000101\rangle + |00000110\rangle + |00000111\rangle + |00001000\rangle \quad + |00001010\rangle + |00001011\rangle$

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What about this?

Quantum entanglement:

feasibility for us to describe a quantum state

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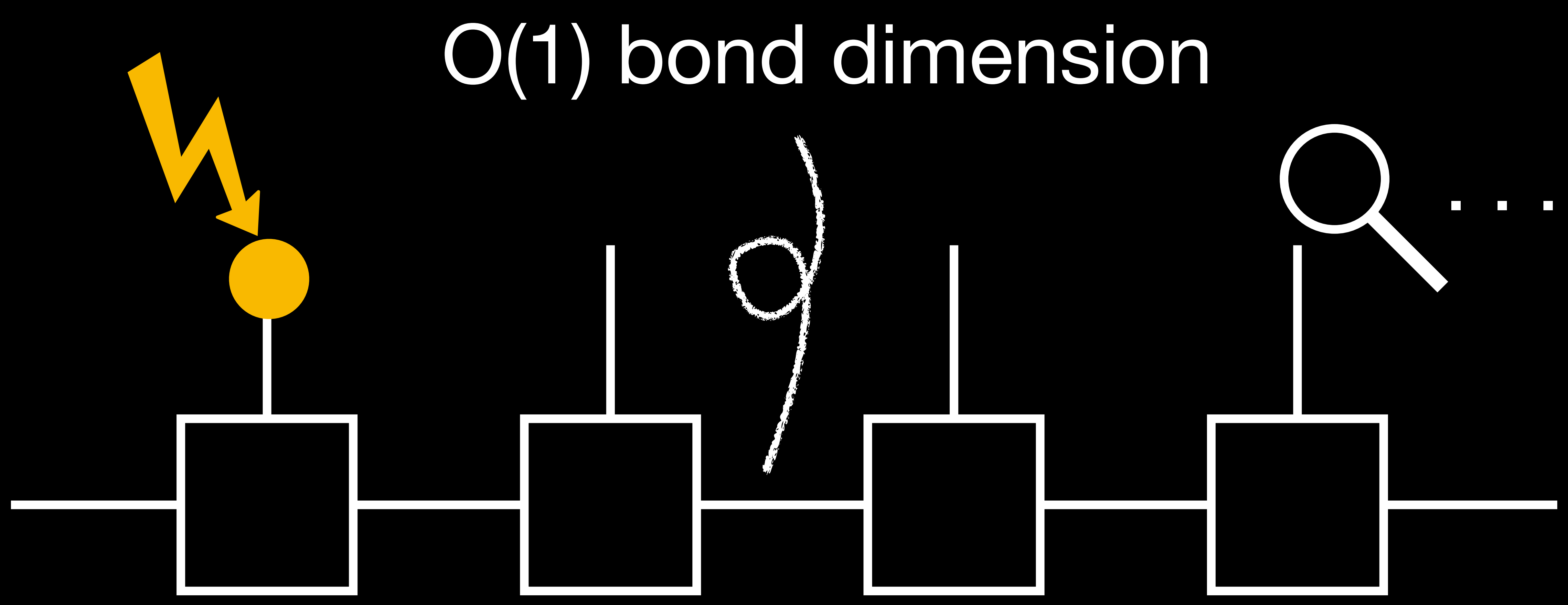
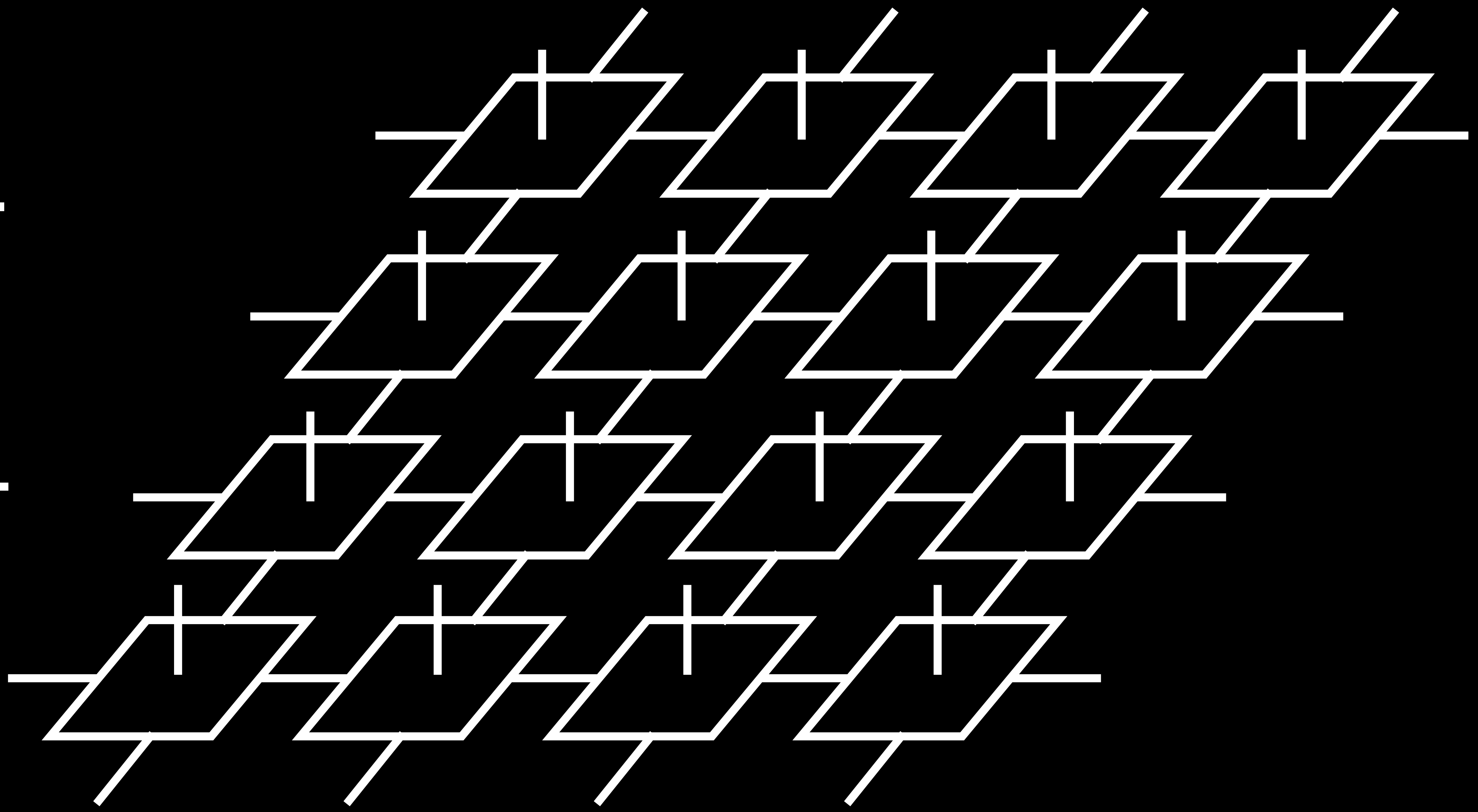


Matrix Product State (MPS)

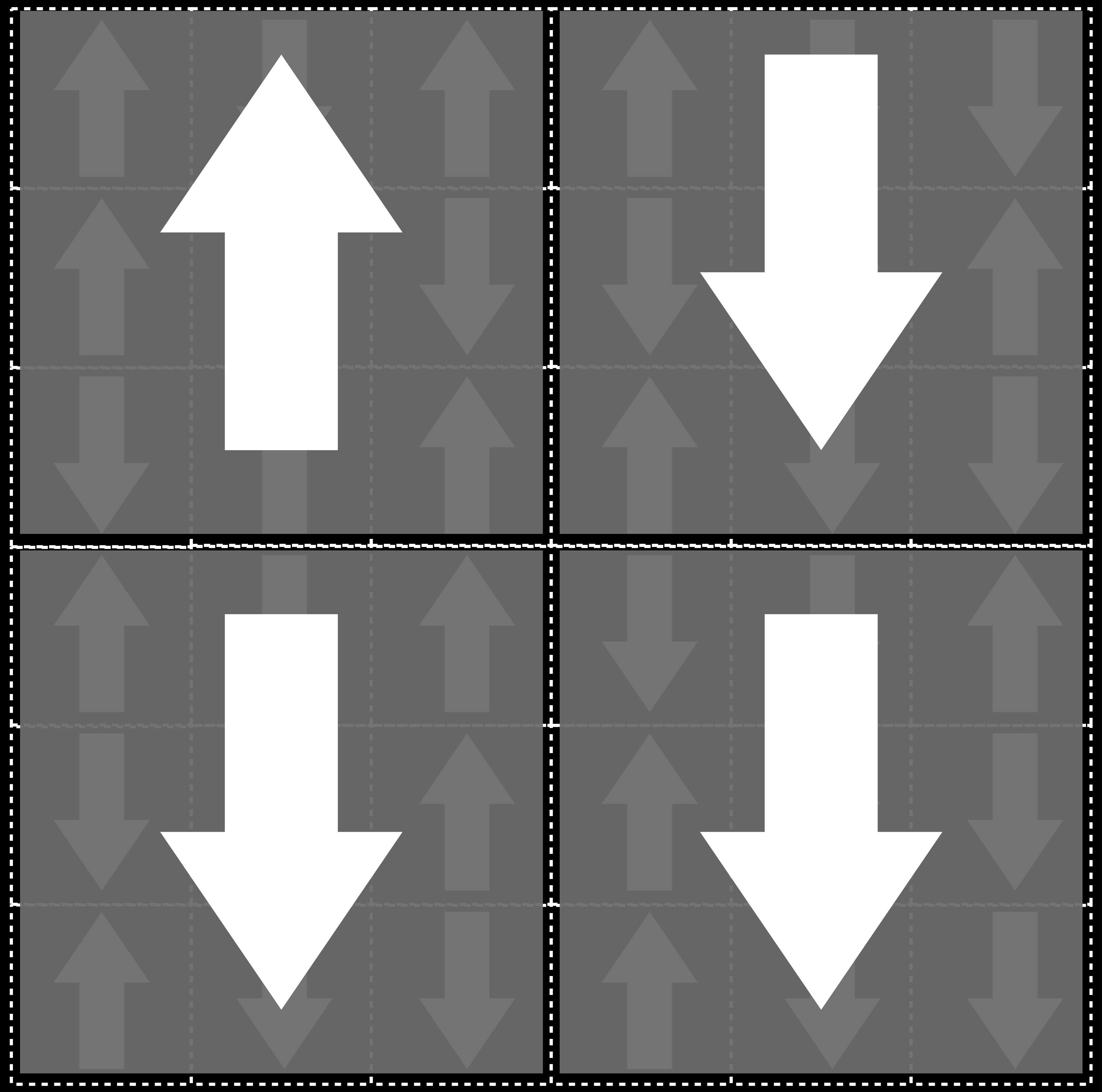


Projected Entangled Pair State (PEPS)

Dilemma: states we can feasibly describe are not necessarily the most interesting

That's why we need RG (renormalization group)

the exponential solution to the exponential problem



System size:

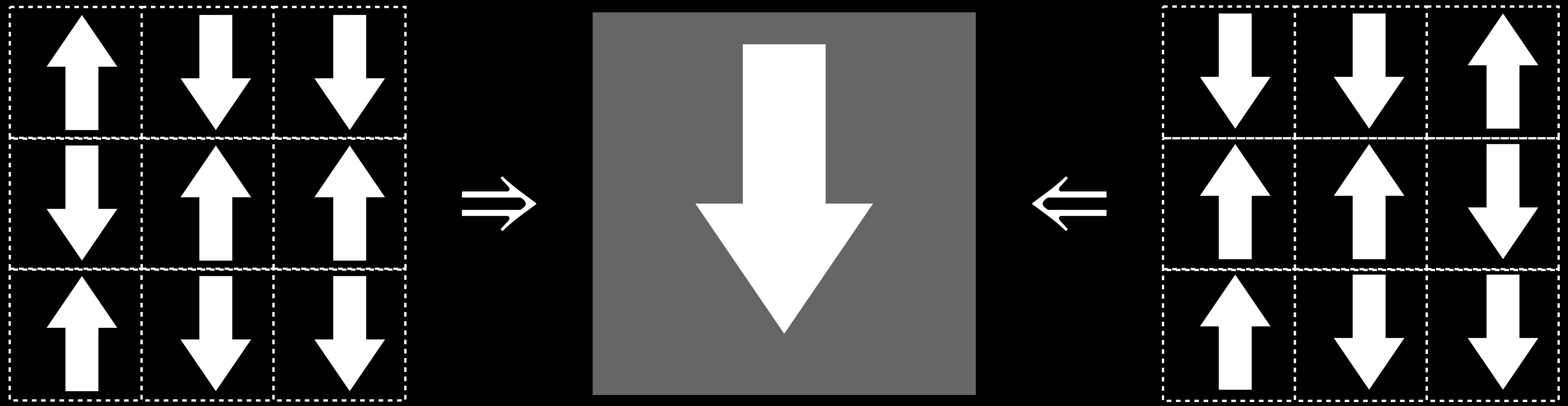
$$V \mapsto V/\chi^d$$

Length scale:

$$l \mapsto l\chi$$

That's why we need RG (renormalization group)

the exponential solution to the exponential problem



Renormalization through coarse-graining is "forgetful"

RG the other way

integrating out high-energy modes

also forgetful

Until we are left with low-energy

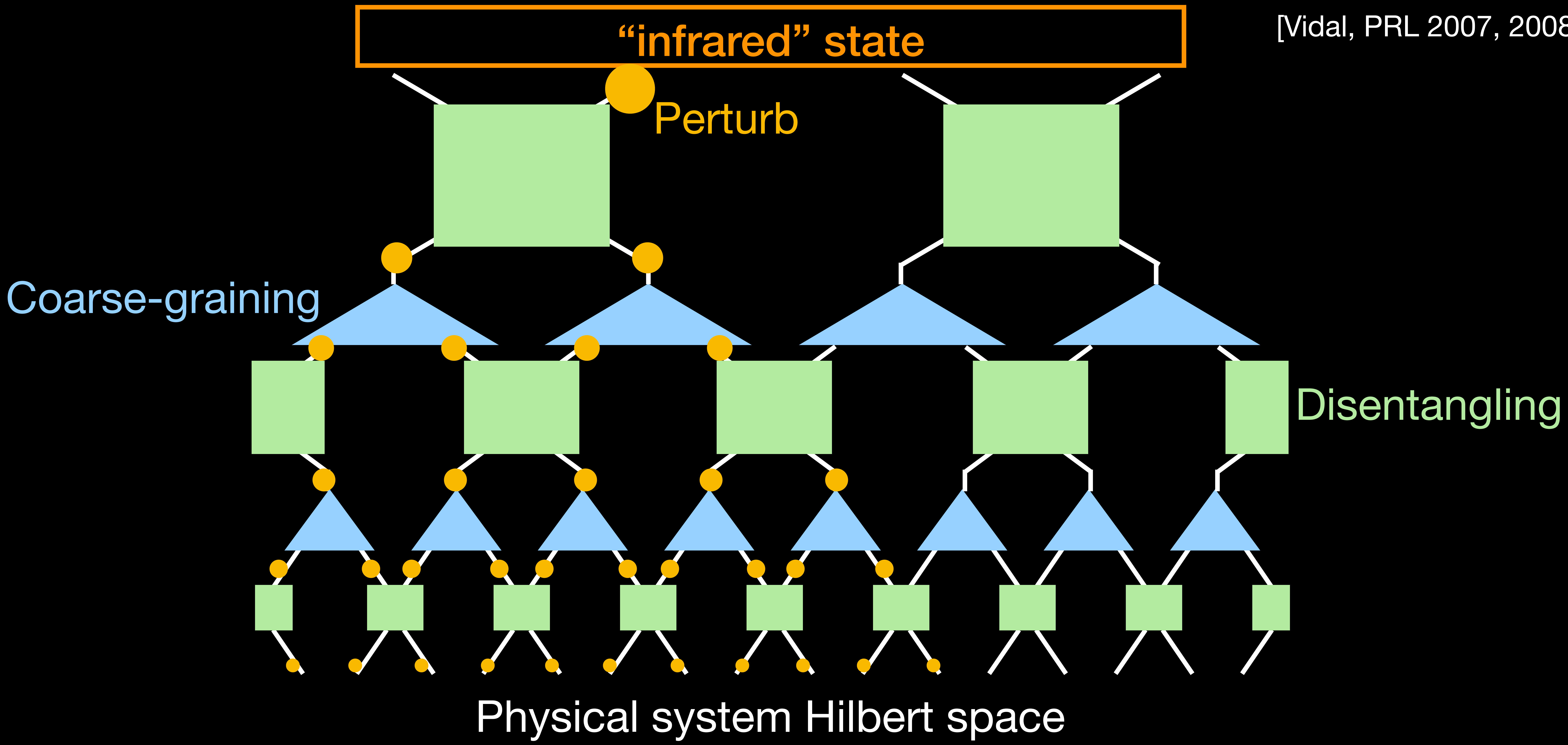
Fermi level

$\simeq$ long-wavelength degrees of freedom

RG yet the other way:

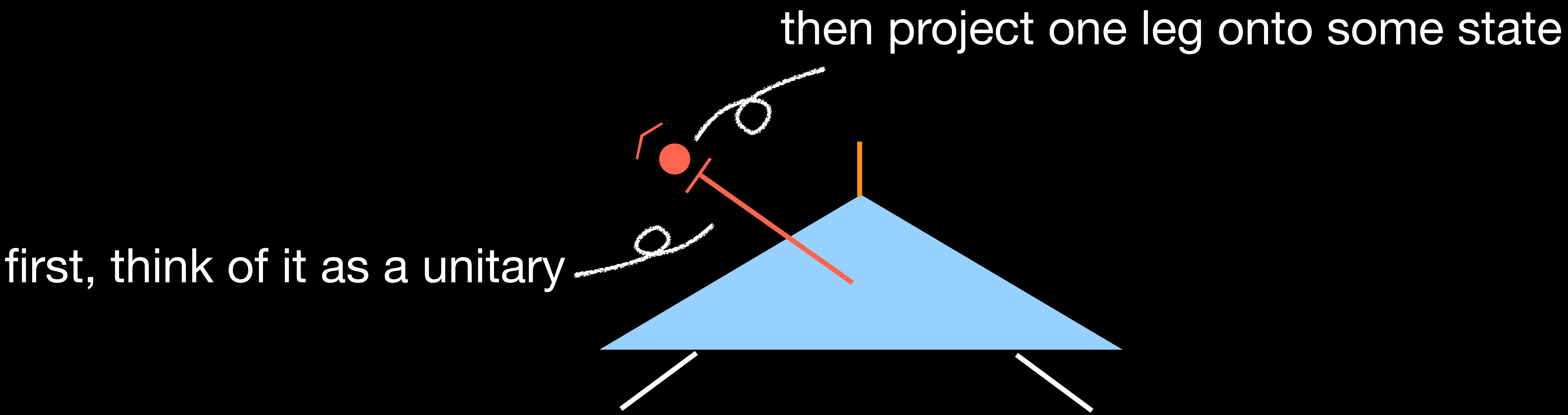
Entanglement renormalization, e.g., MERA

[Vidal, PRL 2007, 2008]



A new future:

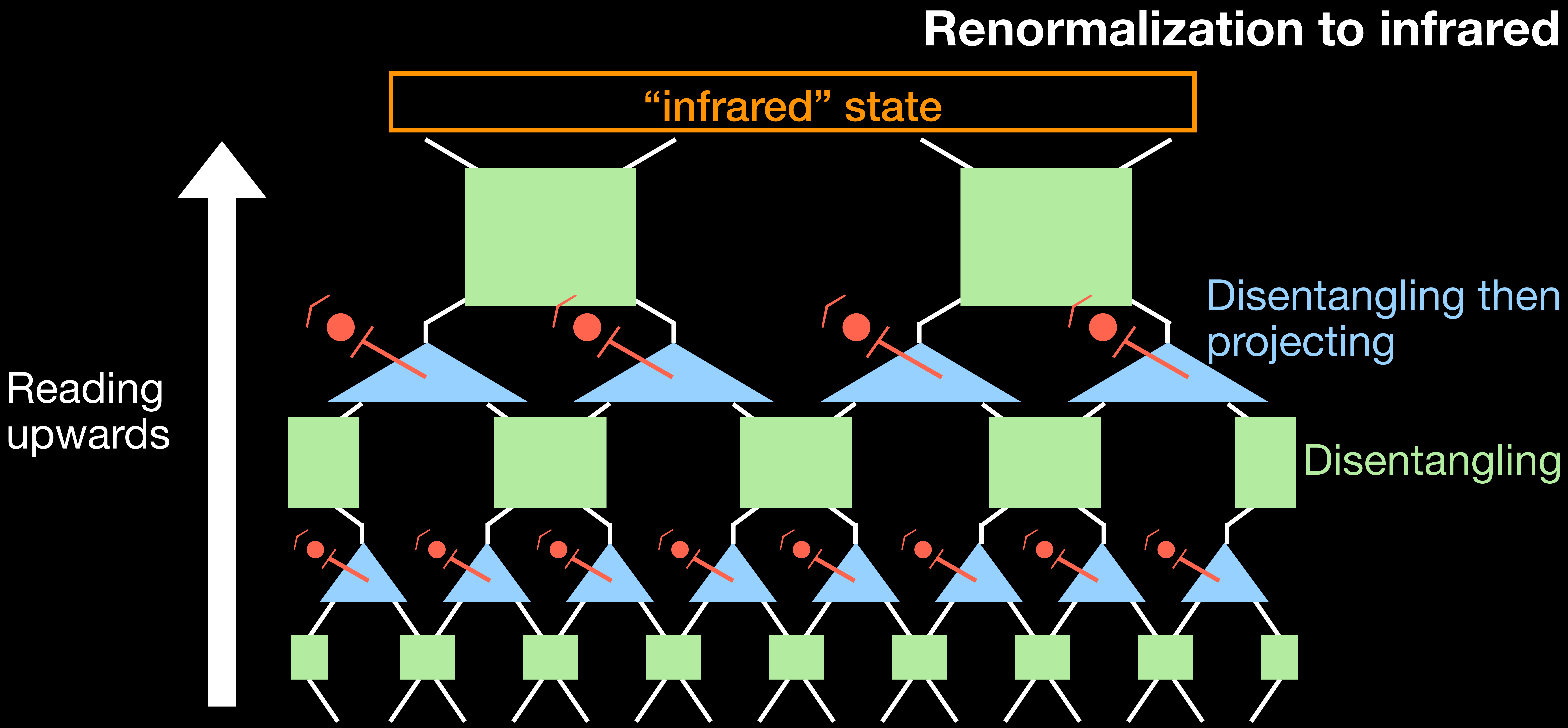
renormalization doesn't have to be forgetful!



Caveat: the state we want to describe may not be factorable in to red vs orange!

This is a variational approach: we do our best!

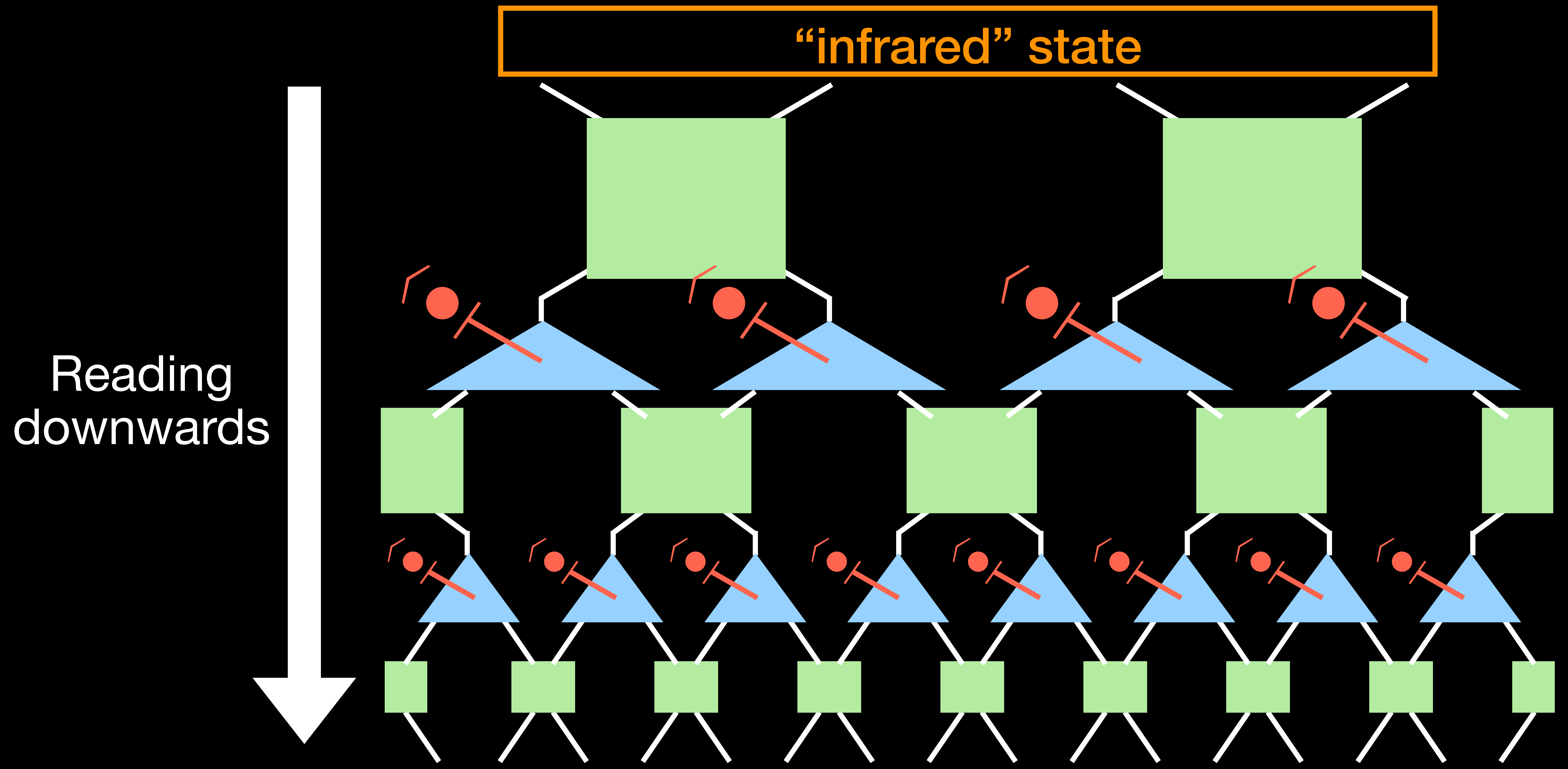
Different ways to look at MERA



Different ways to look at MERA

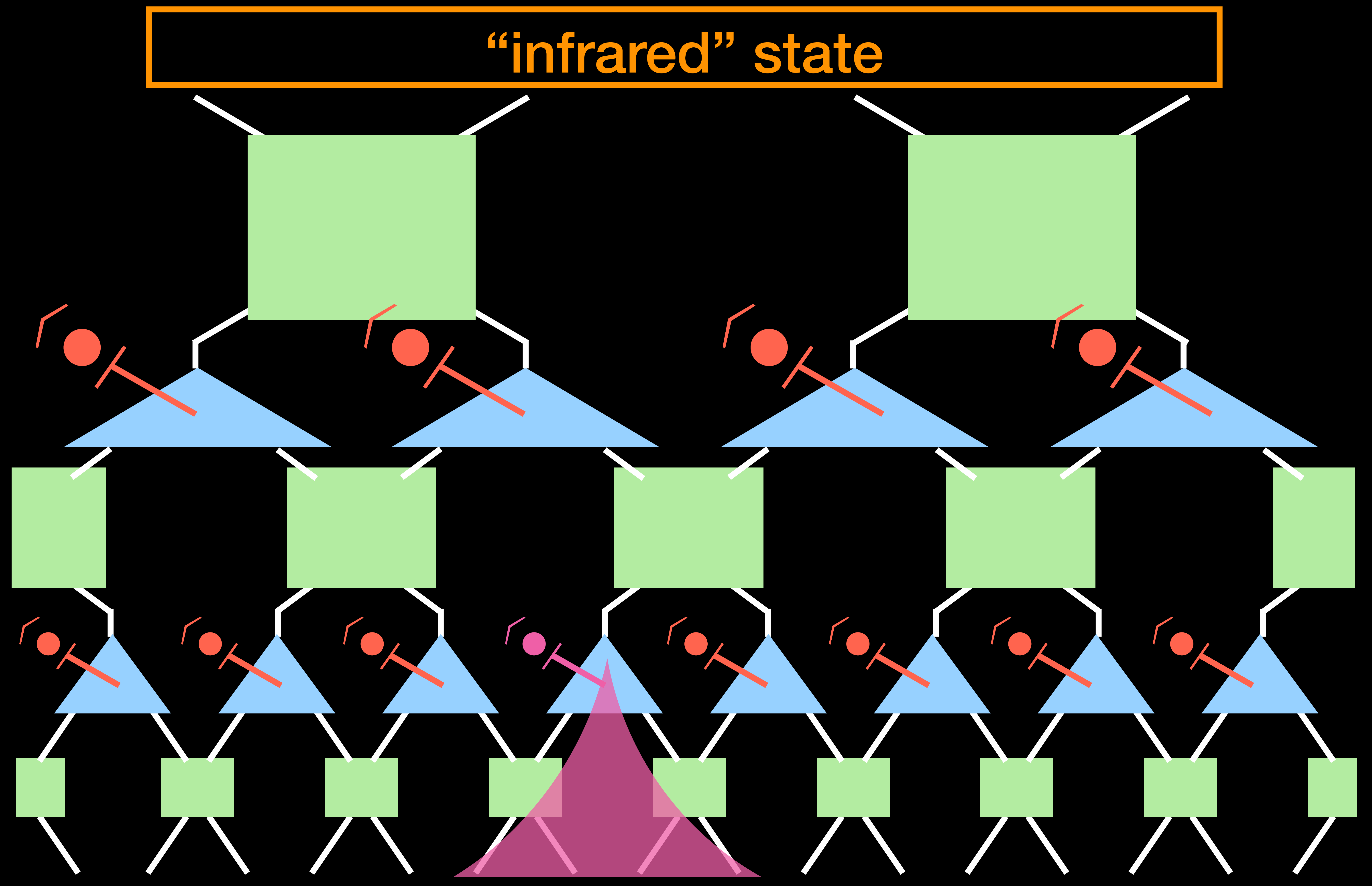
[Swingle & McGreevy, PRB$^{\times 3}$ 2016]

Growing a state with resources



Entanglement in MERA

Casting the product states to the physical space



a compactly supported state that is distilled out early in RG

Entanglement renormalization is a powerful way to describe/ understand/ probe interesting quantum states

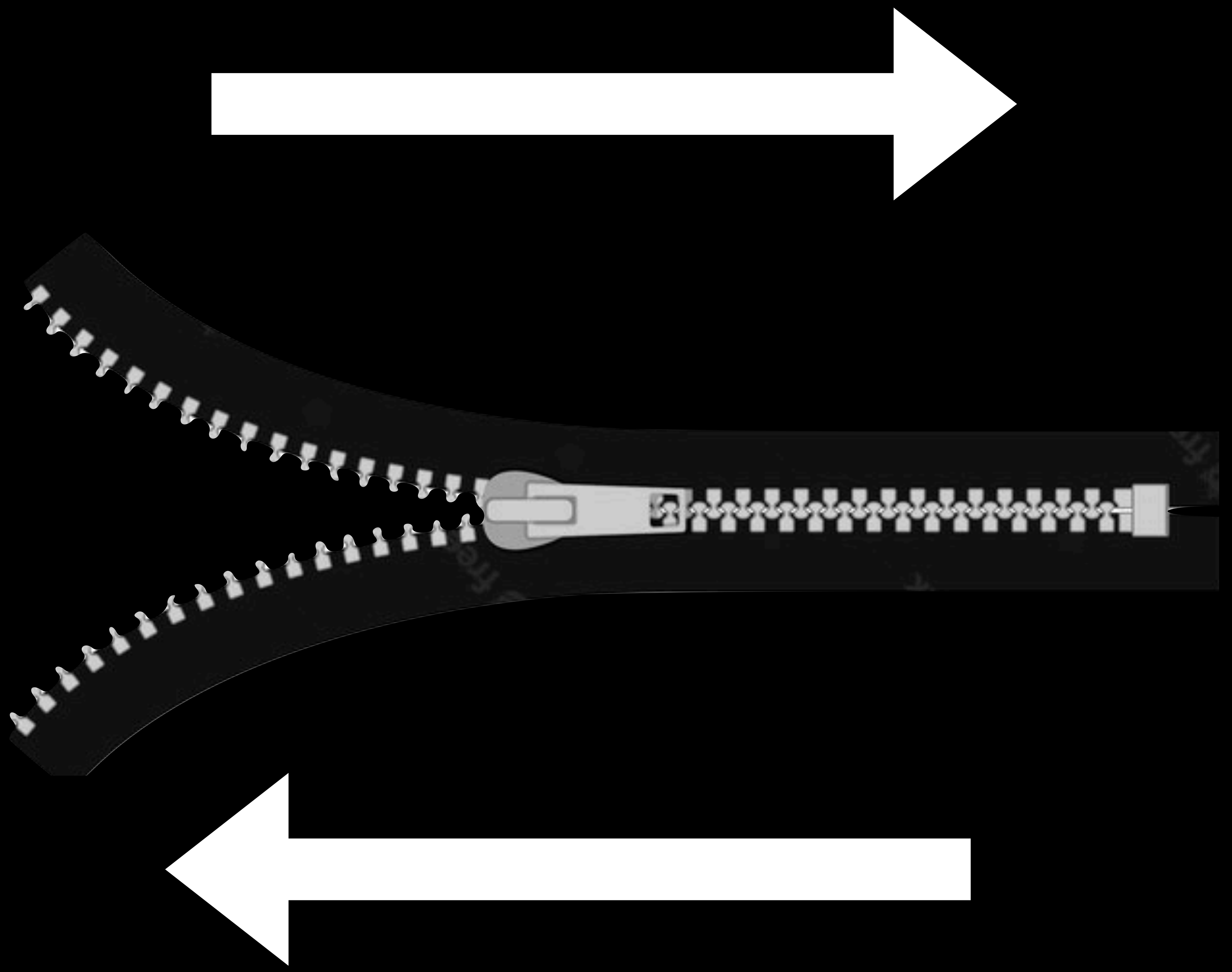
Outline

- Entanglement, renormalization, & entanglement renormalization
- “Zipper”
  - What is it?
  - What could it do?
  - How is it constructed?
- Going beyond 1D!…

Zipper Entanglement Renormalization:

What is it?

unzip

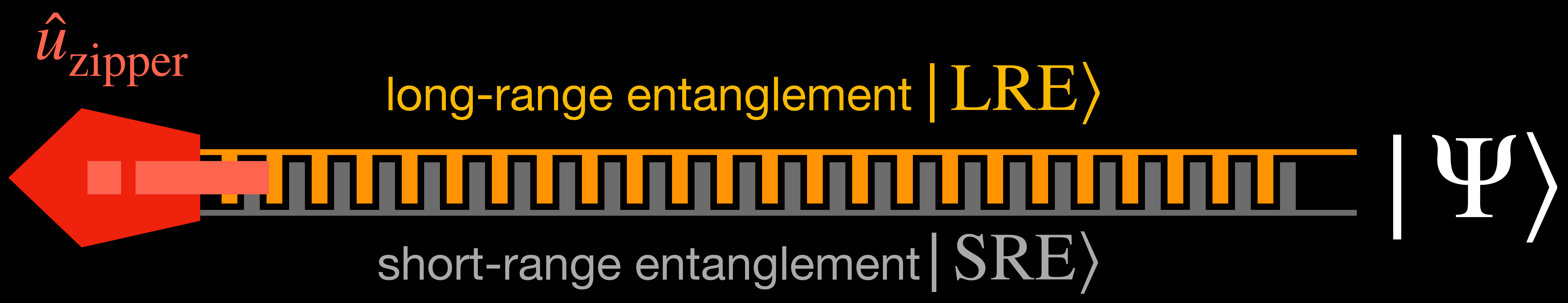


zip

[pic from oxyvector on freepik]

$|\Psi\rangle$

For our purpose here, we always refer to entanglement across *space*



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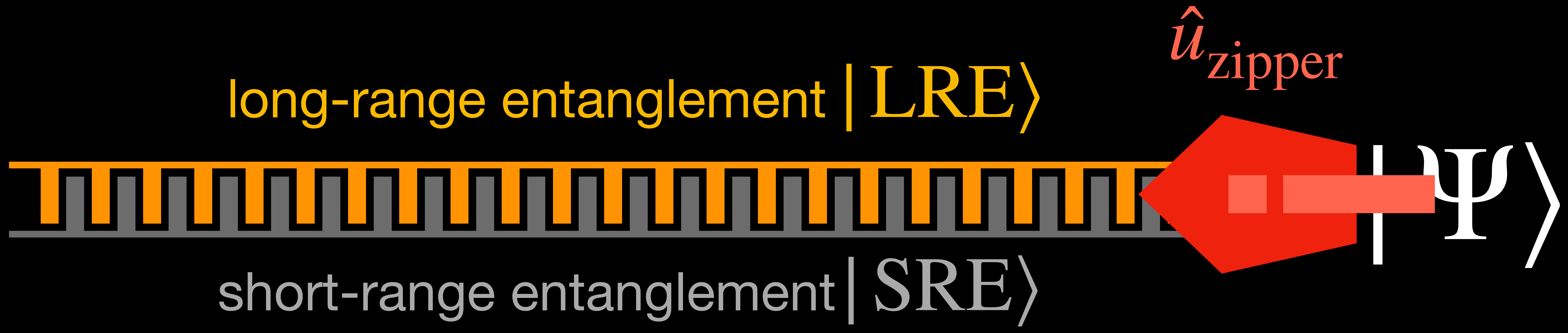

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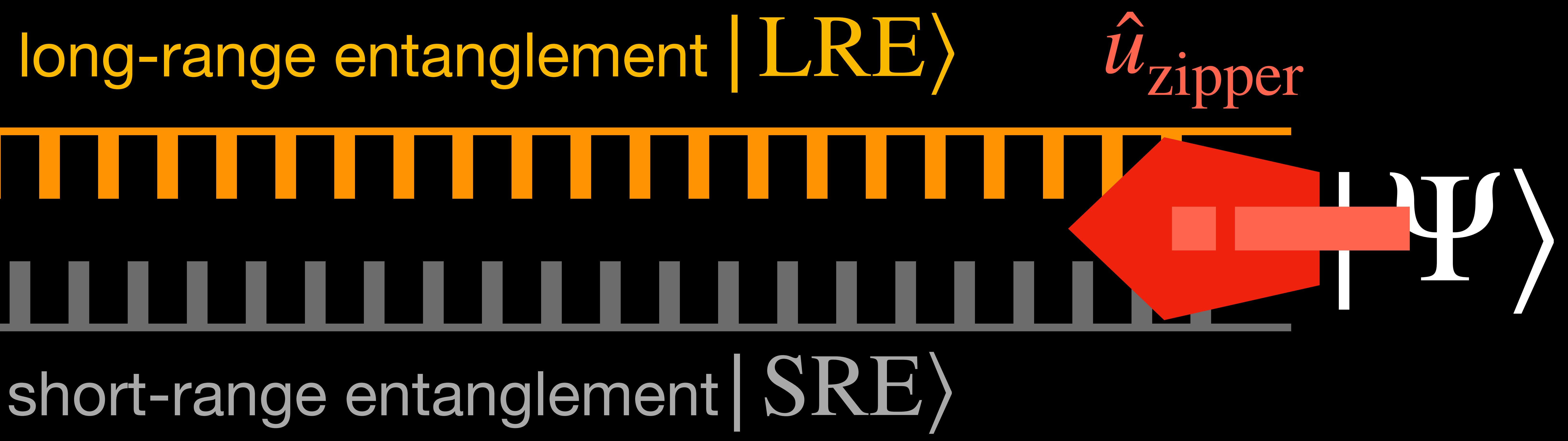
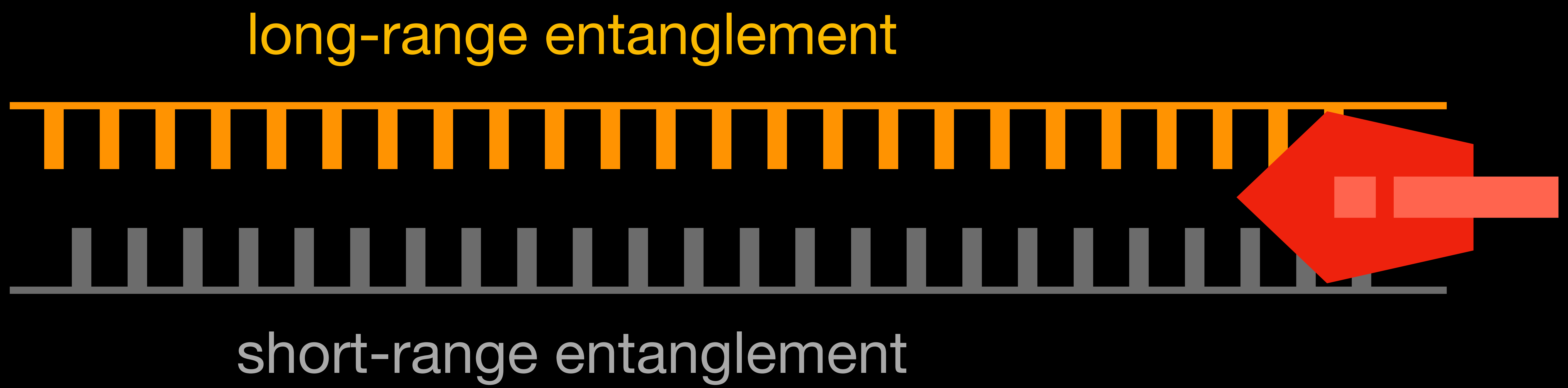
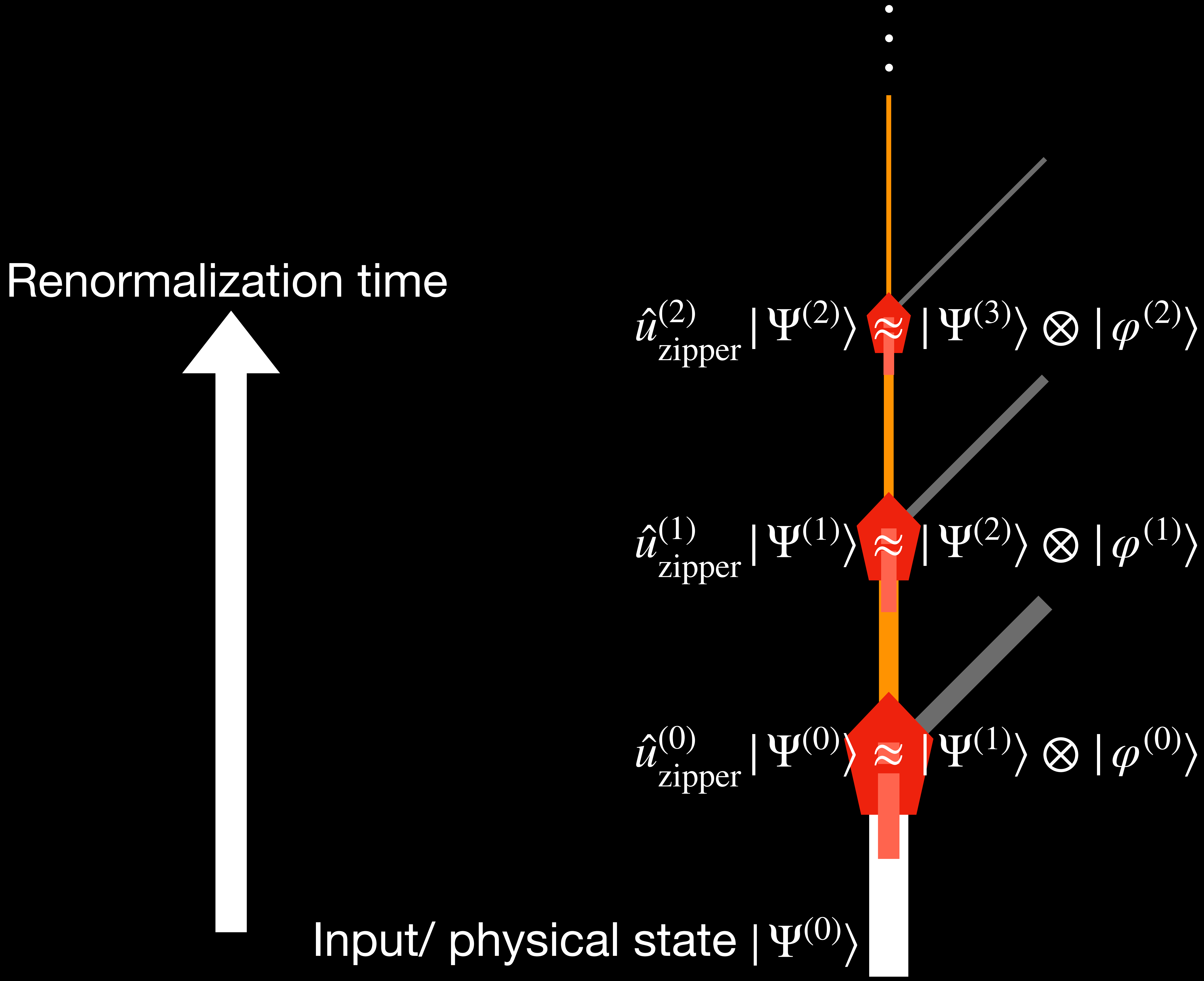


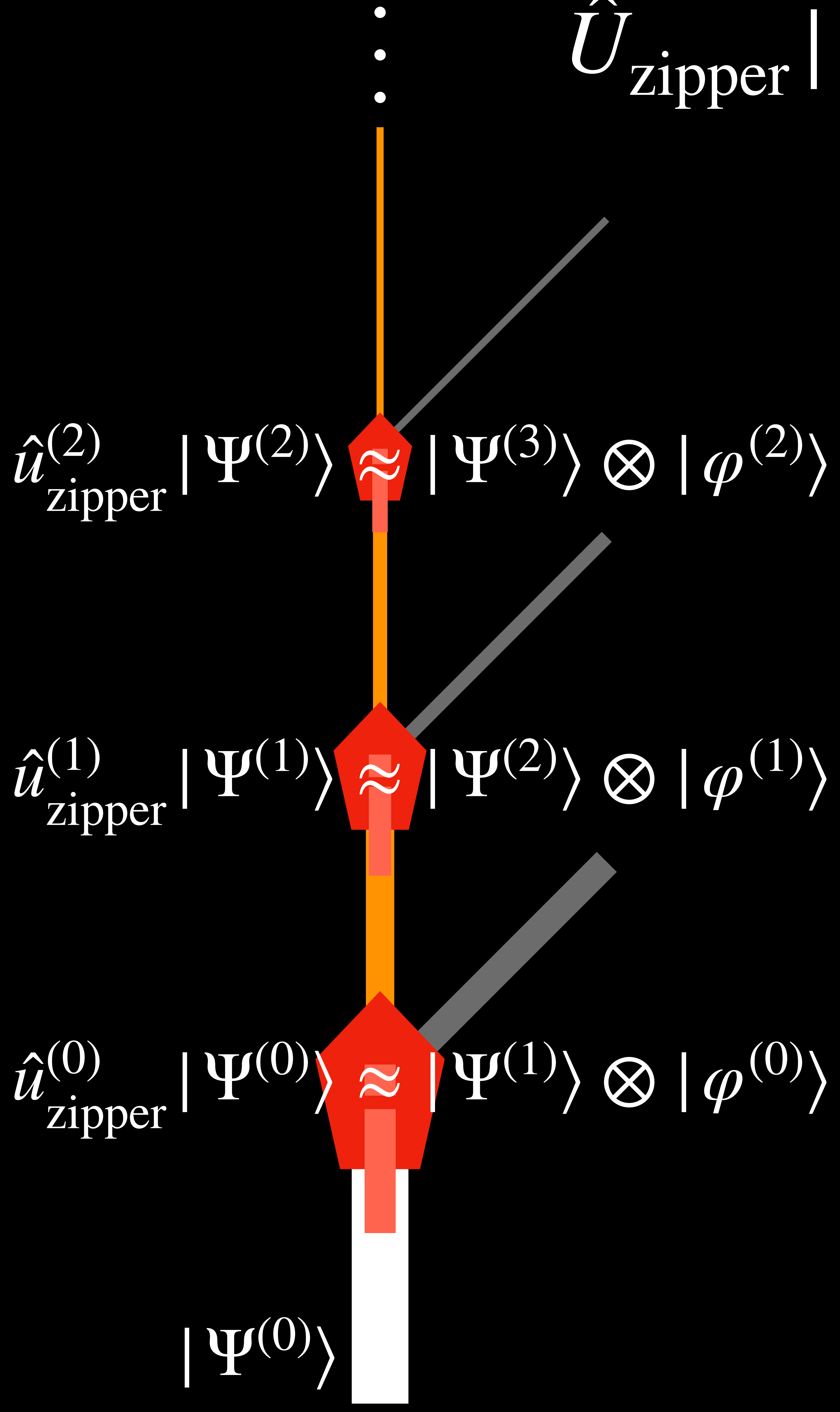
For our purpose here, we always refer to entanglement across *space*

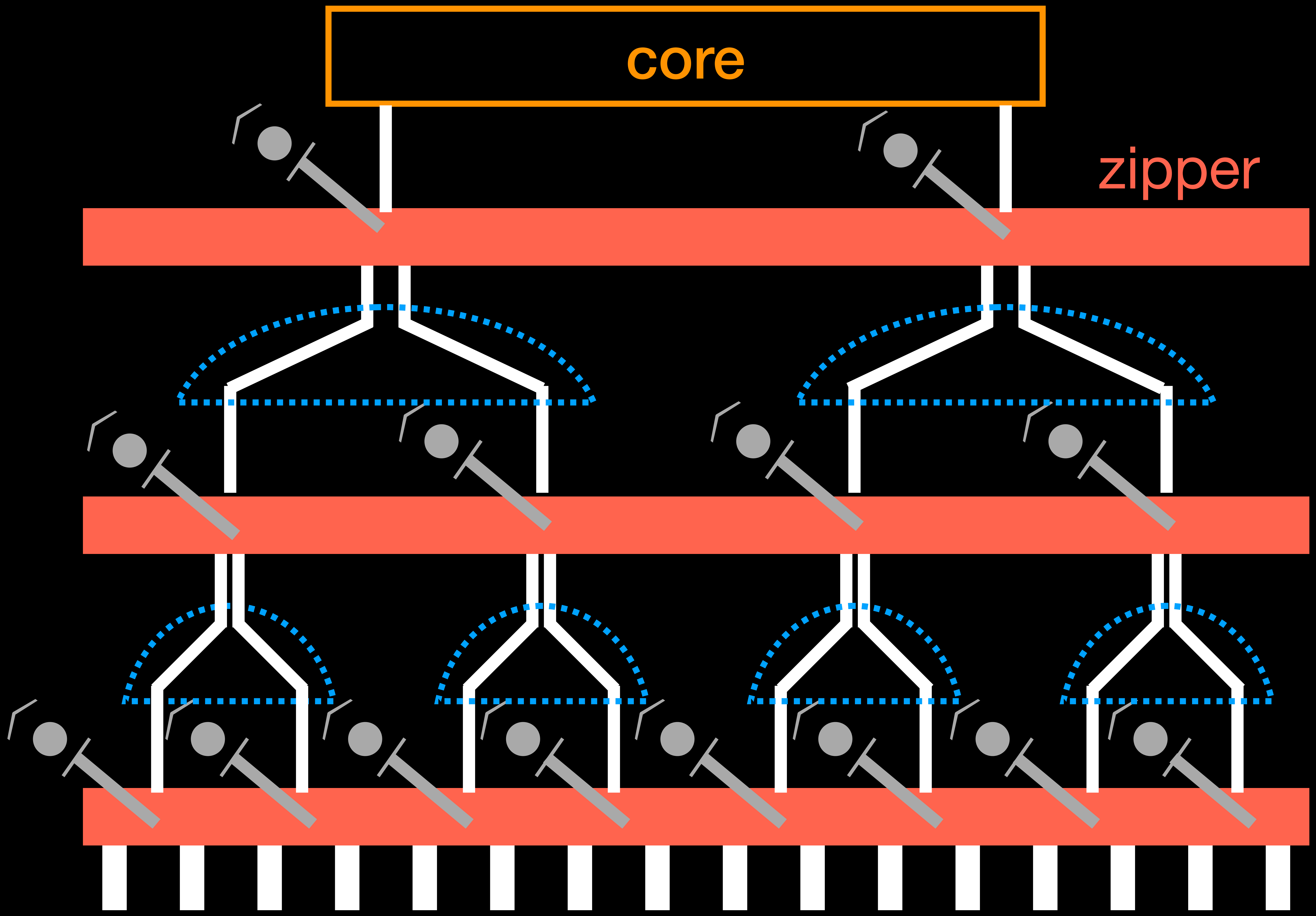
$$\hat{u}_{\text{zipper}}\,|\Psi\rangle \approx |\text{LRE}\rangle \otimes |\text{SRE}\rangle$$

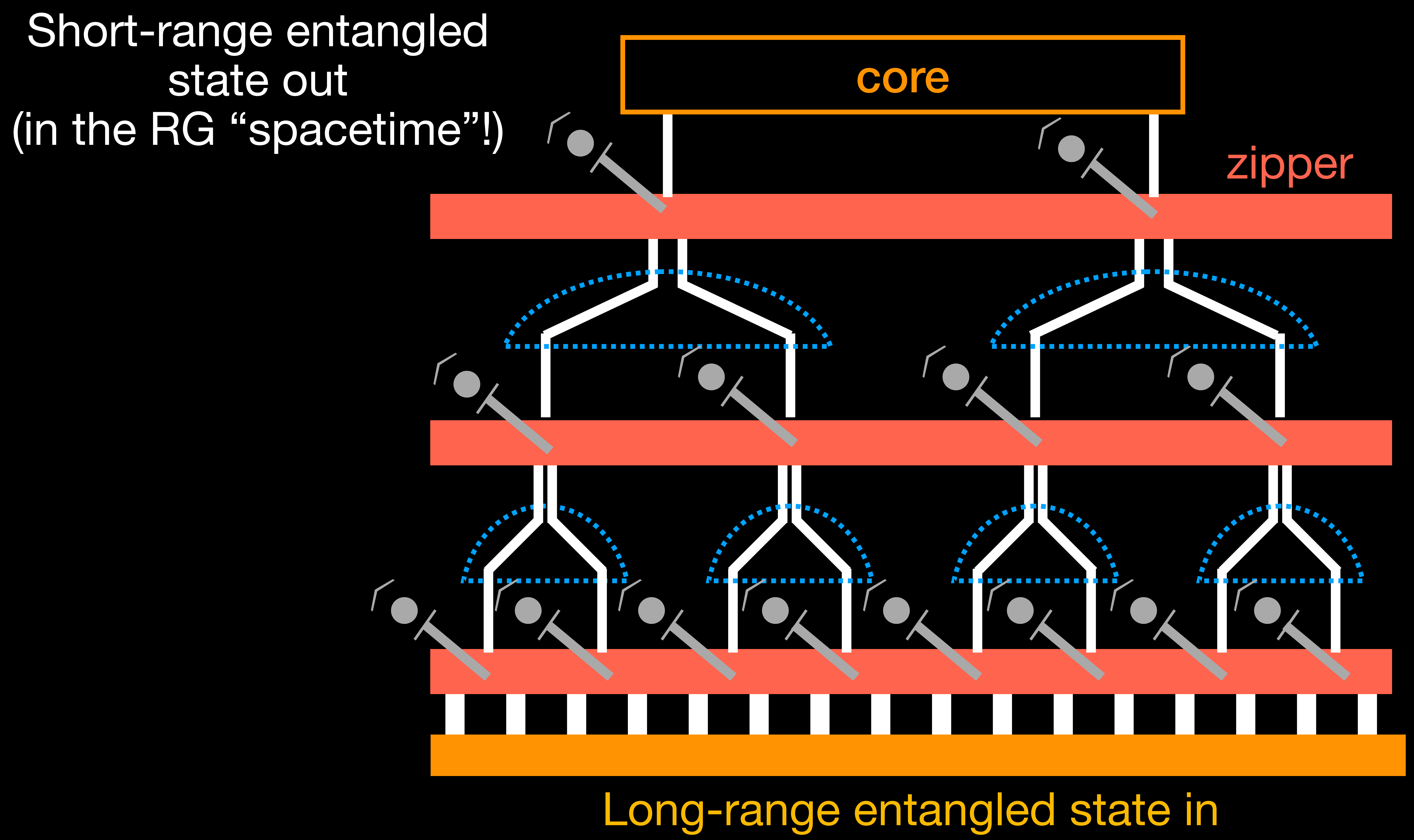


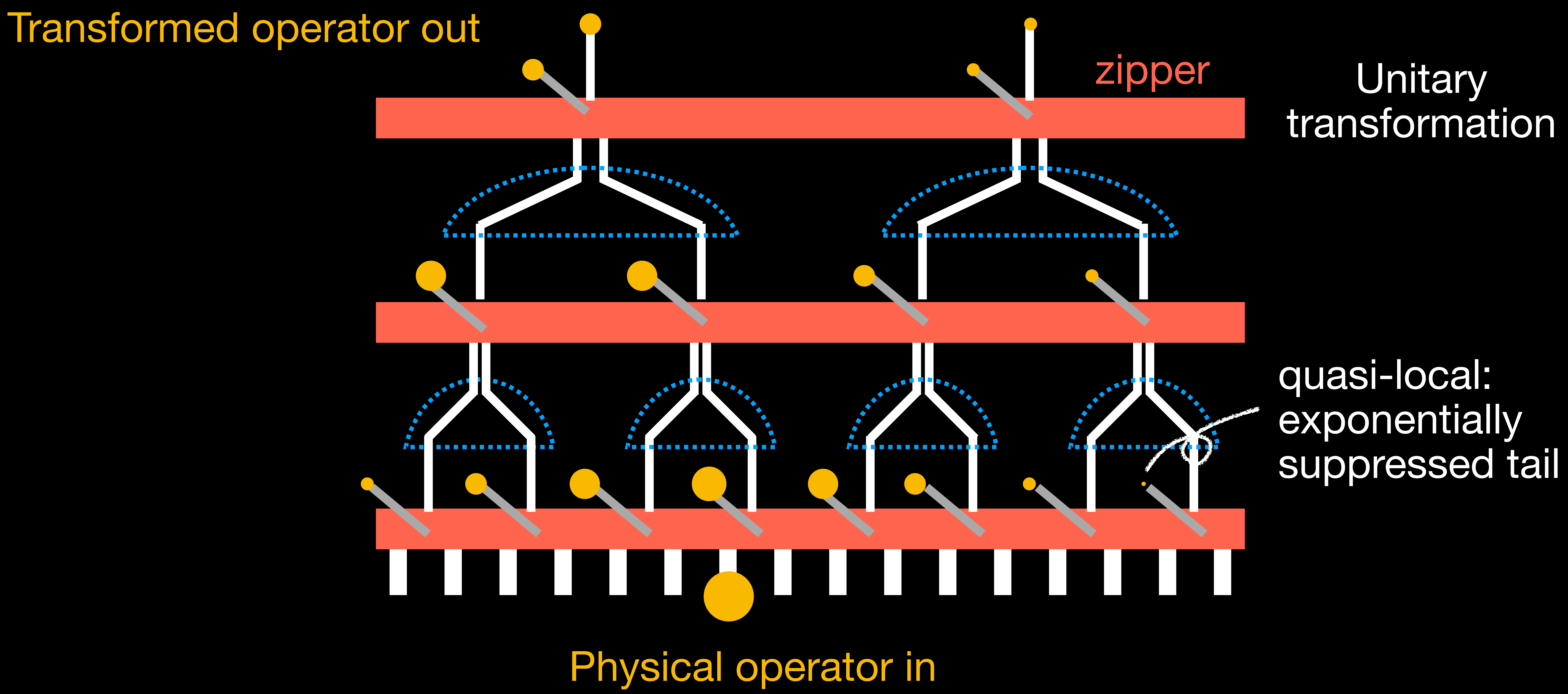

Renormalization time
$\hat{u}^{(2)}_{\text{zipper}} |\Psi^{(2)}\rangle \approx |\Psi^{(3)}\rangle \otimes |\varphi^{(2)}\rangle$
$\hat{u}^{(1)}_{\text{zipper}} |\Psi^{(1)}\rangle \approx |\Psi^{(2)}\rangle \otimes |\varphi^{(1)}\rangle$
$\hat{u}^{(0)}_{\text{zipper}} |\Psi^{(0)}\rangle \approx |\Psi^{(1)}\rangle \otimes |\varphi^{(0)}\rangle$
Input/ physical state $|\Psi^{(0)}\rangle$

$$\hat{U}_{\text{zipper}}|\Psi^{(0)}\rangle = \cdots \hat{u}^{(2)}_{\text{zipper}}\hat{u}^{(1)}_{\text{zipper}}\hat{u}^{(0)}_{\text{zipper}}|\Psi^{(0)}\rangle$$
$$\approx \boxed{|\Psi^{\text{core}}\rangle} \otimes \cdots \otimes |\varphi^{(2)}\rangle \otimes |\varphi^{(1)}\rangle \otimes |\varphi^{(0)}\rangle$$






Short-range entangled
state out
(in the RG “spacetime”!)
core
zipper
Unitary
transformation
Long-range entangled state in


Transformed operator out
zipper
Unitary
transformation
quasi-local:
exponentially
suppressed tail
Physical operator in

Compared to other schemes…

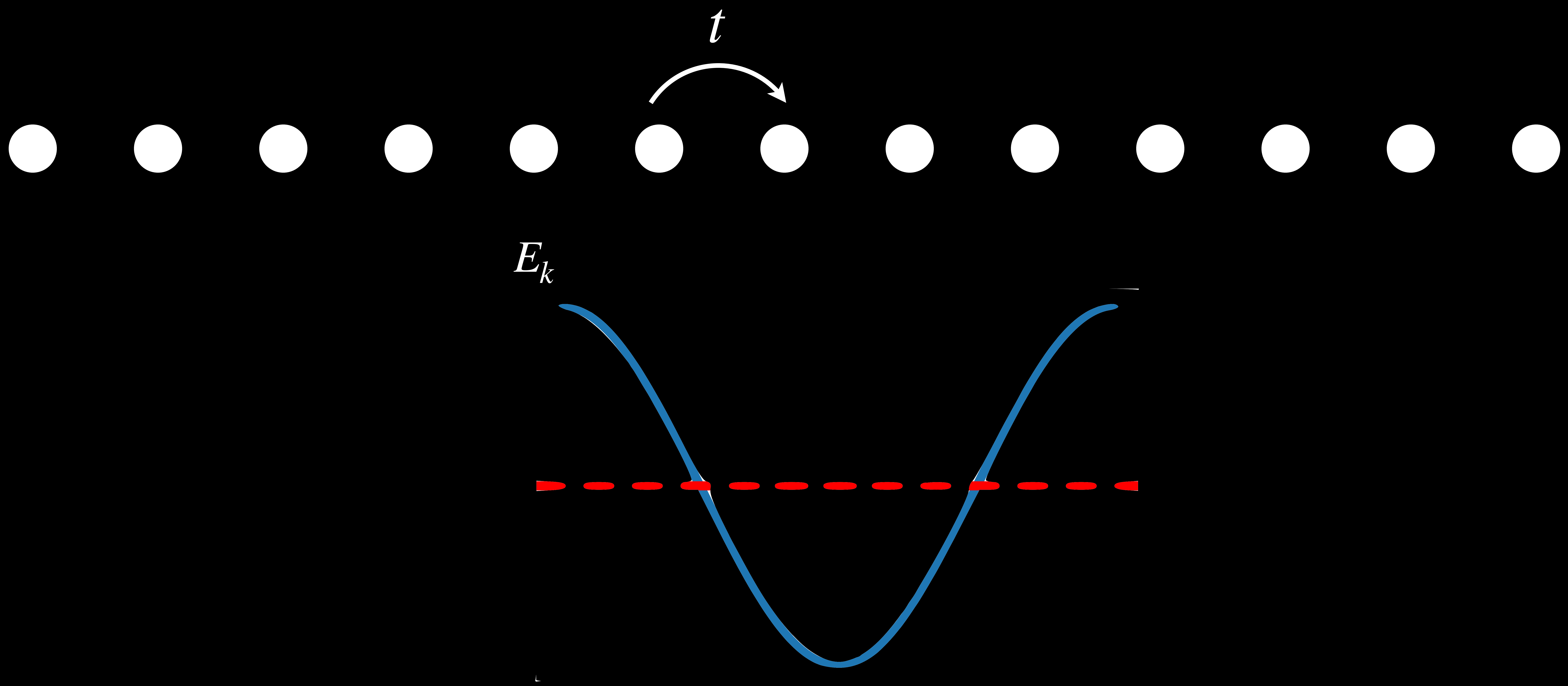
Zipper Entanglement Renormalization is

- Unitary
- Designed for free fermions (at least for now)
- state-based, i.e. not variational/ more deterministic
- Quasi-local, i.e., with exponentially decaying tail
- More versatile!

Zipper: what could it do?

“Proof of principle”: free fermions in 1D

Usual benchmark: uniform chain



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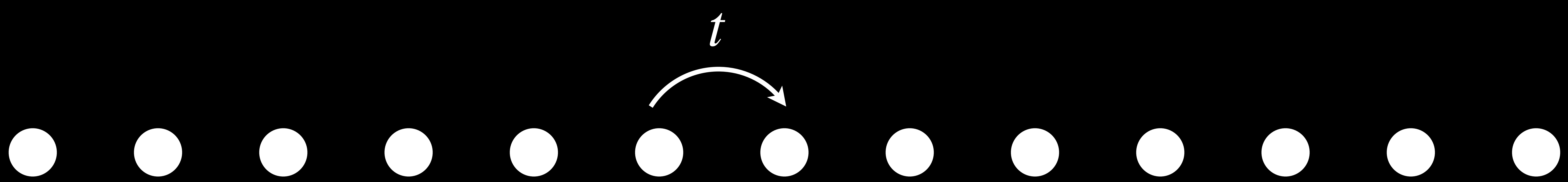

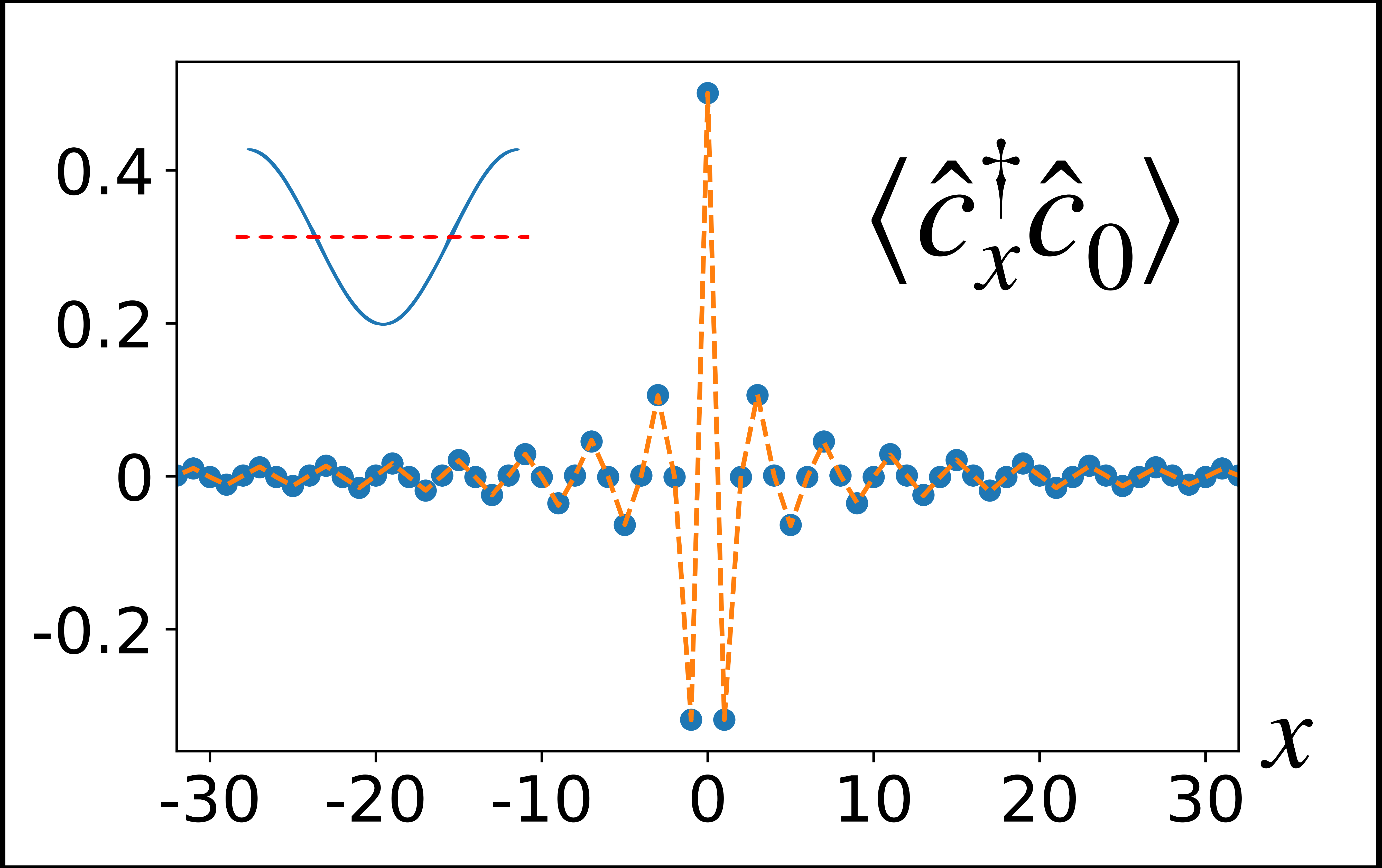

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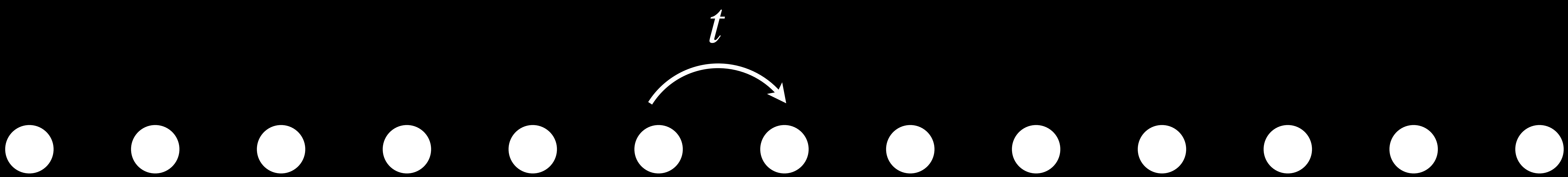
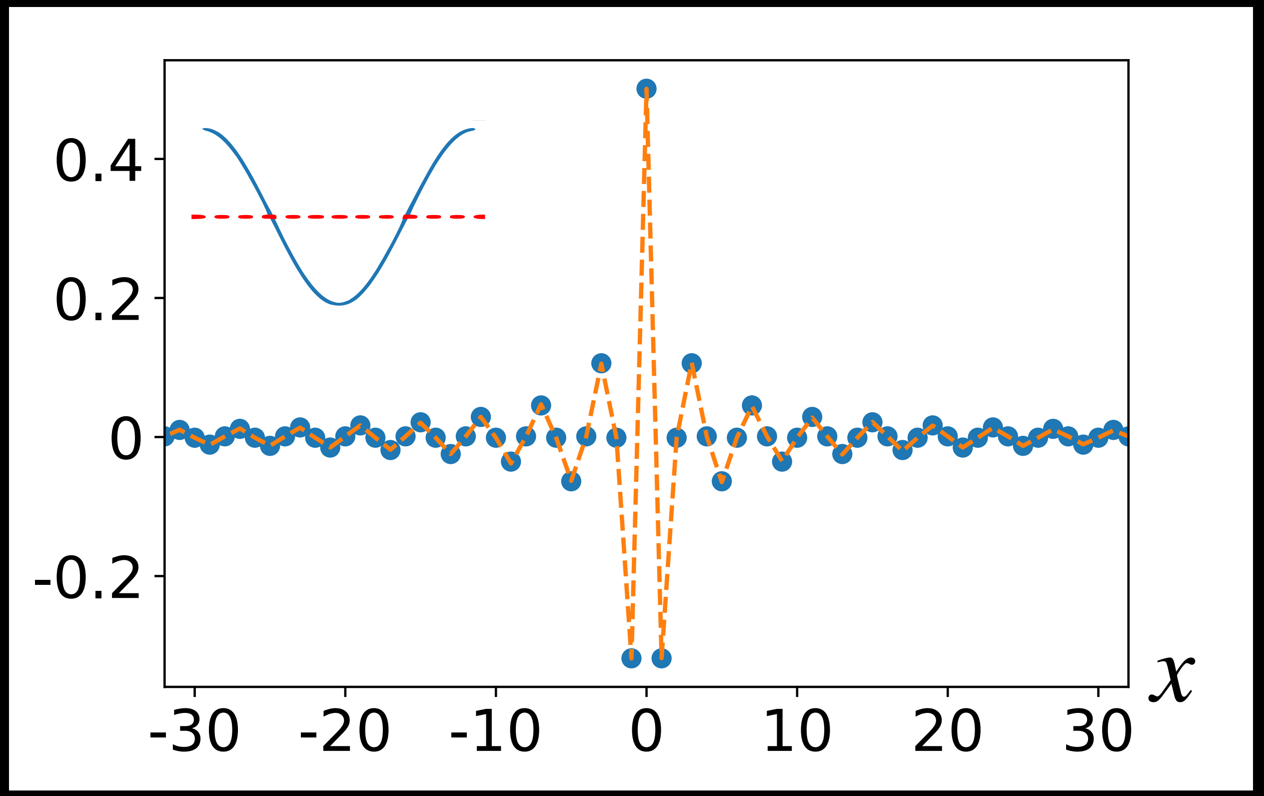
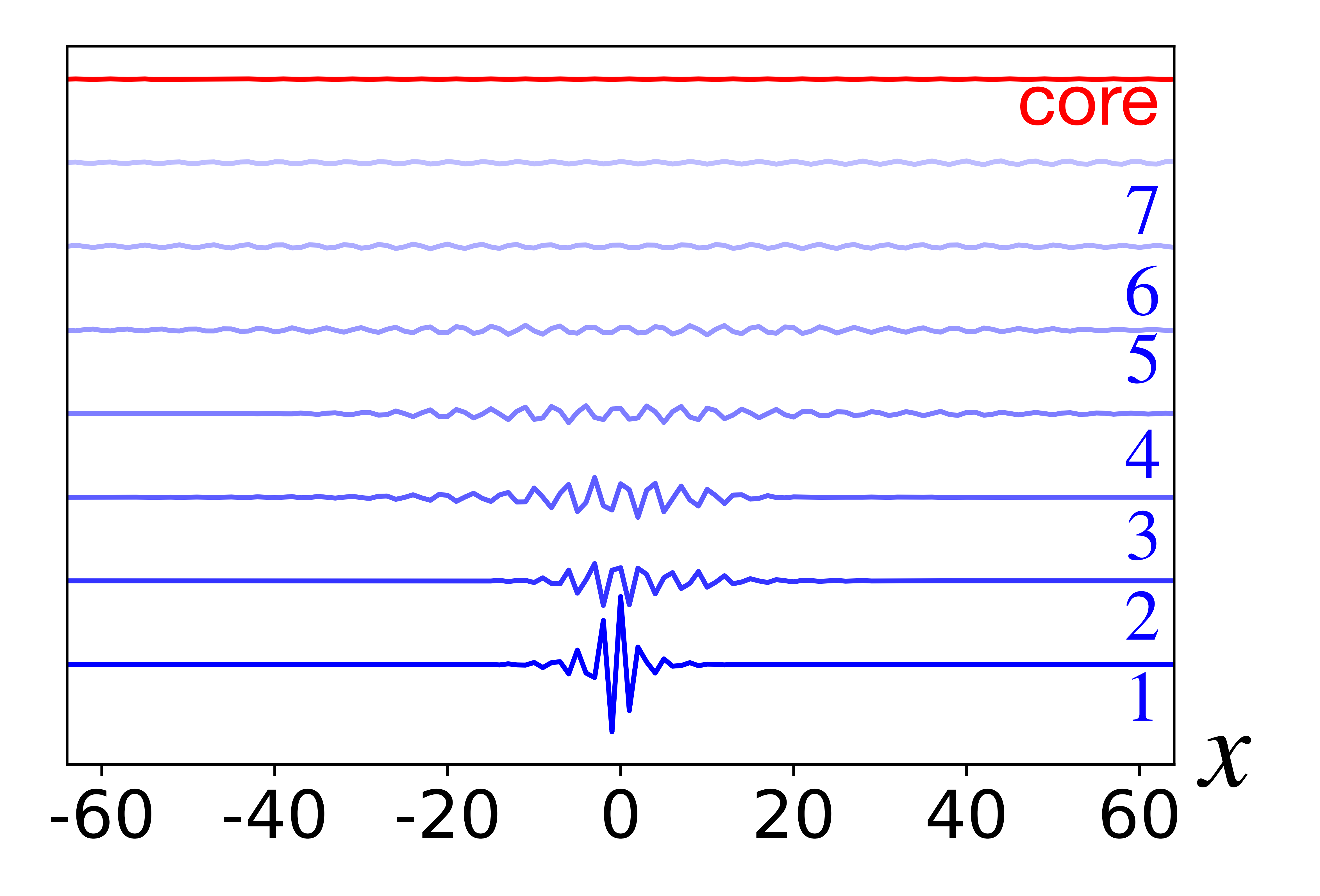




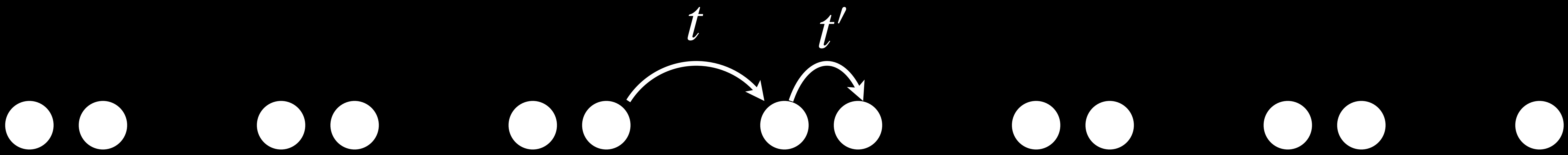


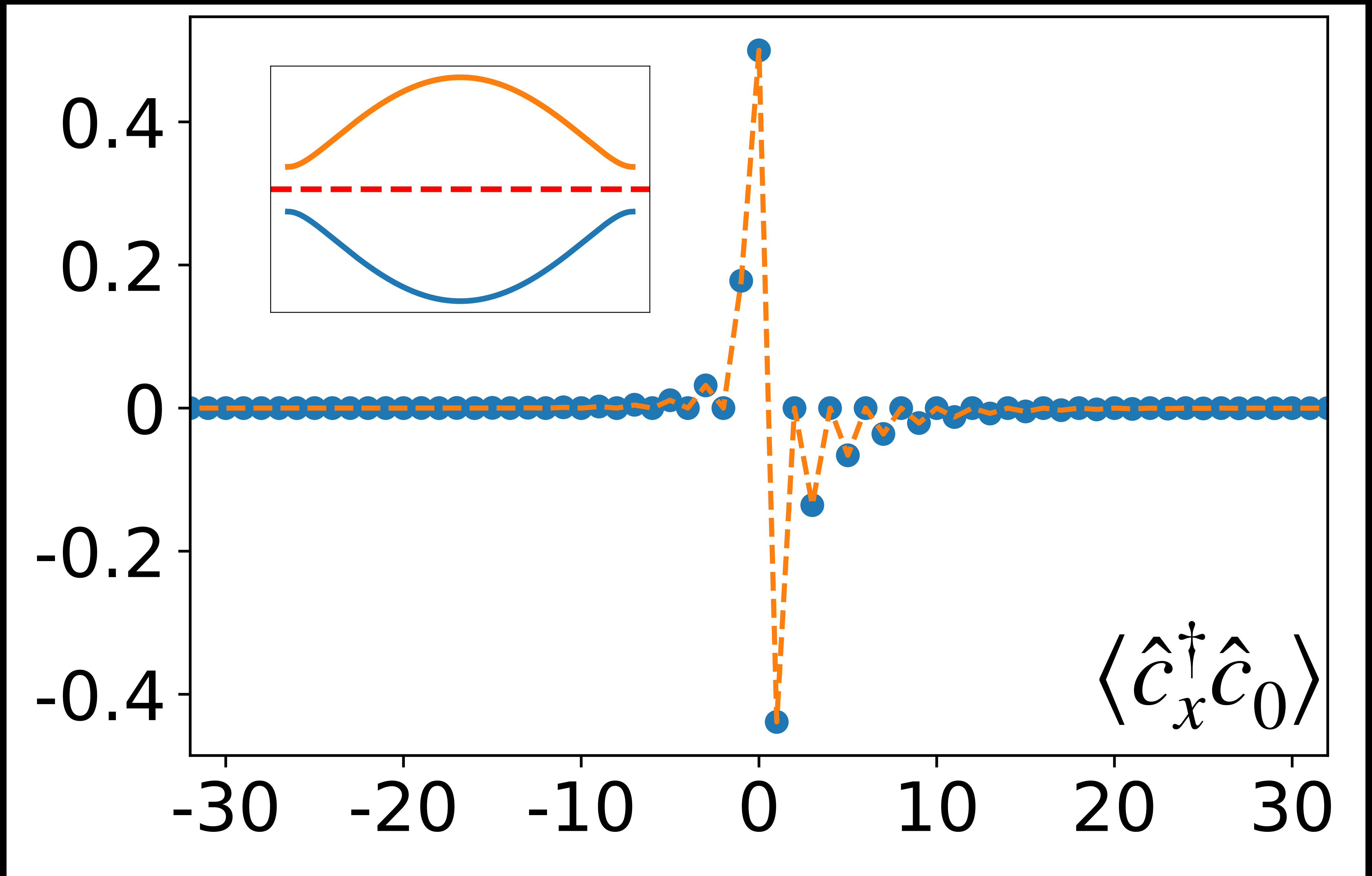
$\langle \hat{c}_x^\dagger \hat{c}_0 \rangle$

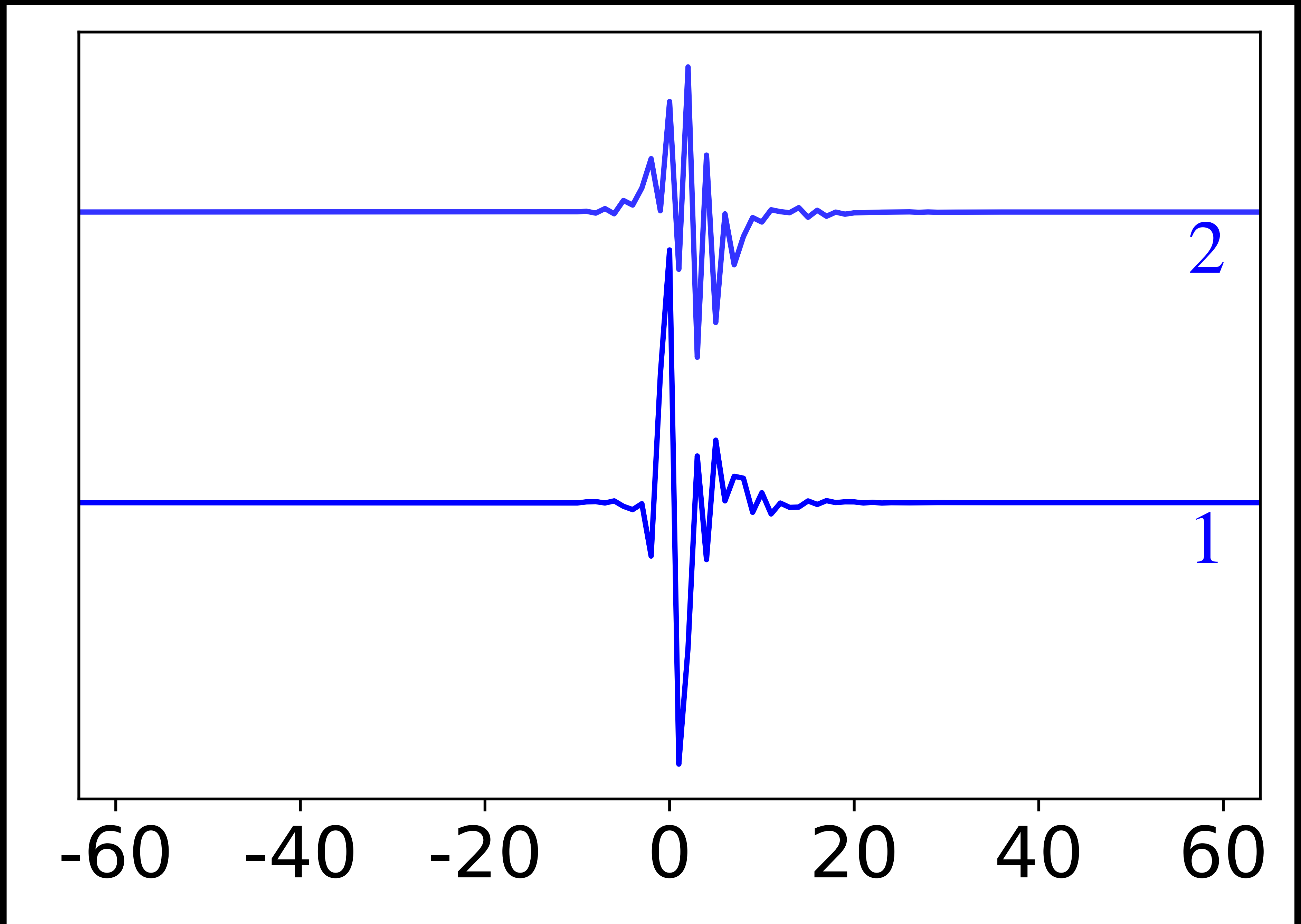
Contribution form different RG levels

[c.f. Evenly & White, PRL 2016, PRA 2018; Haegeman et al, PRX 2018]

Contrasting example: Su-Schrieffer-Heeger

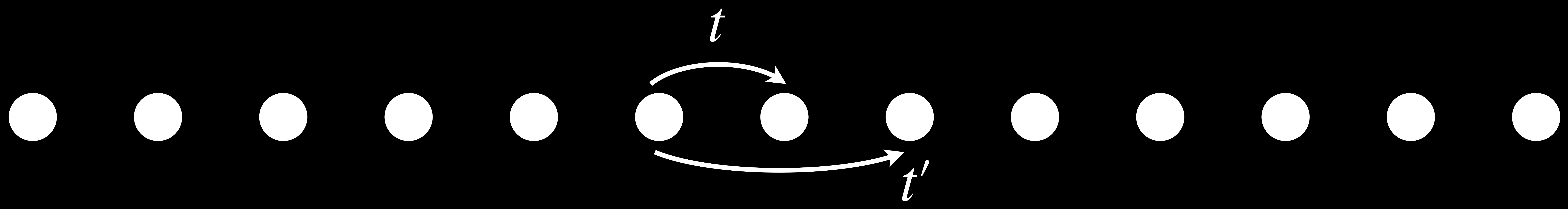


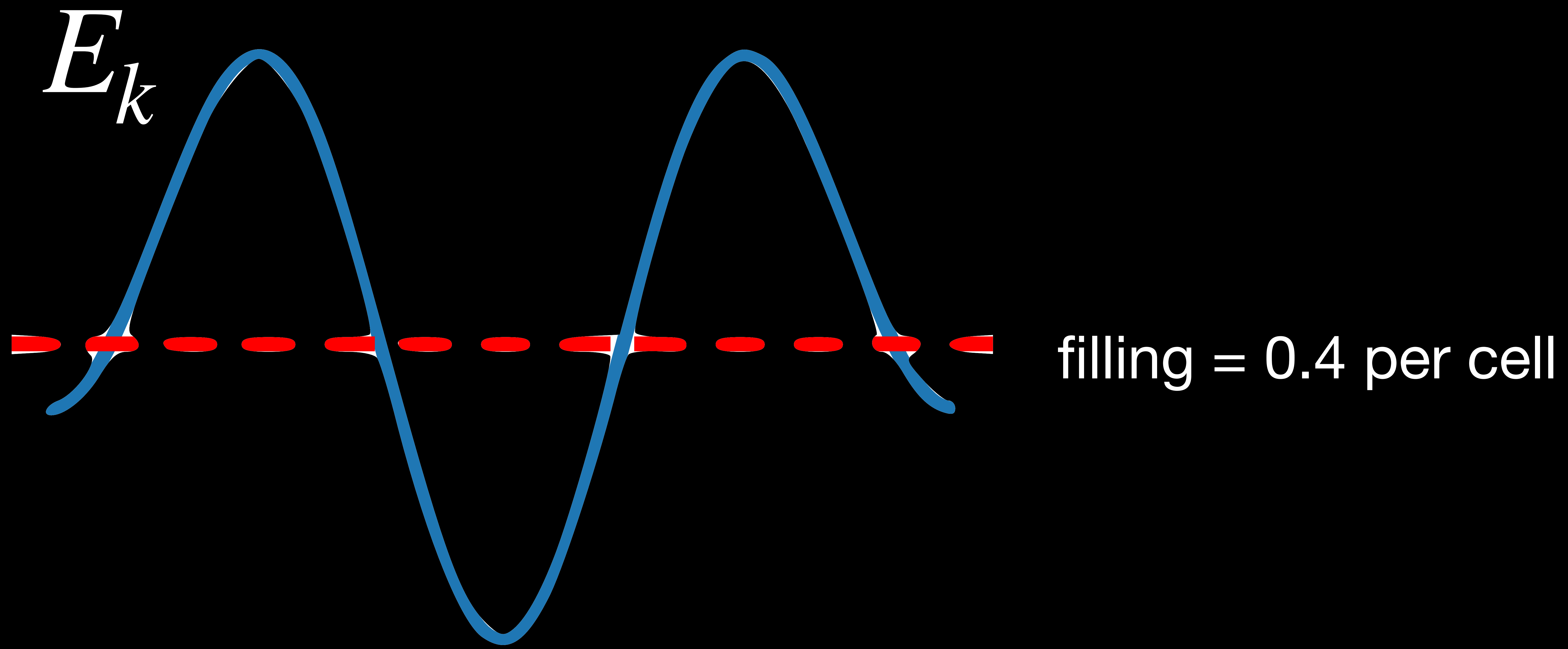




A more interesting example:

multiple Fermi points & away from half filling





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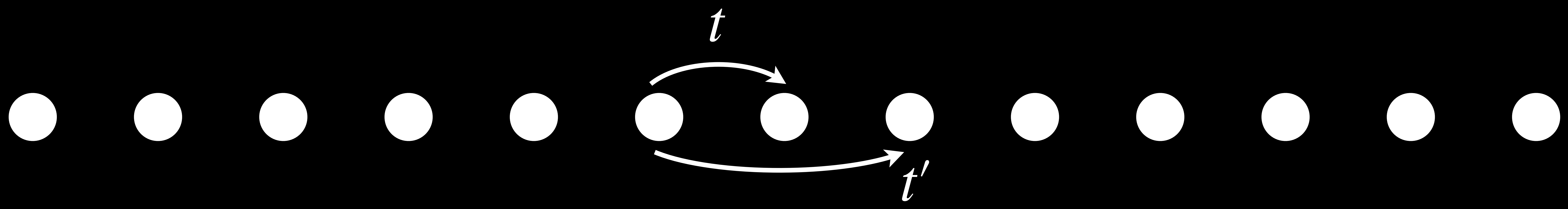
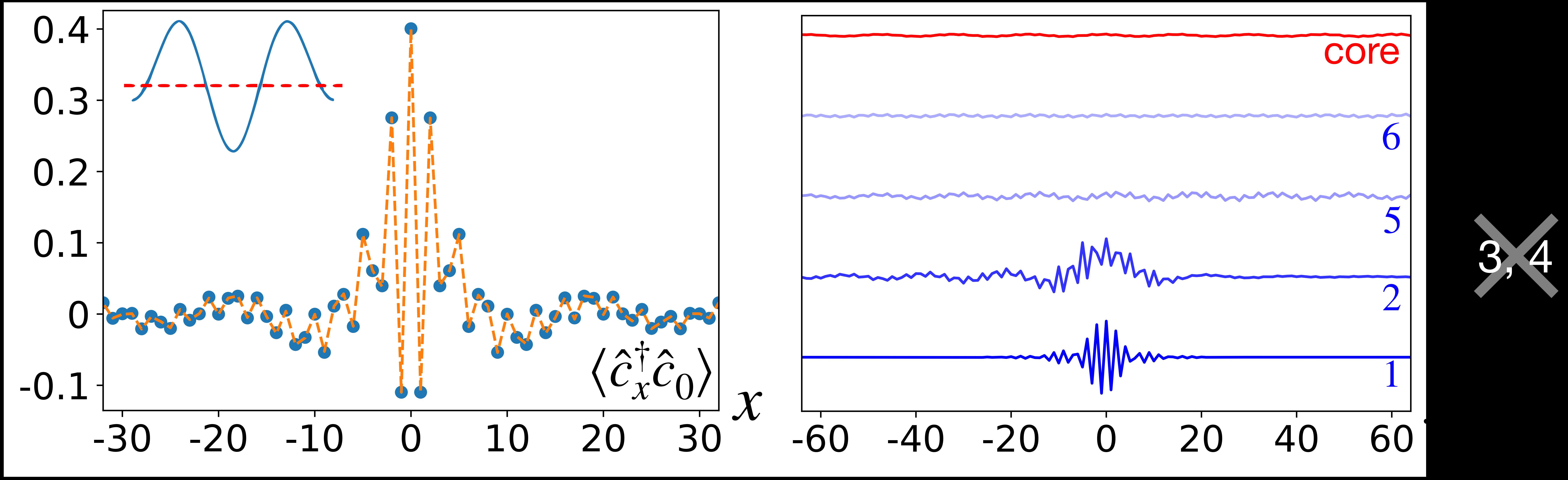
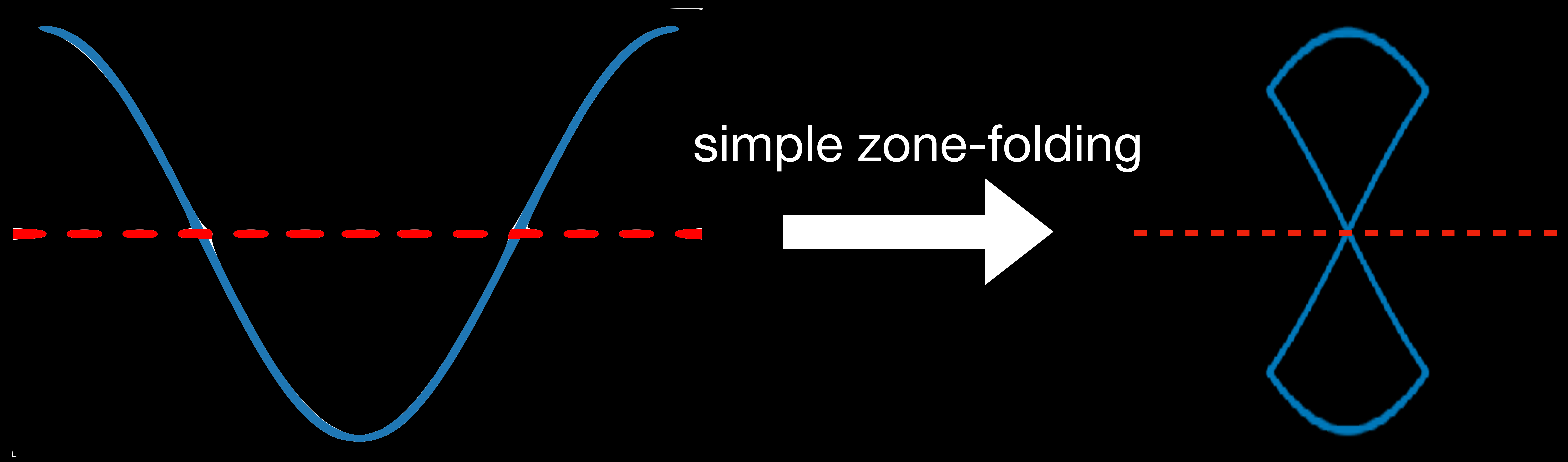




Physical picture—

connecting with conventional RG



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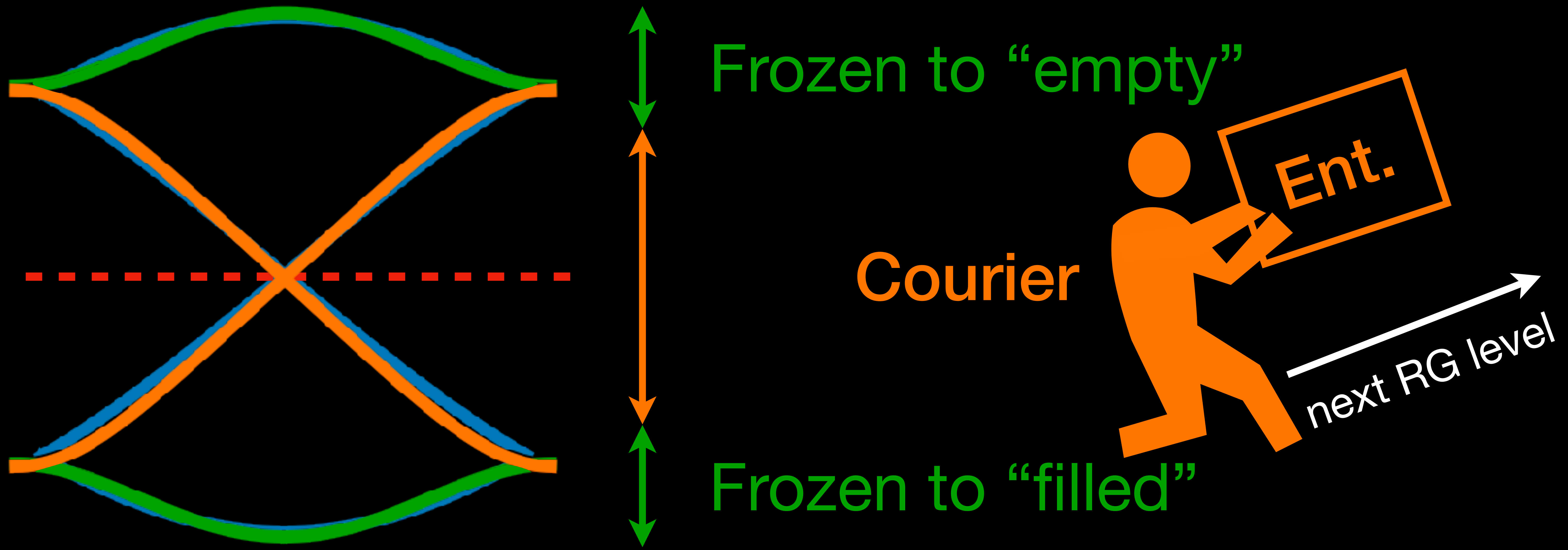

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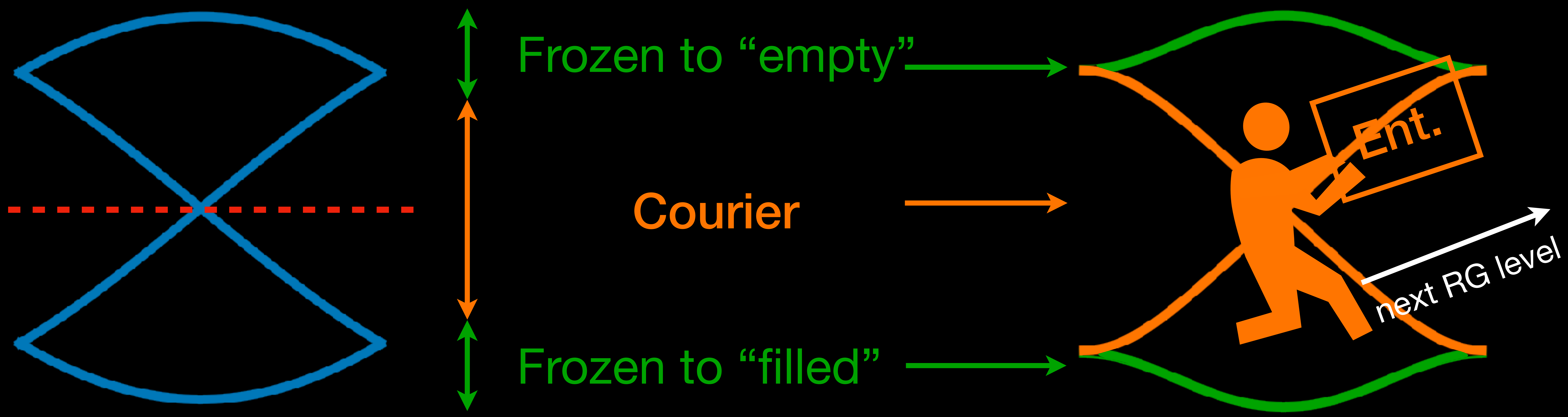

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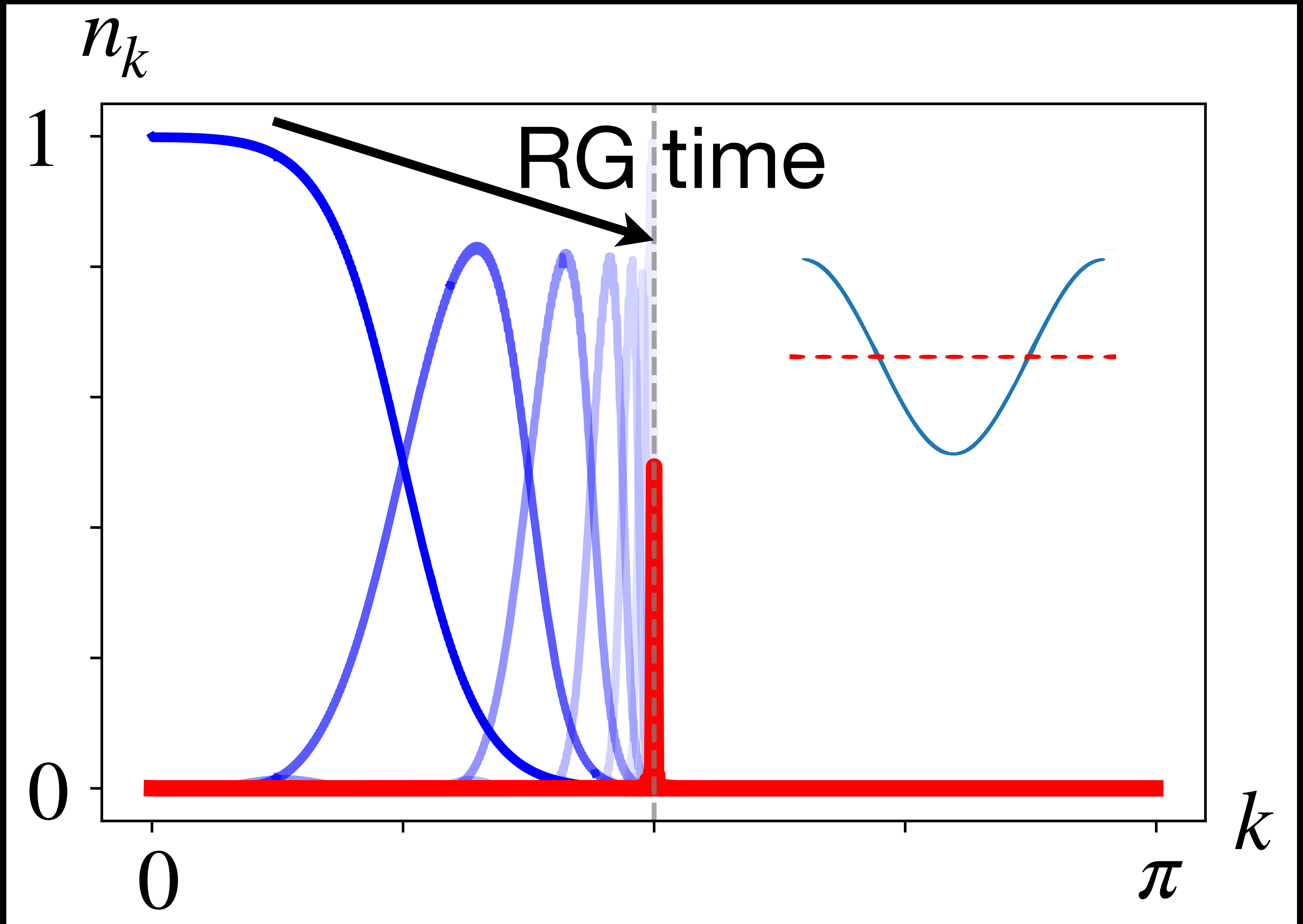
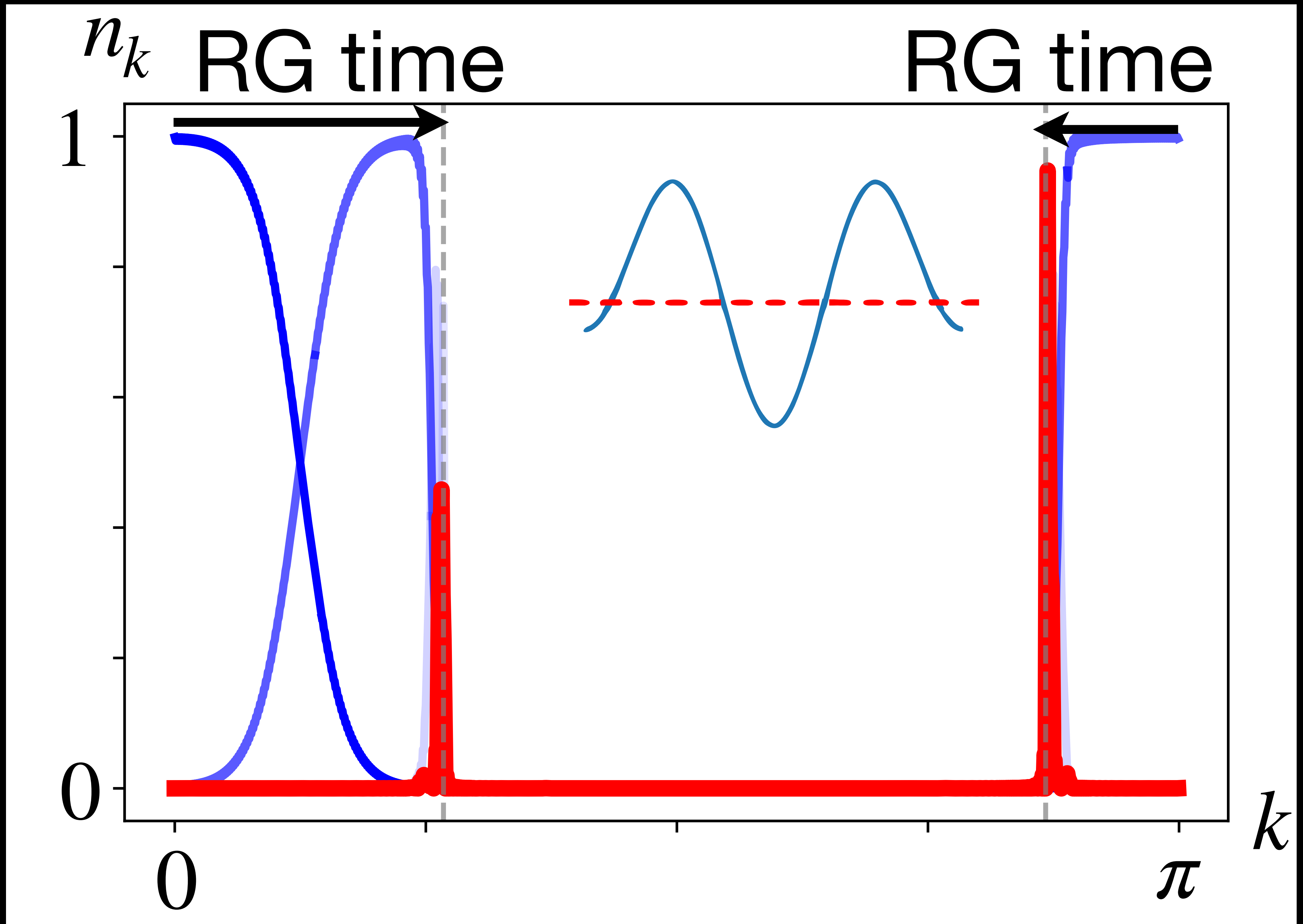




Momentum occupation n_k of the distilled modes against RG time

Zipper: how was it done?

In the interest of time, please see 2206.11761 :)

Free-fermion ZER:

algorithm

repeat until nothing left

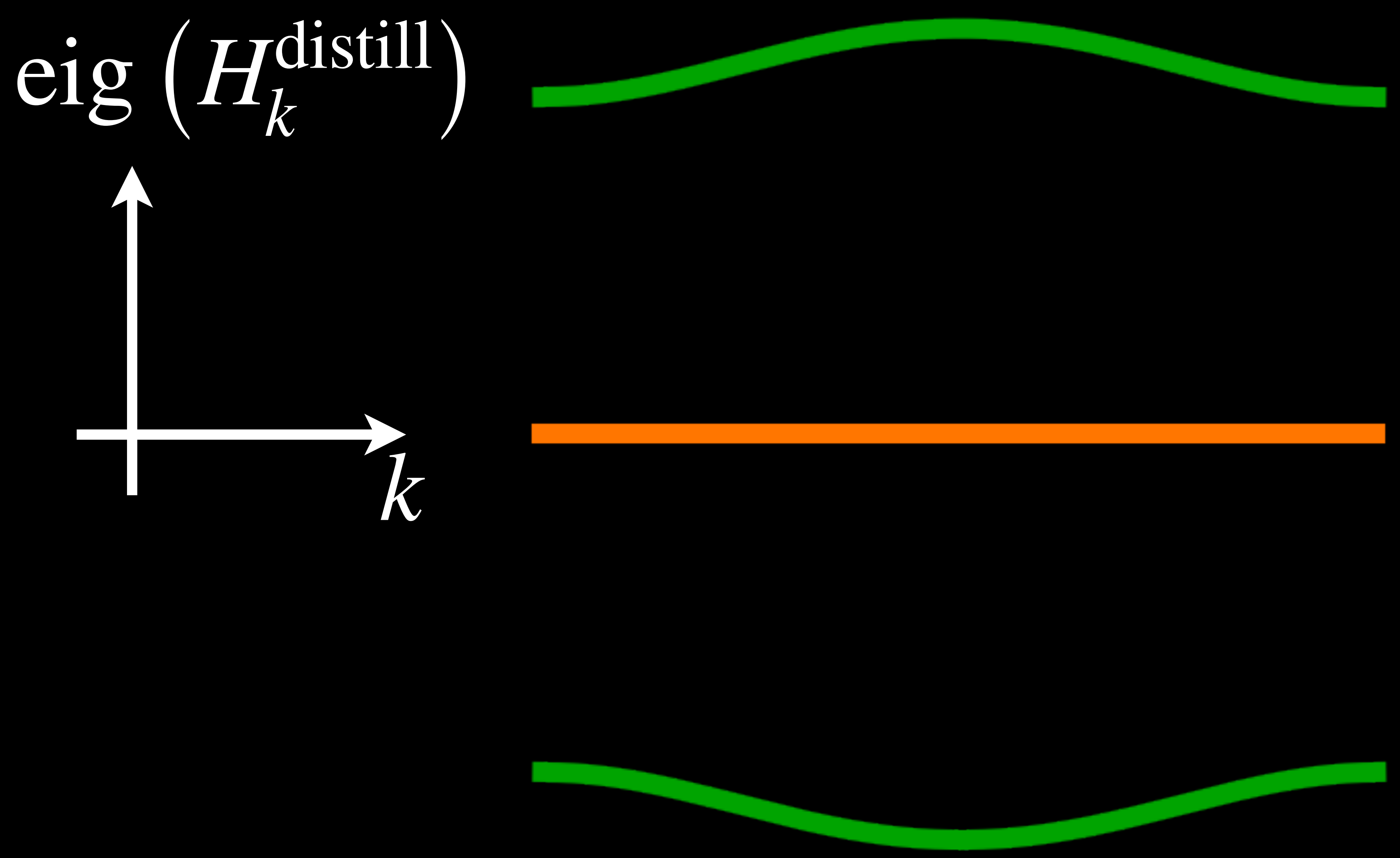
Input: a free-fermion state (+ a O(1) length scale ξ & a small threshold ϵ)	
try:	
‣ Distillation	# distill frozen from courier
‣ Local distillation (over scale ξ & controled by ϵ)	# through entanglement of local regions
‣ Combine to perform global distillation	# by “combining” the local modes
‣ Re-localized courier modes	# with the help of Wannier
except Distillation Error:	# no frozen modes/ cannot Wannier
‣ Simple blocking	# i.e., zone folding
output: a new free-fermion state	# defined in the courier/ blocked modes

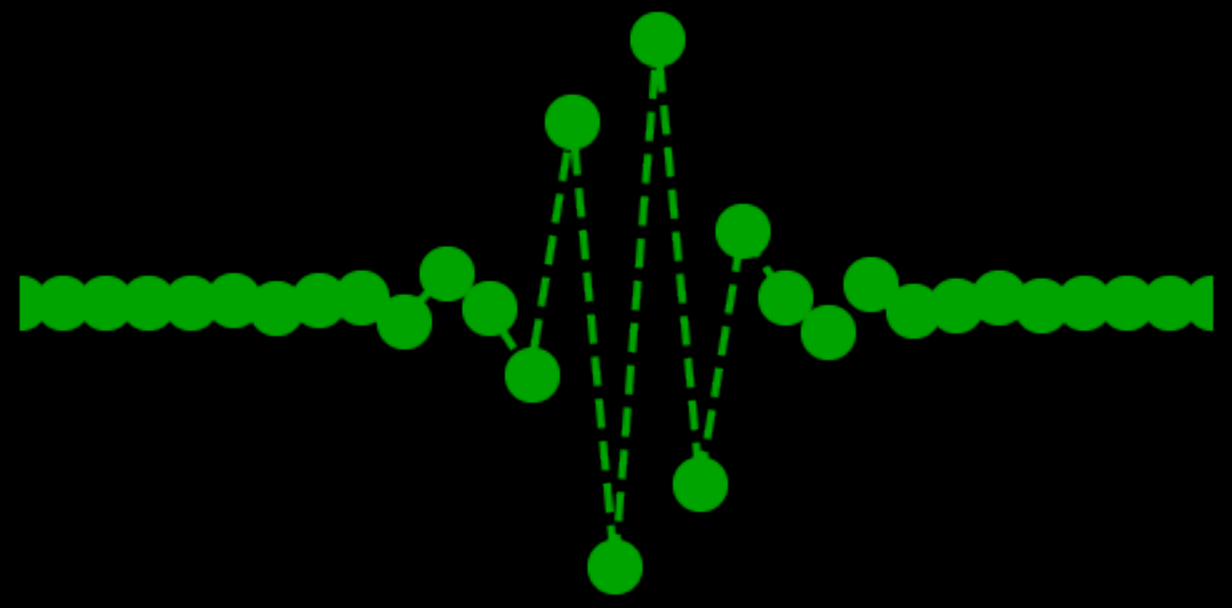
Key technical construction:

the global distiller & Wannierization

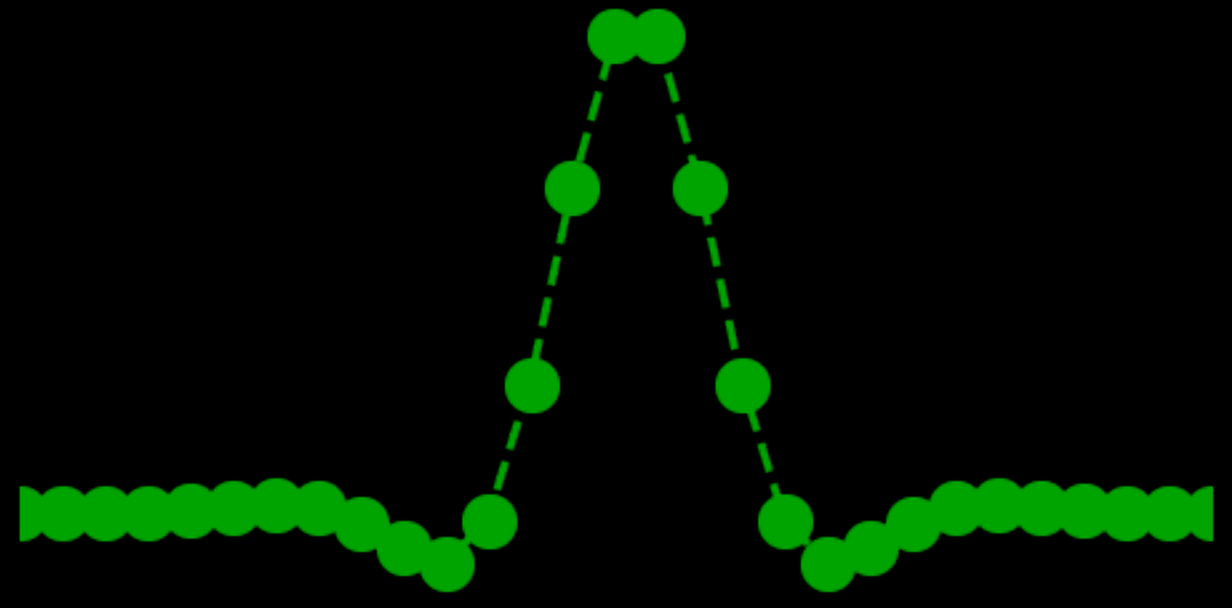
[c.f. Evenly & White, PRL 2016, PRA 2018;
Haegeman et al, PRX 2018]

$$H^{\mathrm{distill}} = \sum_x h_x^{\mathrm{distill}} \Rightarrow$$



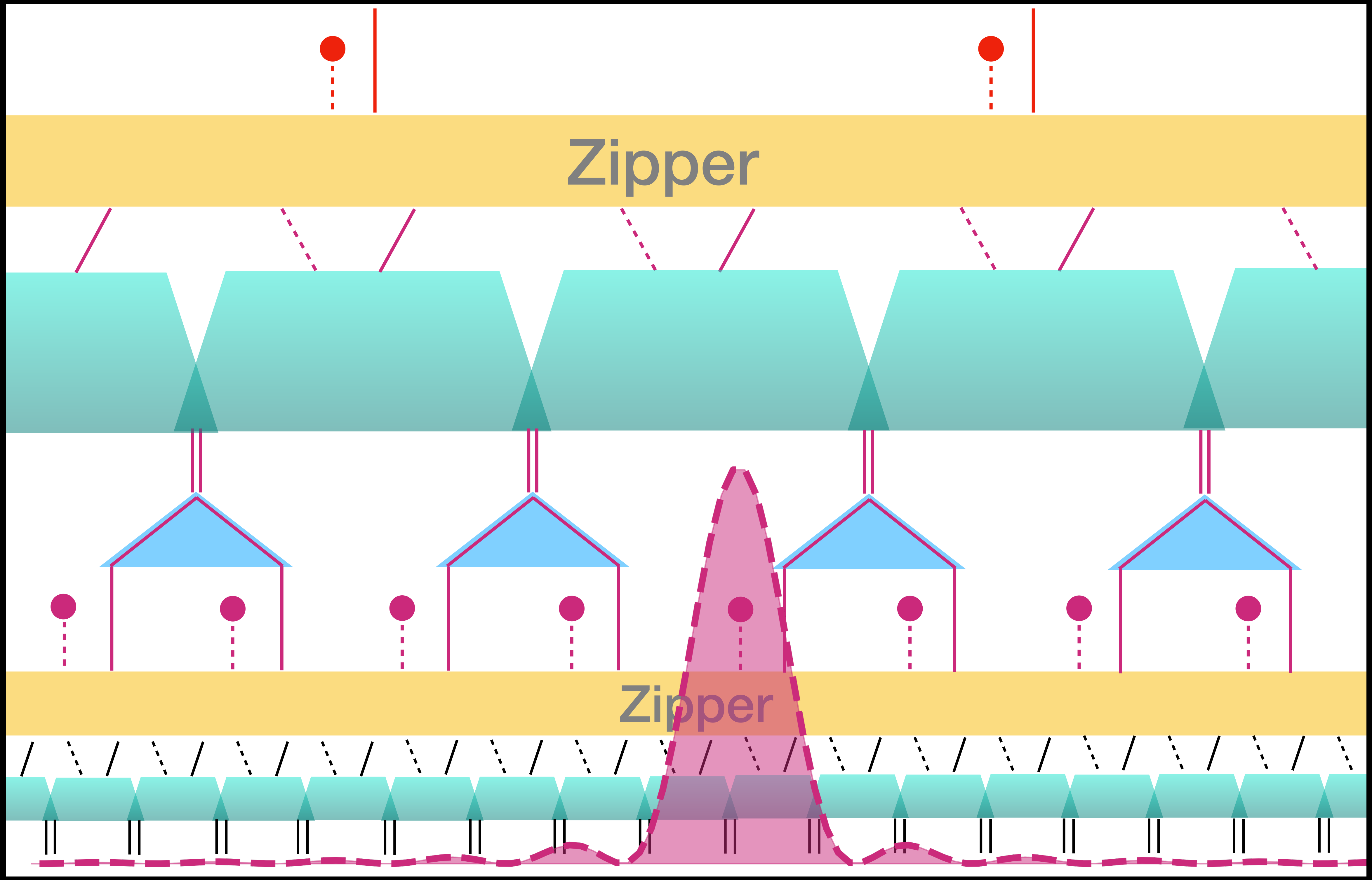


$\Rightarrow$ Wannierize



Summary

- Zipper entanglement renormalization (ZER) is
  - Unitary
  - Quasi-local
  - State-based
  - Versatile



- ZER could
  - Work in any dimension
  - Reveal the “natrual” RG spacetime associated with a state
  - Be the starting point for attacking more interesting/ challenigng problem