SNS junction of topological superconductors revisited:

Fractional Josephson current, fermion parity, and oscillating wave functions

Shuntaro Sumita (The Univ. of Tokyo, RIKEN)
Collaborator: Akira Furusaki (RIKEN)

SS & A. Furusaki, Phys. Rev. B **104**, 205431 (2021)

CREST

Outline

1. Introduction
2. Model of 1D topological SNS junction
3. Perturbation theory
4. Effective model connecting on- & off-resonance
5. Exact diagonalization

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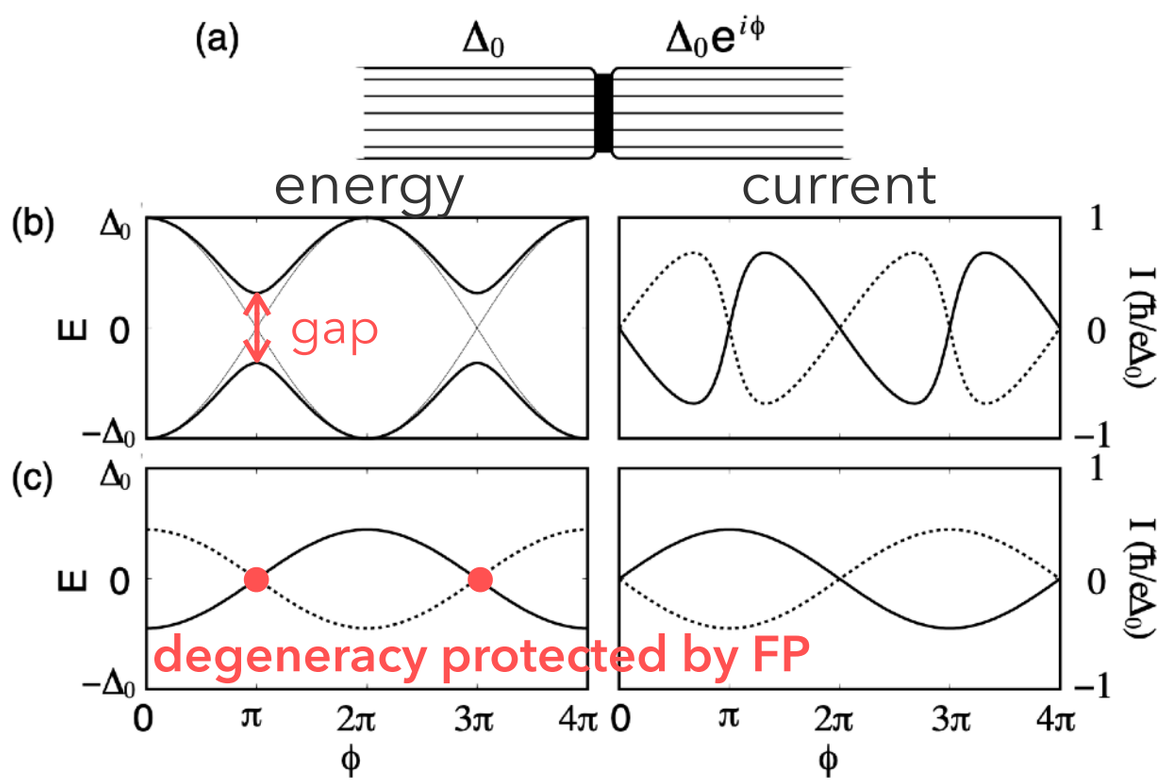
Introduction

- **Majorana zero modes** in topological superconductor
  - expectation for quantum memory & computing
- **Fractional Josephson effect** in topological junction
  - a promising phenomenon for MZMs A. Y. Kitaev (2001) L. Fu & C. L. Kane (2009)
  - ground-state **fermion parity (FP)** switches every 2π

s-*s* junction
(2π periodicity)

p-*p* junction
(4π periodicity)



H.-J. Kwon *et al.*,
Eur. Phys. J. B **37**, 349 (2004)

Semi-infinite Kitaev chain

S. S. Hegde & S. Vishveshwara, PRB **94**, 115166 (2016)

- **Kitaev chain**: a prototype of 1D TSC A. Y. Kitaev, Phys. Usp. **44**, 131 (2001)

$$H = -2\sum_{j=1}^{L}\mu\left(c_j^\dagger c_j - \frac{1}{2}\right) + \sum_{j=1}^{L-1}\left(-tc_j^\dagger c_{j+1} + \Delta e^{i\varphi}c_j^\dagger c_{j+1}^\dagger + \text{H.c.}\right)$$

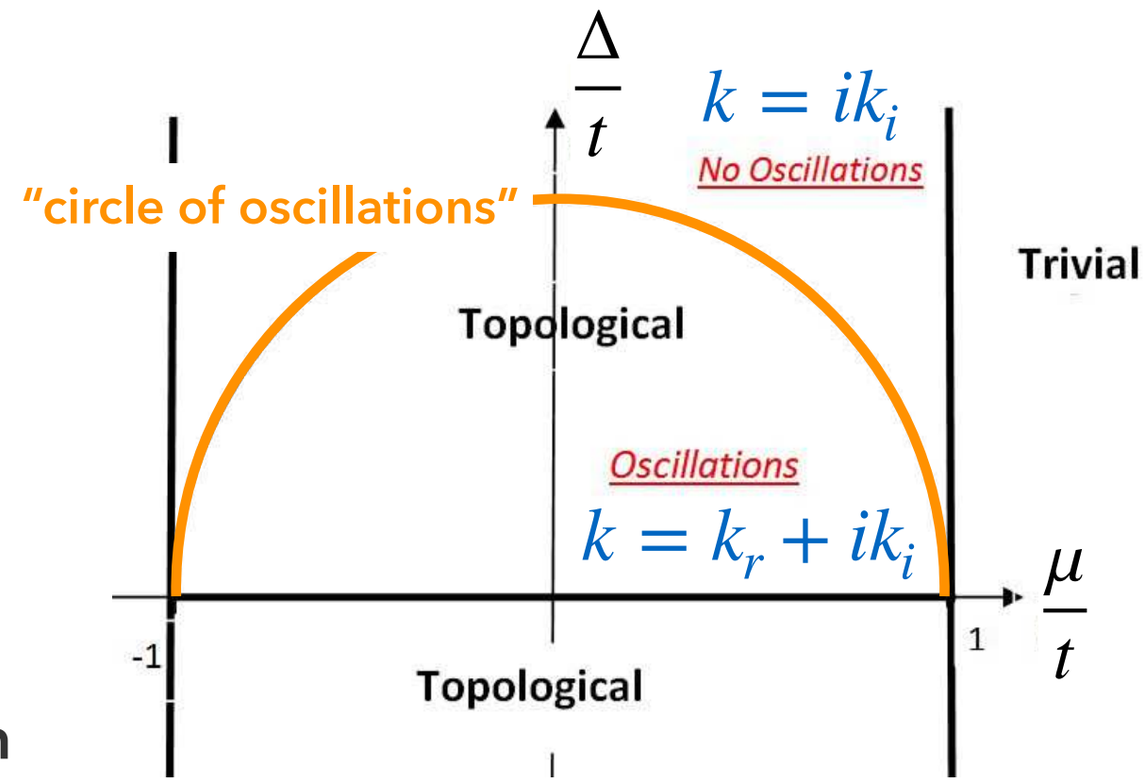
- Left-localized Majorana w.f. for $\underline{L \to \infty}$ & $|\mu/t| < 1$

$$|\psi\rangle = \sum_{j=1}^{\infty}(z_1^j - z_2^j)\,|j\rangle \begin{bmatrix} 1 \\ -e^{-i\varphi} \end{bmatrix}$$

$$z_{1,2} = \frac{-\mu \pm \sqrt{\mu^2 + \Delta^2 - t^2}}{t + \Delta}$$

- $z \sim e^{ik}$: **complex wave #**

  cf. damped oscillation / overdamping in harmonic oscillation with friction

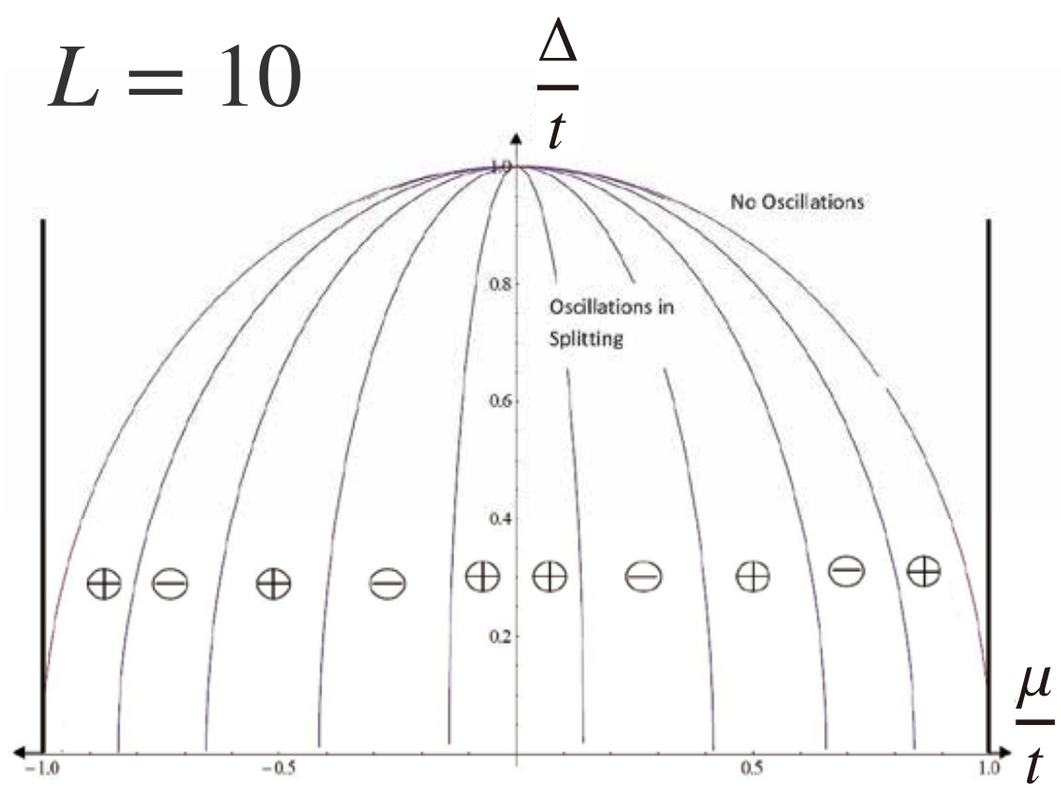


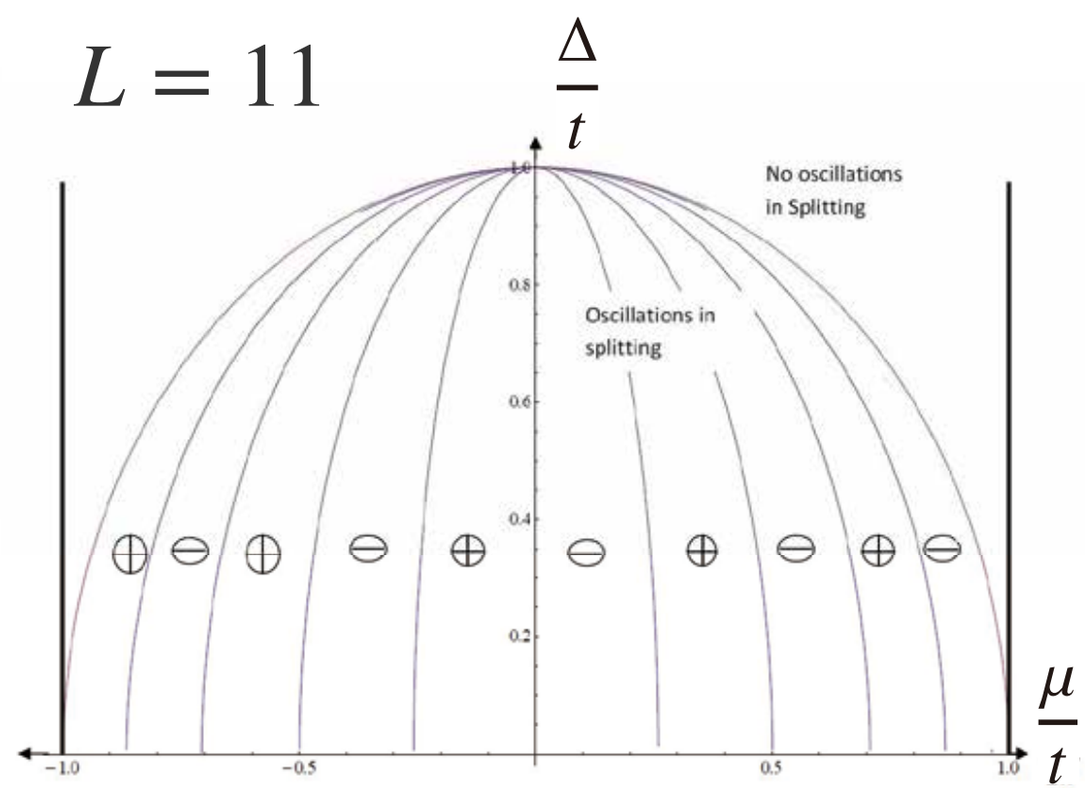
Finite-size effect on Kitaev chain

S. S. Hegde & S. Vishveshwara, PRB **94**, 115166 (2016)

- In general, a tiny energy gap for finite L
- Zero energy solution on: H.-C. Kao, Phys. Rev. B **90**, 245435 (2014)
  S. S. Hegde *et al.* New J. Phys. **17**, 053036 (2015)

$$\left(\Delta^2 + \mu^2 \sec^2\left(\frac{\pi p}{L+1}\right) = t^2, \quad p = \begin{cases} 1, 2, \ldots, \frac{L}{2} & L = \text{even} \\ 1, 2, \ldots, \frac{L-1}{2} & L = \text{odd} \end{cases}\right) \quad \& \quad (\mu = 0 \text{ for } L = \text{odd})$$

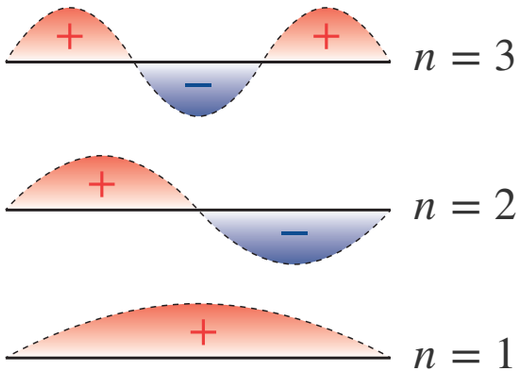




Oscillating behavior → switching of ground-state FP!

Finite-size effect on topological SNS junction

- # of nodes in wavefunction of nth lowest level in N region depends on n
  → **# of oscillations controlled by μ**



- **Fractional Josephson current** A. Y. Kitaev (2001) / H.-J. Kwon *et al.* (2004) L. Fu & C. L. Kane (2009)
  ∝ the first power of the electron tunneling amplitude
  → **reflect oscillating behavior of the N wavefunction**
- $\varepsilon^{\mathrm{N}} \simeq \varepsilon^{\mathrm{MZM}} = 0$ → **strong enhancement of supercurrent**

Question

Can the fractional Josephson current be switched by μ? → **No!**

- **SNS junction w/ intermediate-length** (≠ short-, long-junction)

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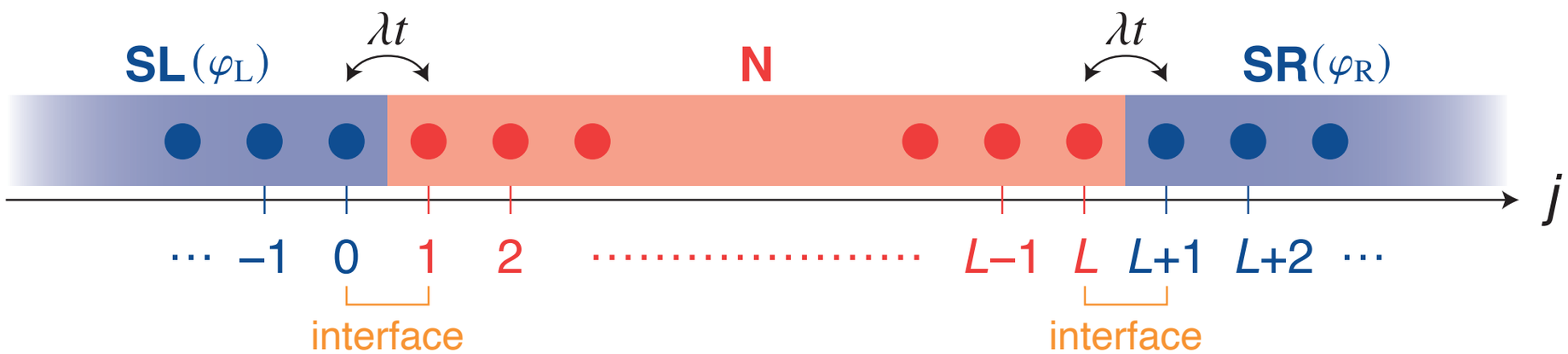
Model

- **An SNS junction using Kitaev chains** (normal: L sites)



- Hamiltonian ($t, \Delta > 0$, $\underbrace{0 \leq \lambda \ll 1}_{\text{SINIS}}$, $\underbrace{|\mu/t| < 1}_{\text{Majorana}}$)

$$H = H_{\mathrm{SL}} + H_{\mathrm{N}} + H_{\mathrm{SR}} + H_{\mathrm{T}}$$

$$H_{\mathrm{SL}} = -2 \sum_{j=-\infty}^{0} \mu \left(c_j^\dagger c_j - \frac{1}{2} \right) - \sum_{j=-\infty}^{-1} (t c_j^\dagger c_{j+1} - \Delta e^{i\varphi_{\mathrm{L}}} c_j^\dagger c_{j+1}^\dagger + \mathrm{H.c.})$$

$$H_{\mathrm{N}} = -2 \sum_{j=1}^{L} \mu \left(c_j^\dagger c_j - \frac{1}{2} \right) - \sum_{j=1}^{L-1} (t c_j^\dagger c_{j+1} + \mathrm{H.c.})$$

$$H_{\mathrm{SR}} = -2 \sum_{j=L+1}^{\infty} \mu \left(c_j^\dagger c_j - \frac{1}{2} \right) - \sum_{j=L+1}^{\infty} (t c_j^\dagger c_{j+1} - \Delta e^{i\varphi_{\mathrm{R}}} c_j^\dagger c_{j+1}^\dagger + \mathrm{H.c.})$$

$$H_{\mathrm{T}} = -\lambda (t c_0^\dagger c_1 + t c_L^\dagger c_{L+1} + \mathrm{H.c.})$$

t: hopping parameter

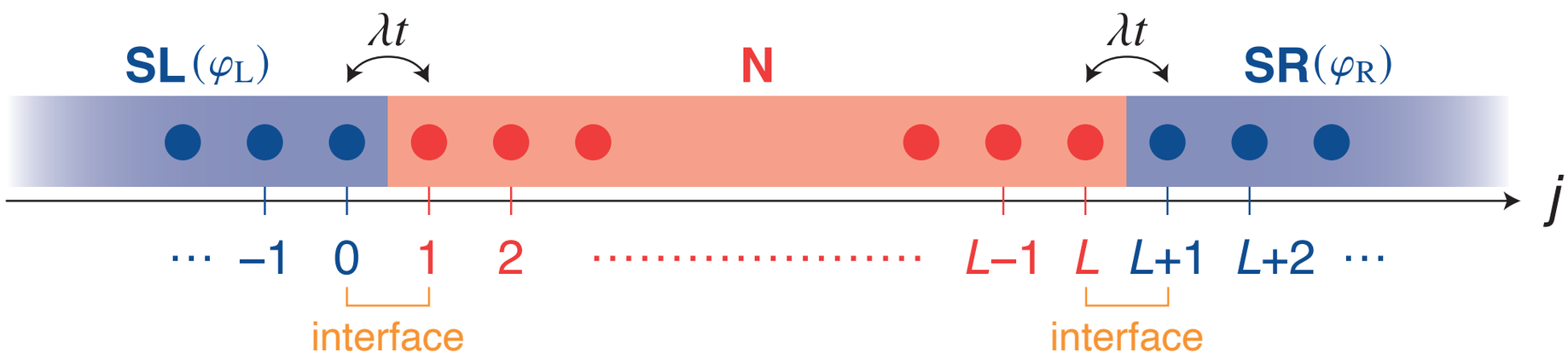
μ: chemical potential

Δ: p-wave order parameter

$\varphi_{\mathrm{L}}, \varphi_{\mathrm{R}}$: SC U(1) phase

Model

- **First-quantized form** of the Hamiltonian



$$H_{\mathrm{SL}} = \mathbb{1}_{\mathrm{SL}} \otimes h_0^{\mathrm{S}} + \left(T_{\mathrm{SL}} \otimes h_1^{\mathrm{S}}(\varphi_{\mathrm{L}}) + T_{\mathrm{SL}}^{\dagger} \otimes h_1^{\mathrm{S}}(\varphi_{\mathrm{L}})^{\dagger}\right)$$

$$H_{\mathrm{N}} = \mathbb{1}_{\mathrm{N}} \otimes h_0^{\mathrm{N}} + \left(T_{\mathrm{N}} \otimes h_1^{\mathrm{N}} + T_{\mathrm{N}}^{\dagger} \otimes h_1^{\mathrm{N}\dagger}\right)$$

$$H_{\mathrm{SR}} = \mathbb{1}_{\mathrm{SR}} \otimes h_0^{\mathrm{S}} + \left(T_{\mathrm{SR}} \otimes h_1^{\mathrm{S}}(\varphi_{\mathrm{R}}) + T_{\mathrm{SR}}^{\dagger} \otimes h_1^{\mathrm{S}}(\varphi_{\mathrm{R}})^{\dagger}\right)$$

$$H_{\mathrm{T}} = \lambda \left[(|0\rangle\langle 1| + |L\rangle\langle L+1|) \otimes h_1^{\mathrm{N}} + \mathrm{H.c.} \right]$$

$$h_0^{\mathrm{S}} = h_0^{\mathrm{N}} = \begin{bmatrix} -\mu & 0 \\ 0 & \mu \end{bmatrix},$$

$$h_1^{\mathrm{S}}(\varphi) = \frac{1}{2}\begin{bmatrix} -t & \Delta e^{i\varphi} \\ -\Delta e^{-i\varphi} & t \end{bmatrix}, \quad h_1^{\mathrm{N}} = \frac{1}{2}\begin{bmatrix} -t & 0 \\ 0 & t \end{bmatrix}$$

$$\mathbb{1}_{\mathrm{SL}} = \sum_{j=-\infty}^{0} |j\rangle\langle j|, \quad T_{\mathrm{SL}} = \sum_{j=-\infty}^{-1} |j\rangle\langle j+1|$$

$$\mathbb{1}_{\mathrm{N}} = \sum_{j=1}^{L} |j\rangle\langle j|, \quad T_{\mathrm{N}} = \sum_{j=1}^{L-1} |j\rangle\langle j+1|$$

$$\mathbb{1}_{\mathrm{SR}} = \sum_{j=L+1}^{\infty} |j\rangle\langle j|, \quad T_{\mathrm{SR}} = \sum_{j=L+1}^{\infty} |j\rangle\langle j+1|$$

Assumption

The SNS junction is not coupled to electron reservoirs

→ **The total FP is conserved**

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Perturbation theory

- **Treat H_{T} as perturbation** by assuming $\lambda \ll 1$
- Exact solutions of non-perturbative Hamiltonian
  - Majorana zero modes for SL/SR ($|\mu/t| < 1$)

$$|\psi^{\mathrm{SL}}\rangle = N^{\mathrm{S}} \sum_{j=-\infty}^{0} \left[(\mathsf{z}_1^{\mathrm{S}})^{-j+1} - (\mathsf{z}_2^{\mathrm{S}})^{-j+1} \right] |j\rangle \begin{bmatrix} e^{i\varphi_{\mathrm{L}}/2} \\ e^{-i\varphi_{\mathrm{L}}/2} \end{bmatrix}$$

$$|\psi^{\mathrm{SR}}\rangle = N^{\mathrm{S}} \sum_{j=L+1}^{\infty} \left[(\mathsf{z}_1^{\mathrm{S}})^{j-L} - (\mathsf{z}_2^{\mathrm{S}})^{j-L} \right] |j\rangle \begin{bmatrix} ie^{i\varphi_{\mathrm{R}}/2} \\ -ie^{-i\varphi_{\mathrm{R}}/2} \end{bmatrix} \qquad \left(\mathsf{z}_{1,2}^{\mathrm{S}} = \frac{-\mu \pm \sqrt{\mu^2 + \Delta^2 - t^2}}{t + \Delta} \right)$$

  - $2L$ energy eigenvalues/states for N

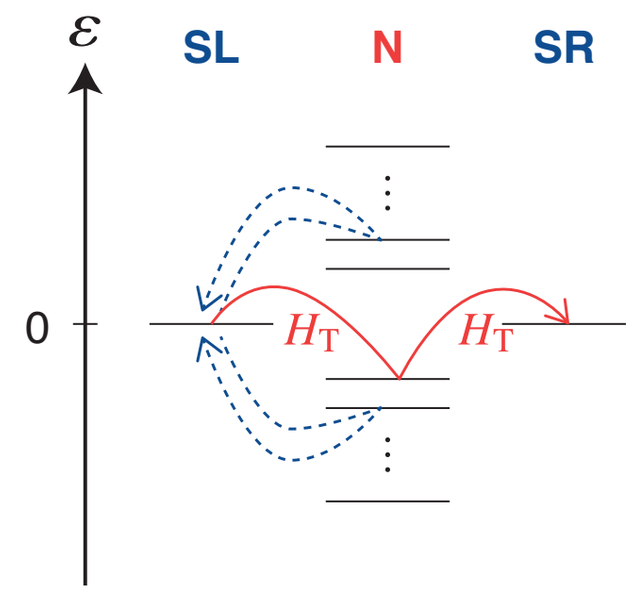
$$\varepsilon_{1,2}^{\mathrm{N}}(q) = \mp \left[\mu + t \cos\left(\frac{\pi q}{L+1} \right) \right] =: \boxed{\pm\varepsilon^{\mathrm{N}}(q)} \quad (q = 1, 2, \ldots, L)$$

$$\left|\psi_{1,2}^{\mathrm{N}}(q)\right\rangle = N^{\mathrm{N}} \sum_{j=1}^{L} \sin\left(\frac{\pi q j}{L+1} \right) |j\rangle \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

2nd- and 1st-order perturbation for off- and on-resonance cases

Perturbation theory (off-resonance case)

- $\varepsilon^{\mathrm{N}}(q) \neq 0$ for any q
- **2nd-order** perturbation theory for two states $\{|\psi^{\mathrm{SL}}\rangle, |\psi^{\mathrm{SR}}\rangle\}$

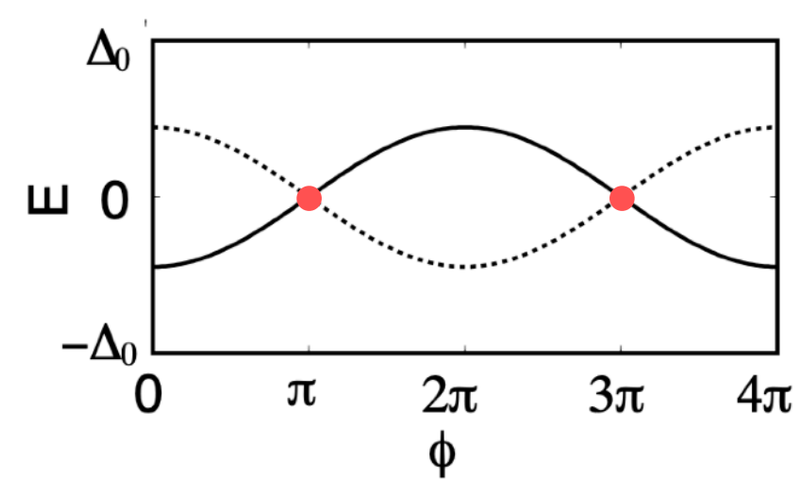


Energy splitting

$$\varepsilon_{\pm}^{(2)} = \pm\lambda^2 t A^{(2)}(\bar{\mu}, \bar{\Delta}, L) \cos\left(\frac{\varphi_{\mathrm{R}} - \varphi_{\mathrm{L}}}{2}\right)$$

consistent with the previous studies

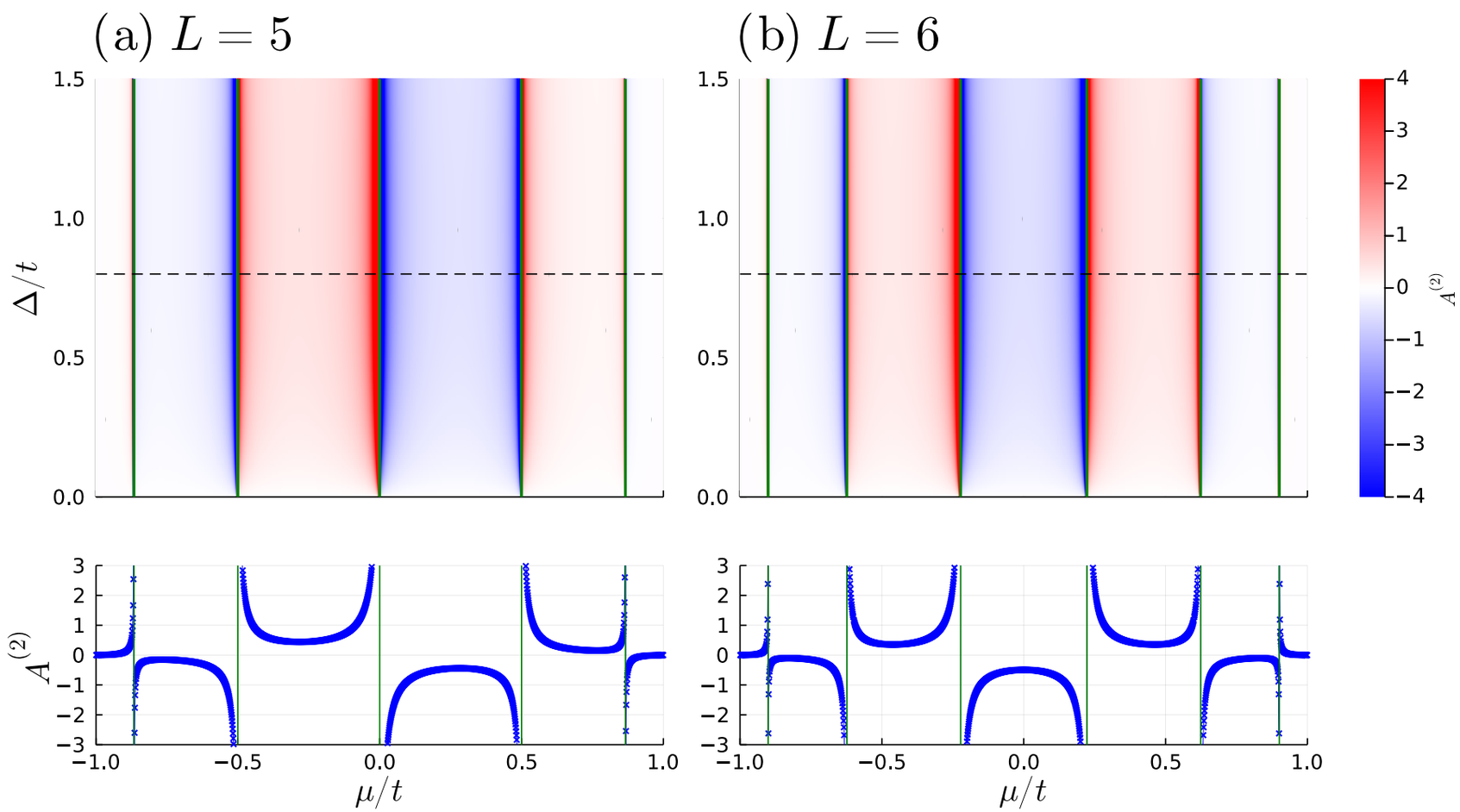
- **degeneracy @ $\varphi_{\mathrm{R}} - \varphi_{\mathrm{L}} = (2n+1)\pi$ protected by the FP**



- "Amplitude" function $A^{(2)}(\bar{\mu}, \bar{\Delta}, L)$
  - $(\bar{\mu} = \mu/t, \bar{\Delta} = \Delta/t, L)$ dependence?

Perturbation theory (off-resonance case)

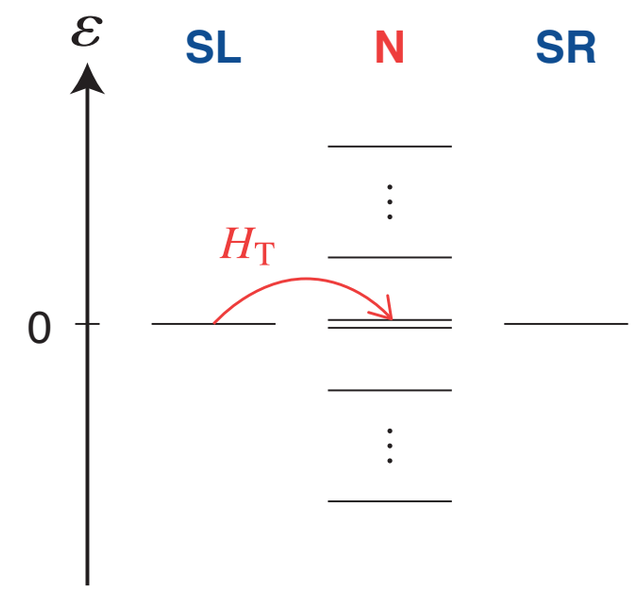
- "Amplitude" $A^{(2)}(\bar{\mu}, \bar{\Delta}, L) = (N^{\mathrm{S}} N^{\mathrm{N}})^2 \frac{|\bar{\mu}^2 + \bar{\Delta}^2 - 1|}{(1+\bar{\Delta})^2} \sum_{q=1}^{L} \frac{2(-1)^q \sin^2\left(\frac{\pi q}{L+1}\right)}{\bar{\mu} + \cos\left(\frac{\pi q}{L+1}\right)}$



- ***L* sign changes → oscillations of the N wavefunctions**
- **Resonance at** $\varepsilon^{\mathrm{N}}(q) = 0 \iff \bar{\mu} = -\cos[\pi q/(L+1)]$ (green lines)
  - breakdown of 2nd-order perturbation → divergent behavior

Perturbation theory (on-resonance case)

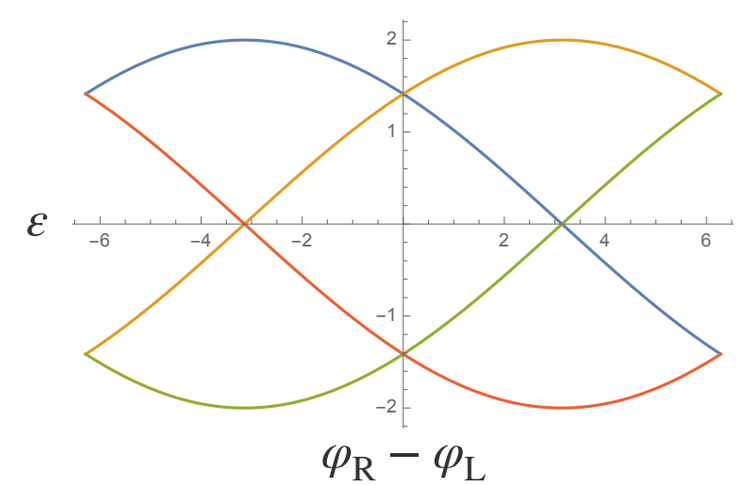
- There exists q_0 s.t. $\varepsilon^{\mathrm{N}}(q_0) = 0$
- **1st-order** perturbation theory for 4 states $[|\psi^{\mathrm{SL}}\rangle, |\psi^{\mathrm{SR}}\rangle, |\psi_1^{\mathrm{N}}(q_0)\rangle, |\psi_2^{\mathrm{N}}(q_0)\rangle]$



Energy splitting

$$\varepsilon^{(1)}_{\sigma_1\sigma_2} = \sigma_1 \lambda t A^{(1)}(q_0, \bar{\Delta}, L) \cos\left(\frac{\varphi_{\mathrm{R}} - \varphi_{\mathrm{L}} + \sigma_2 \pi}{4}\right) \qquad (\sigma_1, \sigma_2 \in \{\pm 1\})$$

- **drastic change of spectrum**



- Nevertheless, Josephson current is 4π-periodic (not 8π-periodic)
  - we will see it using effective model

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Effective four-state model

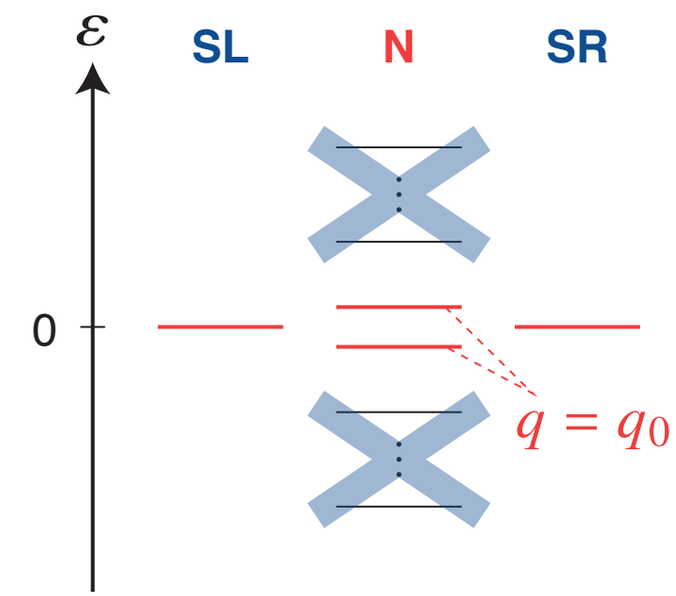
- Four energy levels closest to $\varepsilon = 0$

cf.) Affleck–Giuliano (2014)

$$H_{\mathrm{eff}} = \varepsilon^{\mathrm{N}}\left(c^\dagger c - \frac{1}{2}\right) + H_{\mathrm{AG}}$$

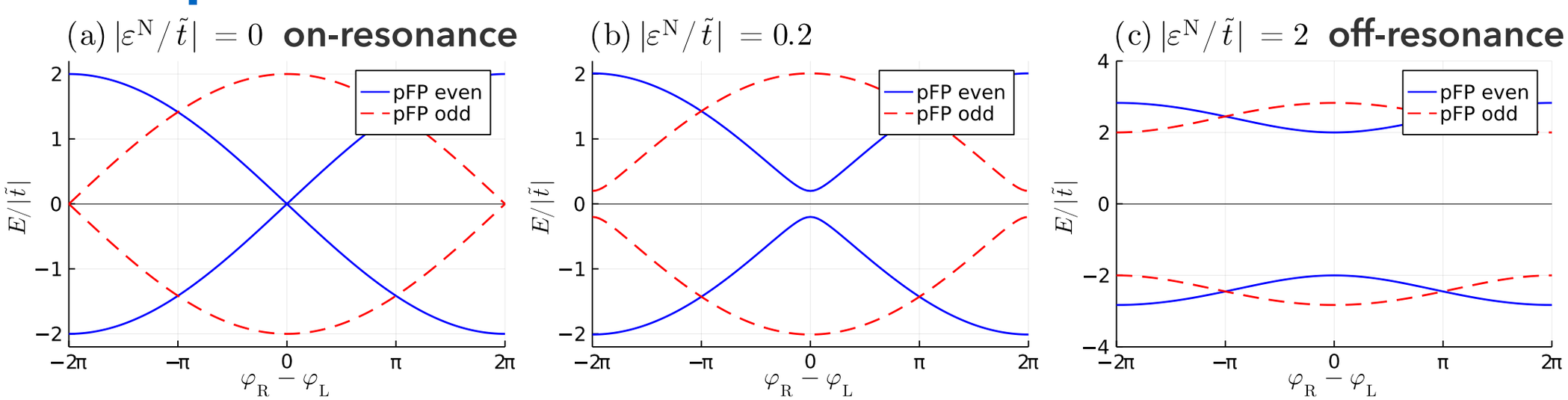
fixed

$$H_{\mathrm{AG}} = \left[c^\dagger \left\langle \psi_1^{\mathrm{N}}(q_0)\right| H_{\mathrm{T}} \left|\psi^{\mathrm{SL}}\right\rangle \gamma_{\mathrm{L}} + \mathrm{H.c.}\right] + \left[c^\dagger \left\langle \psi_1^{\mathrm{N}}(q_0)\right| H_{\mathrm{T}} \left|\psi^{\mathrm{SR}}\right\rangle \gamma_{\mathrm{R}} + \mathrm{H.c.}\right]$$
$$= -\left[\tilde{t}(q_0)\left(e^{i\varphi_{\mathrm{L}}/2} + (-1)^{q_0} e^{i\varphi_{\mathrm{R}}/2}\right) c^\dagger f^\dagger + \mathrm{H.c.}\right]$$
$$- \left[\tilde{t}(q_0)\left(e^{i\varphi_{\mathrm{L}}/2} - (-1)^{q_0} e^{i\varphi_{\mathrm{R}}/2}\right) c^\dagger f + \mathrm{H.c.}\right]$$



- nonlocal fermion: $f = \dfrac{\gamma_{\mathrm{L}} + i\gamma_{\mathrm{R}}}{2}$, $f^\dagger = \dfrac{\gamma_{\mathrm{L}} - i\gamma_{\mathrm{R}}}{2}$

- Many-body spectrum (q_0: odd)
  - **interpolation between on- and off-resonance cases**



Effective four-state model

- **dc Josephson current** can be derived
  - ex.) off-resonance $I = (-1)^{F_{\rm eff}+q_0} \frac{e|\tilde{t}(q_0)|^2}{\hbar|\varepsilon^{\rm N}|} \sin\left(\frac{\varphi_{\rm R}-\varphi_{\rm L}}{2}\right)$
  - **partial FP** for the effective model

$$[H_{\rm eff}, (-1)^{F_{\rm eff}}] = 0, \quad (-1)^{F_{\rm eff}} = e^{i\pi(c^\dagger c + f^\dagger f)} = (1-2c^\dagger c)(1-2f^\dagger f)$$

→ sign change of $(-1)^{q_0}$ by varying μ?

- **Total FP** $(-1)^F = (-1)^{F_{\rm eff}}(-1)^{q_0-1}$ for the original model
  - conserved quantity (no electron reservoir)

    → **the current is not switched** even when μ is varied
  - FP in the N region compensated by **FP of the MZMs**

Current-Phase relation

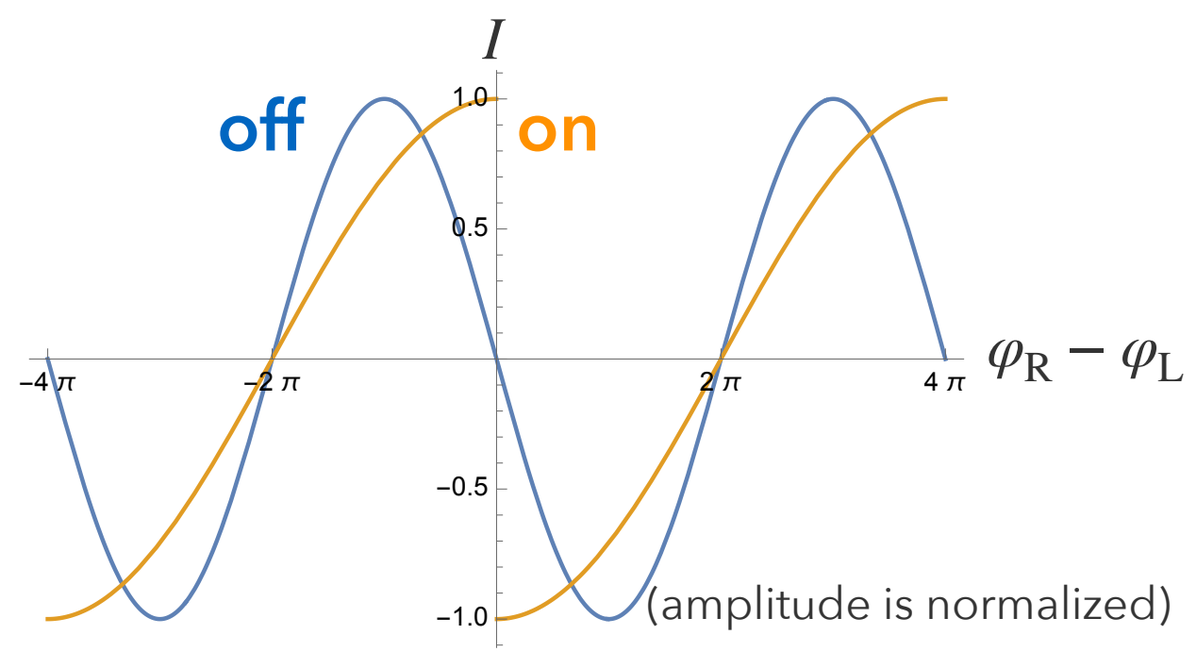
- The dc Josephson current
  - off-resonance (integrating out the nonzero levels)

$$I^{\mathrm{off}} = (-1)^{1+F}\frac{e\lambda^2 t}{\hbar}\left|A^{(2)}(\bar{\mu},\bar{\Delta},L)\right|\sin\left(\frac{\varphi_{\mathrm{R}}-\varphi_{\mathrm{L}}}{2}\right)$$

  - on-resonance

$$I^{\mathrm{on}} = \begin{cases} -\frac{e|\tilde{t}(q_0)|}{\hbar}\,\mathrm{sgn}\left[\sin\left(\frac{\varphi_{\mathrm{R}}-\varphi_{\mathrm{L}}}{4}\right)\right]\cos\left(\frac{\varphi_{\mathrm{R}}-\varphi_{\mathrm{L}}}{4}\right) & (-1)^F = +1 \\ +\frac{e|\tilde{t}(q_0)|}{\hbar}\,\mathrm{sgn}\left[\cos\left(\frac{\varphi_{\mathrm{R}}-\varphi_{\mathrm{L}}}{4}\right)\right]\sin\left(\frac{\varphi_{\mathrm{R}}-\varphi_{\mathrm{L}}}{4}\right) & (-1)^F = -1 \end{cases}$$

  - **4π-periodic (fractional)** for both cases



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Generalized Bloch's theorem

A. Alase *et al.*, PRB **96**, 195133 (2017) / E. Cobanera *et al.*, PRB **98**, 245423 (2018)

- Our model: interface, short-range, & translation sym.
- Exact (and numerical) diagonalization of **corner-modified banded block-Toeplitz matrix**
  - no finite-size effect of SL/SR: good for tiny gap

ex.) 1-dimensional tight-binding model with L sites

$$\hat{H}_L + \hat{W}, \quad \hat{H}_L = -t\sum_{j=1}^{L-1}(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)$$

$$\hat{W} = 0 \qquad \hat{W} = -t(c_L^\dagger c_1 + c_1^\dagger c_L) \qquad \hat{W} = w(c_1^\dagger c_1 + c_L^\dagger c_L)$$

$$\begin{bmatrix} 0 & -t & & 0 & 0 \\ -t & 0 & & & 0 \\ & & \ddots & & \\ 0 & & & 0 & -t \\ 0 & 0 & & -t & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & -t & & 0 & -t \\ -t & 0 & & & 0 \\ & & \ddots & & \\ 0 & & & 0 & -t \\ -t & 0 & & -t & 0 \end{bmatrix} \quad \begin{bmatrix} w & -t & & 0 & 0 \\ -t & 0 & & & 0 \\ & & \ddots & & \\ 0 & & & 0 & -t \\ 0 & 0 & & -t & w \end{bmatrix}$$

Calculation flow

A. Alase *et al.*, PRB **96**, 195133 (2017) / E. Cobanera *et al.*, PRB **98**, 245423 (2018)

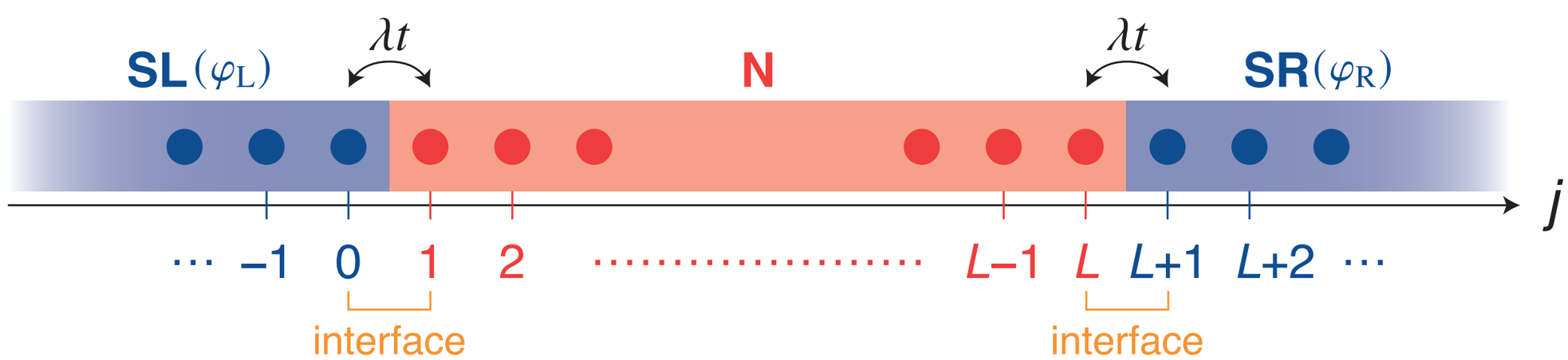
- Fix a certain initial energy value ε
- Construct **bulk solutions** using "translation sym."

$$\left|\psi_m^{\mathrm{SL}}(\varepsilon;\varphi_{\mathrm{L}})\right\rangle = \sum_{j=-\infty}^{0} \left[z_m^{\mathrm{S}}(\varepsilon)\right]^{-j} |j\rangle \left|u_m^{\mathrm{S}}(\varepsilon;\varphi_{\mathrm{L}})\right\rangle$$

$$\left|\psi_m^{\mathrm{SR}}(\varepsilon;\varphi_{\mathrm{R}})\right\rangle = \sum_{j=L+1}^{\infty} \left[z_m^{\mathrm{S}}(\varepsilon)\right]^{j-(L+1)} |j\rangle \left|u_m^{\mathrm{S}}(\varepsilon;\varphi_{\mathrm{R}})\right\rangle \quad (m=1,2;|z_m^{\mathrm{S}}|<1)$$

$$\left|\psi_{l\sigma}^{\mathrm{N}}(\varepsilon)\right\rangle = \sum_{j=1}^{L} \left[z_l^{\mathrm{N}}(\varepsilon)\right]^{\sigma j} |j\rangle \left|u_l^{\mathrm{N}}\right\rangle \quad (l=1,2;\sigma=\pm)$$

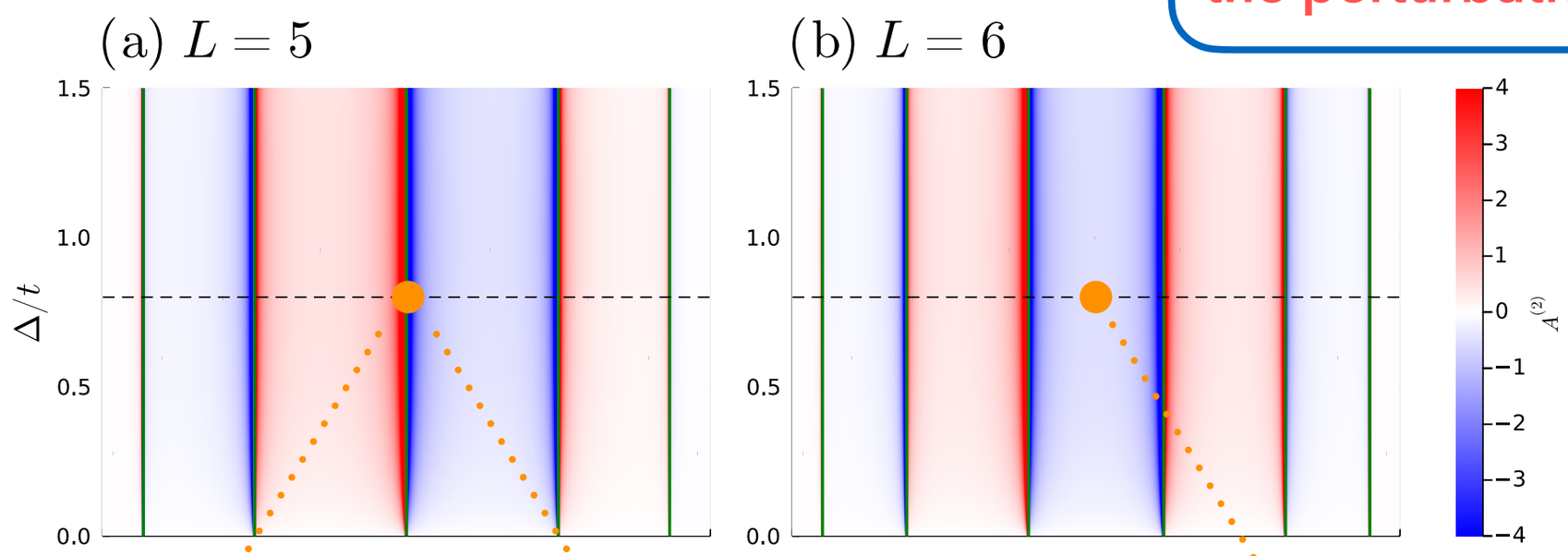
- **Boundary matrix**: $B_{jm}^{\mathrm{reg}}(\varepsilon) = \langle j|(H-\varepsilon\mathbf{1})|\psi_m^{\mathrm{reg}}(\varepsilon)\rangle \quad (j=0,1,L,L+1)$
  - find appropriate ε s.t. $\det B(\varepsilon) = 0$



Exact diagonalization: ($\varphi_R - \varphi_L$)-dependence

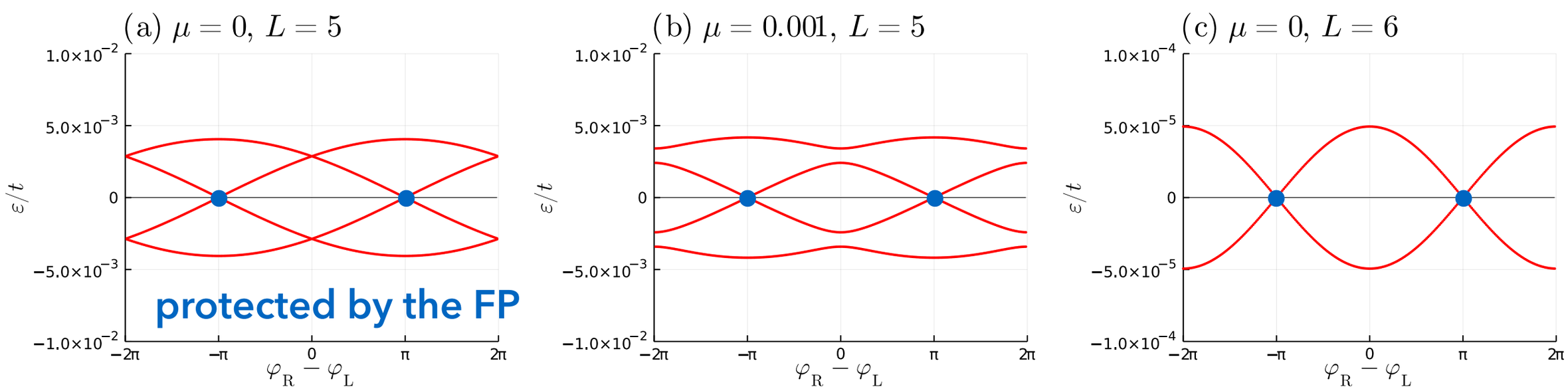
- Calculation for $\bar{\mu} \simeq 0, \bar{\Delta} = 0.8$

consistent with
the perturbation theory!



$$\varepsilon^{(1)} \sim \pm \cos \frac{\varphi \pm \pi}{4}$$

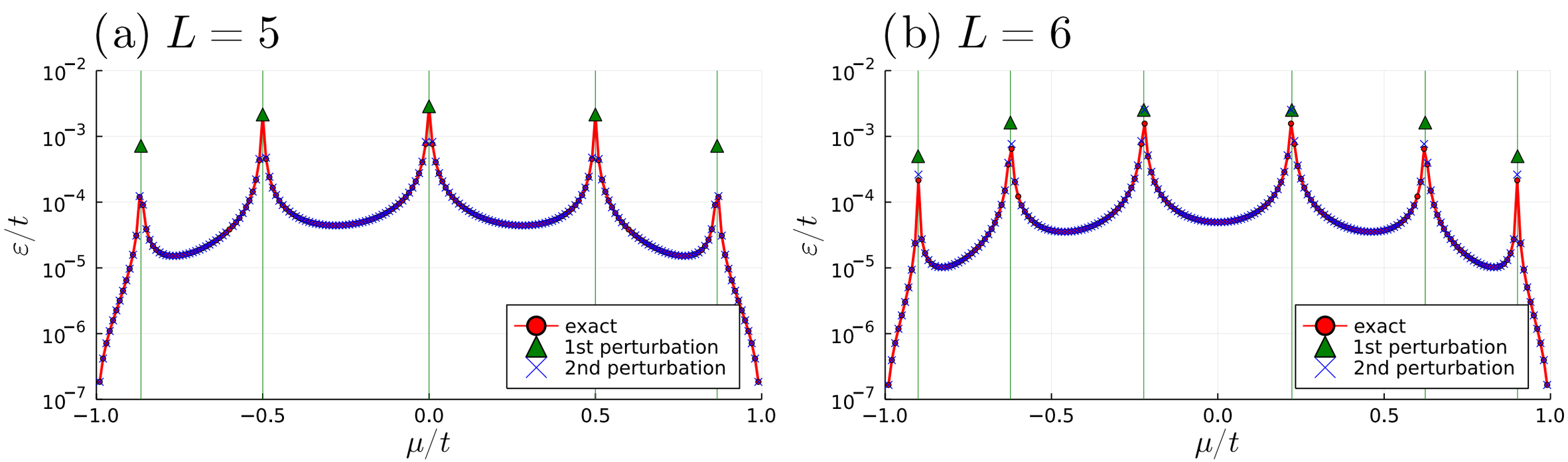
$$\varepsilon^{(2)} \sim \pm \cos \frac{\varphi}{2}$$



Exact diagonalization: μ-dependence

- Logarithm plot of $\min|\varepsilon|$
  for $(\bar{\Delta}, \varphi_{\mathrm{R}} - \varphi_{\mathrm{L}}, \lambda) = (0.8, 0, 0.01)$
  - **quantitatively agree with the perturbation theory**
- Sharp peaks @ the resonance points (green lines)

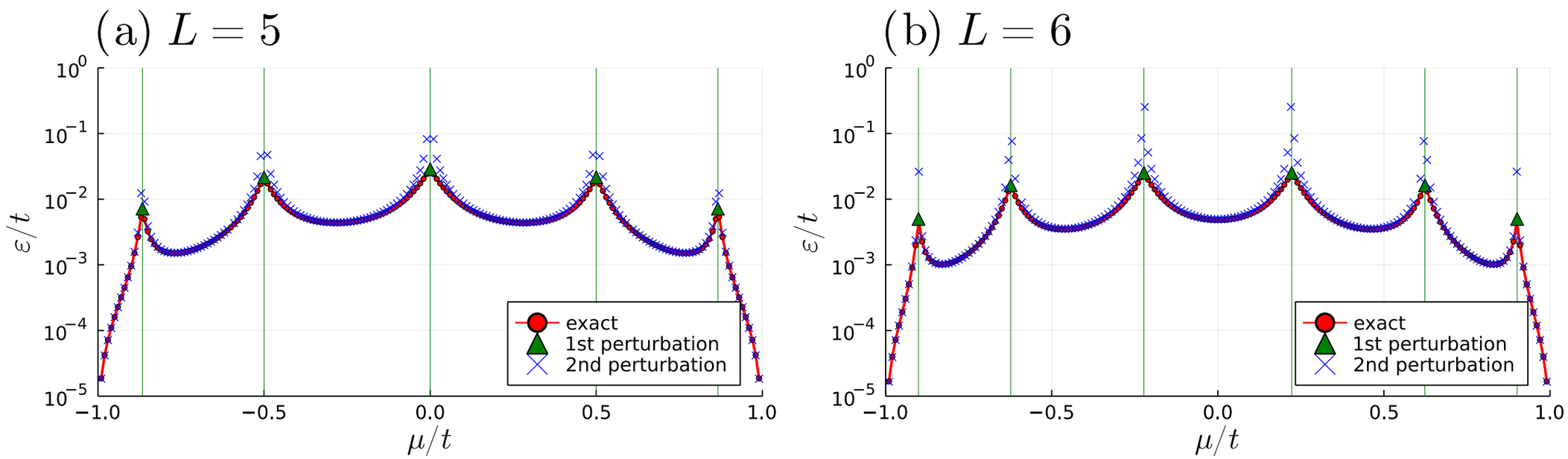
→ **enhancement of the fractional Josephson current**



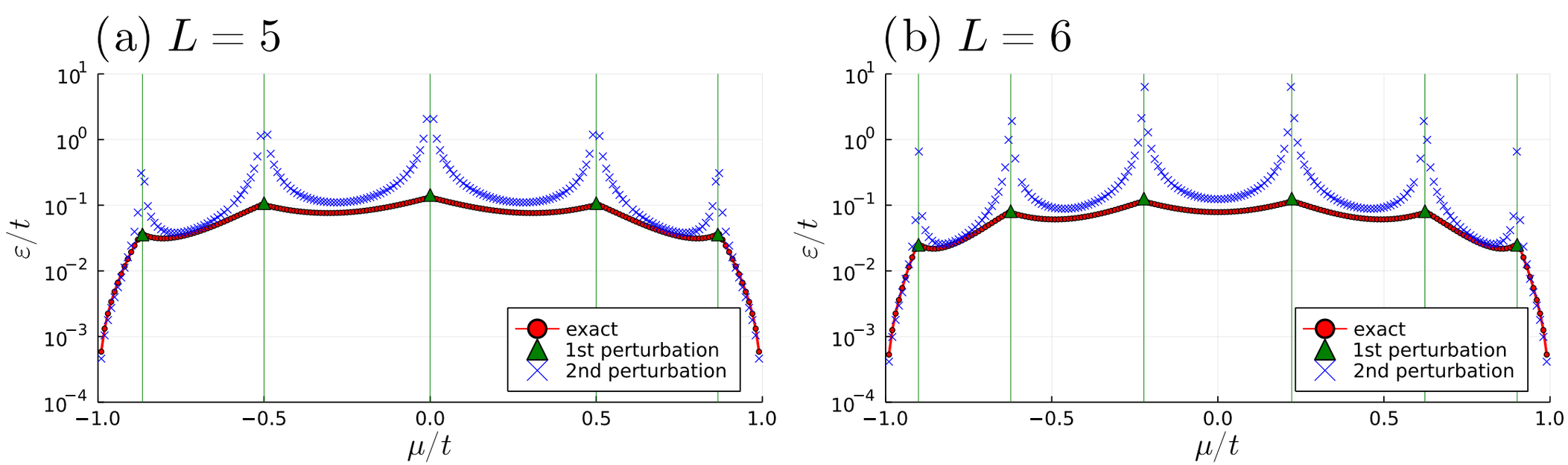
Exact diagonalization: μ-dependence

- Deviation from the perturbation theory for larger λ

$\lambda = 0.1$



$\lambda = 0.5$



Summary

SS & A. Furusaki, Phys. Rev. B **104**, 205431 (2021)

- Finite-size effects on topological SNS junction with a moderate length in the N region
  - perturbation theory & exact diagonalization method
- **Fractional Josephson effect**
  - **4π periodicity** in the phase difference
  - **supercurrent $\propto$ total FP** (occ. N levels & two MZMs)
  - Can the Josephson current be switched by μ? → **No.**
- **Drastic enhancement of fractional Josephson current** when a discrete N level and MZMs are **in resonance**