Liouvillian analysis of the relaxation time in open quantum systems

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joint work with Tatsuhiko Shirai (Waseda)

Novel Quantum States in Condensed Matter 2022

Open quantum many-body systems

cold atoms: highly controllable isolated quantum systems

foundation of statistical mechanics (thermalization)

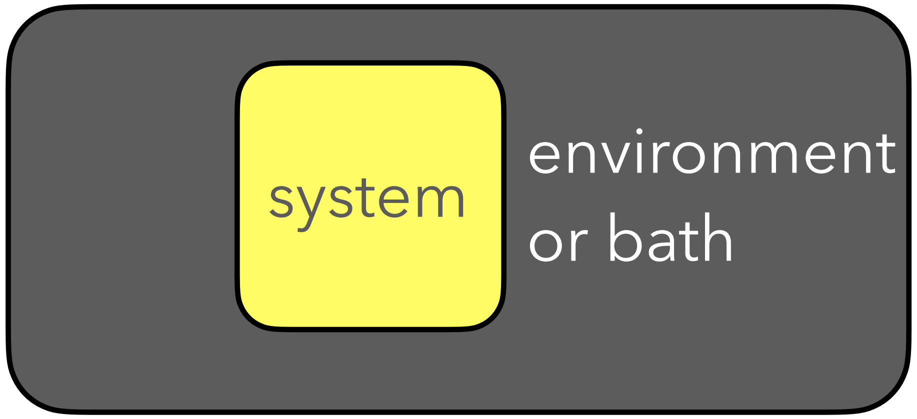
quantum control of many-body systems (Floquet engineering)

important problem: taking into account the effect of dissipation

unavoidable dissipation in condensed matters

controlled dissipation in recent cold-atom experiments

→dissipation engineering



J. T. Barreiro et al., Nature (2011)
G. Barontini et al., Phys. Rev. Lett. (2013)
T. Tomita et al., Sci. Adv. (2017)

Outline

relaxation time τ in open quantum systems

Liouvillian

$$\frac{d\rho_S(t)}{dt} = -\mathscr{L}\rho_S(t) \qquad \mathscr{L} \to \tau?$$

- Gap discrepancy problem and its resolution

  TM and T. Shirai, Phys. Rev. Lett. 125, 230604 (2020)

  g: spectral gap of $\mathscr{L}$ "Liouvillian gap" ~~$\tau \lesssim g^{-1}$~~

- general upper bound on the auto-correlation function in the steady state: "instantaneous decay rate"

  TM and T. Shirai, to appear soon on arXiv

Lindblad equation (GKLS equation)

dynamics of open quantum systems

Markov approximation: Lindblad (GKLS) equation

$$\frac{d\rho(t)}{dt} = -i[H, \rho(t)] + \sum_k \left(L_k \rho(t) L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho(t)\} \right)$$

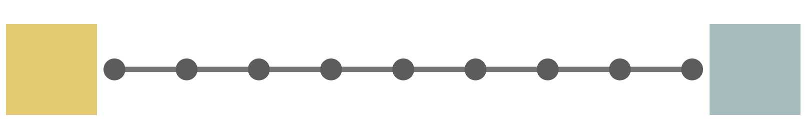
L_k : Lindblad jump operator

particle loss $L_k = b_i$ two-body loss $L_k = (b_i)^2$

particle gain $L_k = b_i^\dagger$ dephasing $L_k = b_i^\dagger b_i$

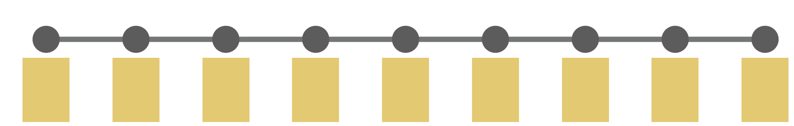
boundary dissipation

dissipation only at the boundaries



bulk dissipation

dissipation in the bulk



Liouvillian and its eigenmodes

$$\frac{d\rho(t)}{dt} = -i[H, \rho(t)] + \sum_k \left(L_k \rho(t) L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho(t)\} \right)$$
$$= -\mathscr{L}\rho(t)$$

$\mathscr{L}$ Liouvillian (superoperator)

eigenvalues and eigenmodes $\mathscr{L}\rho_n = \lambda_n \rho_n$

$$0 = \lambda_0 < \mathrm{Re}\,\lambda_1 \le \mathrm{Re}\,\lambda_2 \le \ldots$$

$$\rho(t) = e^{-\mathscr{L}t}\rho(0) = \rho_{\mathrm{ss}} + \sum_{n\neq 0} C_n e^{-\lambda_n t}\rho_n$$

$n = 0$: unique steady state $\rho_0 = \rho_{\mathrm{ss}}$

$n \neq 0$: decaying eigenmodes; decay rate $= \mathrm{Re}\,\lambda_n$

Liouvillian gap and relaxation time

$$\rho(t) = e^{-\mathscr{L}t}\rho(0) = \rho_{\mathrm{ss}} + \sum_{n\neq 0} C_n e^{-\lambda_n t}\rho_n \qquad 0 = \lambda_0 < \mathrm{Re}\,\lambda_1 \leq \mathrm{Re}\,\lambda_2 \leq \ldots$$

Liouvillian gap g: the smallest decay rate

$$g = \mathrm{Re}\,\lambda_1$$

"asymptotic decay rate"

relaxation time τ: $\|\rho(\tau) - \rho_{\mathrm{ss}}\|_{\mathrm{tr}} = \epsilon$

ϵ: a fixed threshold

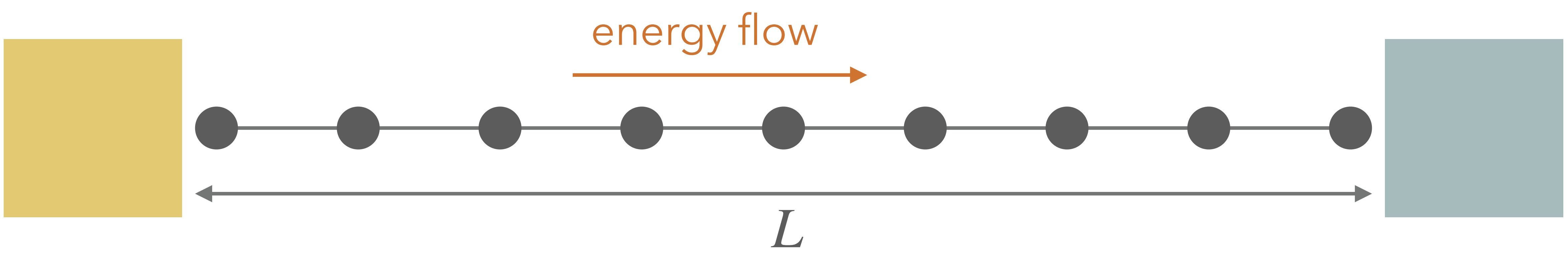
trace norm $\|\hat{A}\|_{\mathrm{tr}} = \mathrm{Tr}\,|\hat{A}\,|$

$$\|\rho(t) - \rho_{\mathrm{ss}}\|_{\mathrm{tr}} \lesssim e^{-gt} \quad (t \to \infty)$$

$$\tau \lesssim g^{-1}$$

boundary dissipated systems

e.g. 1D spin system



bulk Hamiltonian: non-integrable (chaotic)

diffusive transport of energy

relaxation time scales as $\tau \sim L^2$

Liouvillian gap should be scaled as $g \sim L^{-2}$

gap discrepancy problem

numerical evidence of $g \sim L^{-1}$ in chaotic spin chains with boundary dissipation **M. Znidaric, Phys. Rev. E (2015)**

(numerical results) $\tau \gg g^{-1}$ ⟷ $\tau \lesssim g^{-1}$ (naive expectation)

staggered XXZ model

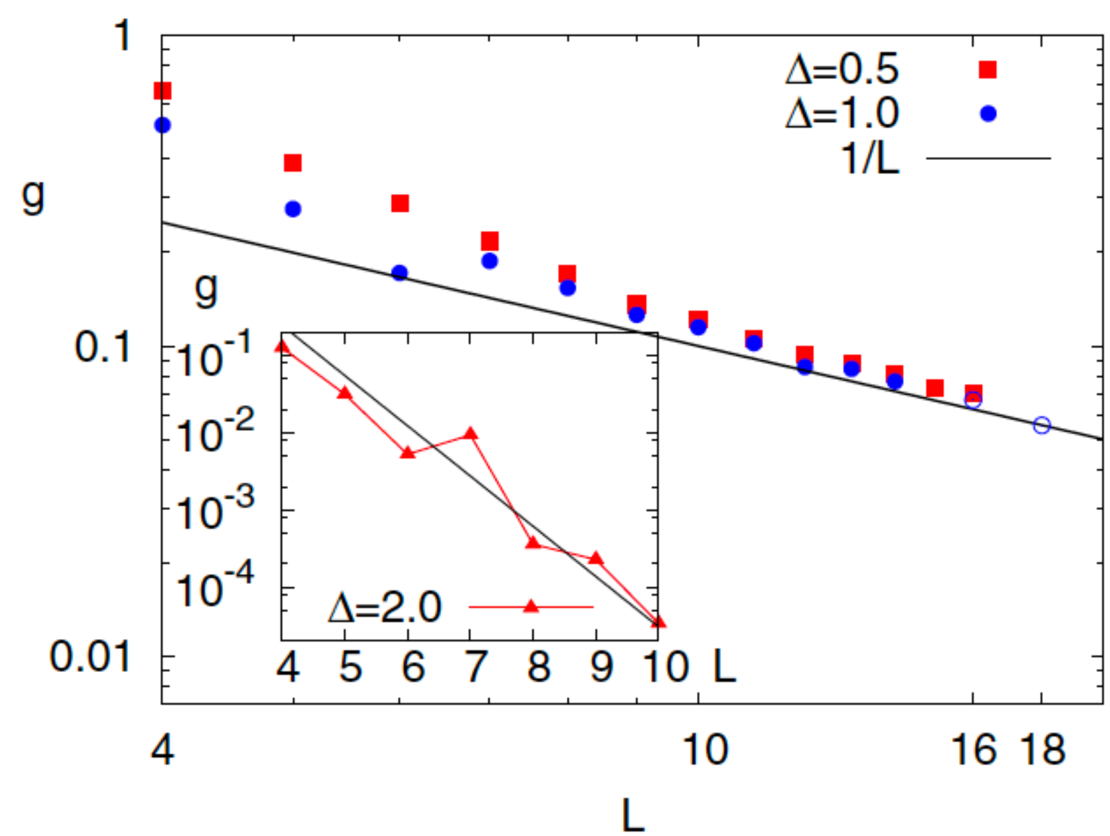
$$H = \sum_{i=1}^{L-1} \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z \right) + \sum_{i=1}^{L} b_i \sigma_i^z \qquad b_i = \left(-1, -\frac{1}{2}, 0, -1, -\frac{1}{2}, 0, \ldots\right)$$

with boundary dephasing

$$\hat{L}_1 = \sqrt{\gamma}\hat{\sigma}_1^z \qquad \hat{L}_2 = \sqrt{\gamma}\hat{\sigma}_L^z$$

$g \sim L^{-1}$ **when** $\Delta < 1$

This is not exceptional:
many models show $g \gg L^{-2}$



model

bulk Hamiltonian: 1D hard-core Bose-Hubbard

$$H = -h\sum_{i=1}^{L-1}\left(b_i^\dagger b_{i+1} + b_{i+1}^\dagger b_i\right) + U\sum_{i=1}^{L-1}\left(n_i - \frac{1}{2}\right)\left(n_{i+1} - \frac{1}{2}\right)$$
$$-h'\sum_{i=1}^{L-2}\left(b_i^\dagger b_{i+2} + b_{i+2}^\dagger b_i\right) + U'\sum_{i=1}^{L-2}\left(n_i - \frac{1}{2}\right)\left(n_{i+2} - \frac{1}{2}\right)$$

$h = U = 1, \quad h' = U' = 0.24$

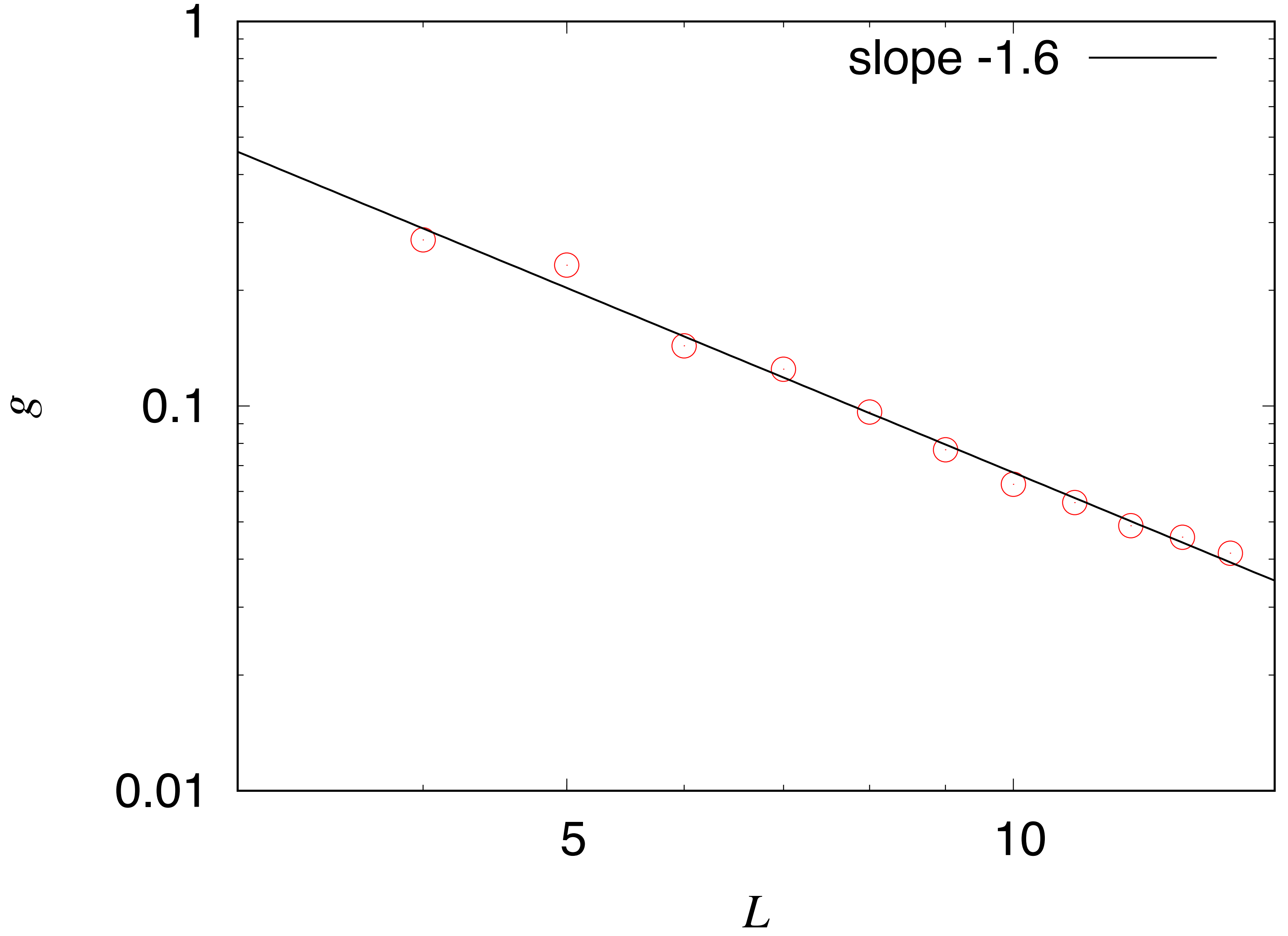
$N = \sum_{i=1}^{L} n_i = \sum_{i=1}^{L} b_i^\dagger b_i$ is conserved: the sector of $N = \frac{L}{2}$ or $\frac{L-1}{2}$

This model is chaotic **L. F. Santos and M. Rigol, Phys. Rev. E (2010)**

boundary dephasing

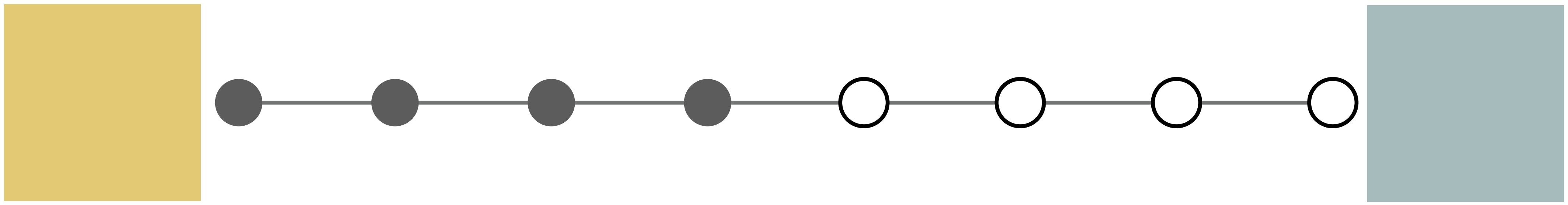
$$L_1 = \sqrt{\gamma}\, b_1^\dagger b_1, \quad L_2 = \sqrt{\gamma}\, b_L^\dagger b_L \qquad \gamma = 4$$

Liouvillian gap



$$g \sim L^{-1.6} \gg L^{-2}$$

time evolution of the trace distance: two timescales



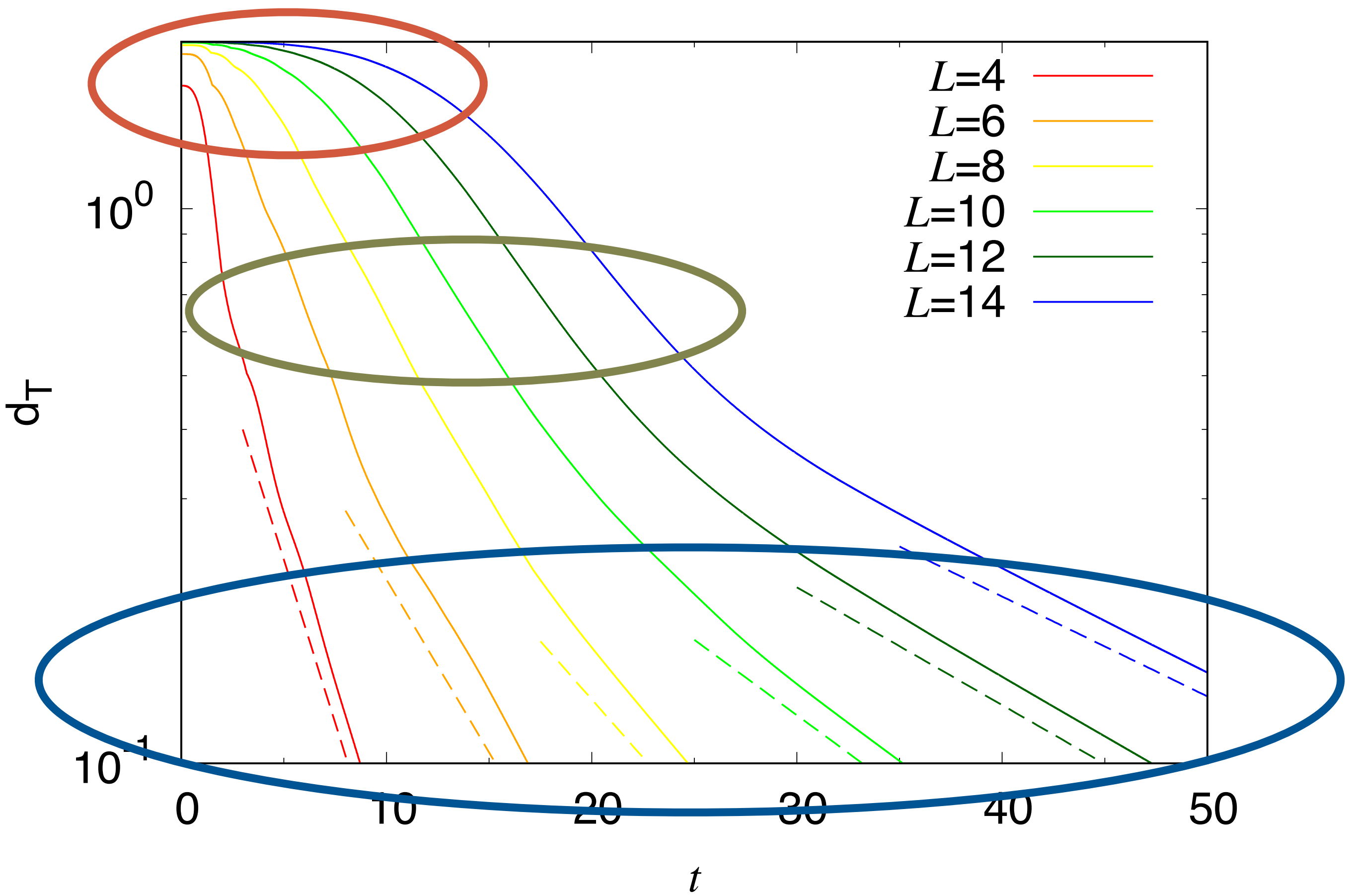
$$d_T(t) = \|\rho(t) - \rho_{ss}\|_{tr}$$

initial plateau $\Delta t \sim L^2$

sudden decay $\Delta t \sim O(1)$

asymptotic decay : $g^{-1} \sim L^{1.6}$

overall relaxation time $\propto L^2$



dashed lines: slope -g

The crossover time to the asymptotic regime diverges in the thermodynamic limit

understanding from eigenmode decays

$$\rho(t) = e^{-\mathscr{L}t}\rho(0) = \rho_{ss} + \sum_{n\neq 0} C_n e^{-\lambda_n t}\rho_n \qquad \|\rho(t) - \rho_{ss}\|_{tr} \leq \sum_{n\neq 0} |C_n| e^{-\mathrm{Re}\,\lambda_n t}$$

normalization for right eigenmodes: $\|\rho_n\|_{tr} = 1$

$$C_n = \frac{\langle \pi_n, \rho(0)\rangle}{\langle \pi_n, \rho_n\rangle}$$

π_n: left eigenmodes $\pi_n^\dagger \mathscr{L} = \lambda_n \pi_n^\dagger$

inner product $\langle A, B\rangle = \mathrm{Tr}\,[A^\dagger B]$

bi-orthogonality $\langle \pi_n, \rho_m\rangle = 0$ for $n \neq m$

normalization for left eigenmodes: $\|\pi_n\|_{op} = 1$

trace norm and operator norm are dual: $|\langle A, B\rangle| \leq \|A\|_{op}\|B\|_{tr}$

upper bounds on expansion coefficients

$|\langle A, B\rangle| \leq \|A\|_{\text{op}}\|B\|_{\text{tr}}$ $\quad |\langle \pi_n, \rho(0)\rangle| \leq \|\pi_n\|_{\text{op}}\|\rho(0)\|_{\text{tr}} = 1$

$$C_n = \frac{\langle \pi_n, \rho(0)\rangle}{\langle \pi_n, \rho_n\rangle} \qquad |C_n| \leq \frac{1}{\langle \pi_n, \rho_n\rangle}$$

vanishingly small eigenmode overlap → diverging expansion coeff.

$$\rho(t) = e^{-\mathscr{L}t}\rho(0) = \rho_{\text{ss}} + \sum_{n\neq 0} C_n e^{-\lambda_n t}\rho_n$$

decay time for nth eigenmode:

$$\tau_n \sim \frac{\ln|C_n|}{\operatorname{Re}\lambda_n} \leq \frac{\ln \Phi_n}{\operatorname{Re}\lambda_n} =: \bar{\tau}_n \qquad \Phi_n = \langle \pi_n, \rho_n\rangle^{-1}$$

maximum relaxation time is identified with $\tau \lesssim \tau_{\max} := \max_{n\neq 0} \bar{\tau}_n$

Comparison with the previous formula

previous formula

$$\tau_{\max} = \max_{n \neq 0} \frac{\ln(1/\epsilon)}{\mathrm{Re}\,\lambda_n} = \frac{\ln(1/\epsilon)}{g}$$

new formula

$$\tau_{\max} = \max_{n \neq 0} \frac{\ln(\Phi_n/\epsilon)}{\mathrm{Re}\,\lambda_n}$$

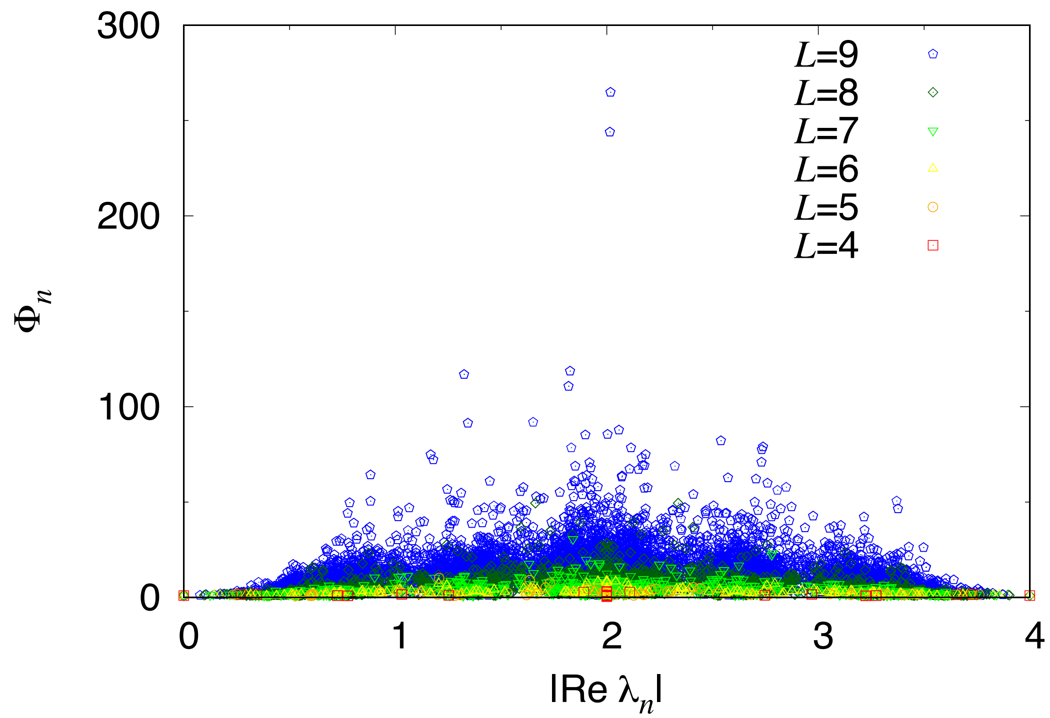
$$\Phi_n = \langle \pi_n, \rho_n \rangle^{-1}$$

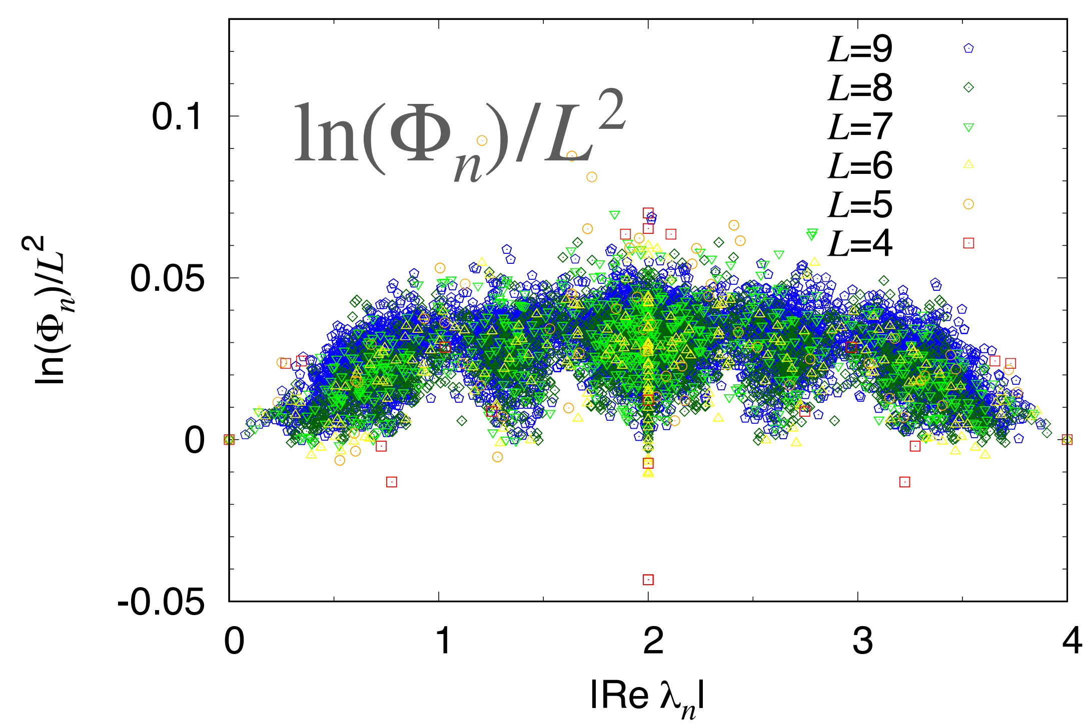
Small eigenmode overlap can alter the relaxation time

If Φ_n depends on the system size L, it even changes the scaling in L

evidence of large C_n (= small eigenmode overlap)

$\Phi_n = \langle \pi_n, \rho_n \rangle^{-1}$ 1D Bose-Hubbard with boundary dephasing





typically $\Phi_n = e^{O(L^2)}$ at $\mathrm{Re}\,\lambda_n = O(1)$

$$\bar{\tau}_n = \frac{\ln \Phi_n}{\mathrm{Re}\,\lambda_n} = O(L^2)$$

diffusive relaxation time!

summary of diffusive relaxation time

TM and T. Shirai, Phys. Rev. Lett. (2020)

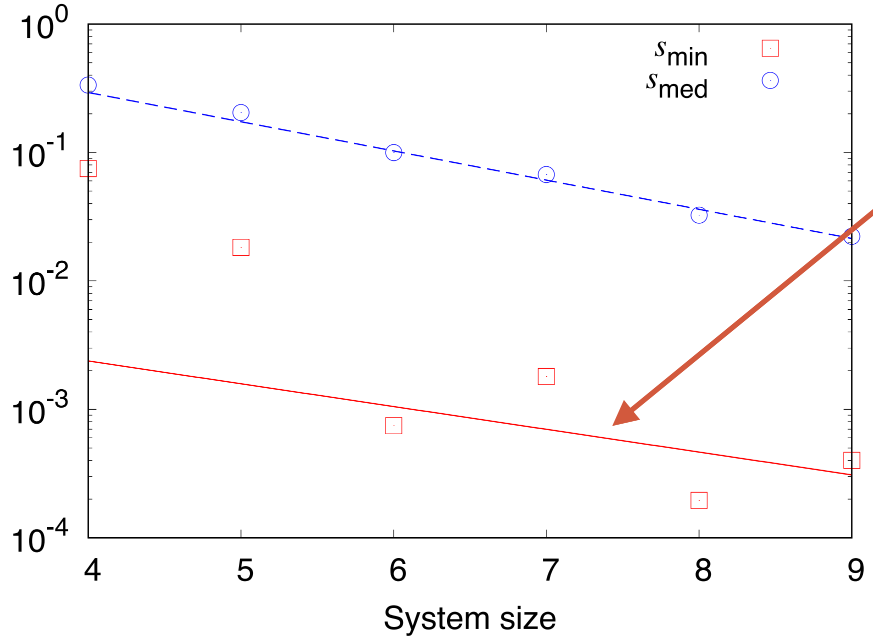
- System size dependence of the relaxation time is not necessarily determined by the Liouvillian gap (gap discrepancy problem)
- Trace distance from the steady state shows an initial plateau over a time interval $\Delta t \sim L^2$, and then rapidly decays and enters the asymptotic regime with the decay rate g
- Explosively large expansion coefficients $|C_n| \sim e^{O(L^2)}$ at $\mathrm{Re}\,\lambda_n = O(1)$ leads to $\bar{\tau}_n \sim L^2$
- Vanishingly small eigenmode overlaps $\langle \pi_n, \rho_n \rangle = e^{-O(L^2)}$ give rise to such large expansion coefficients

comment 1: exceptional points

Large expansion coefficients are typically observed near an exceptional point

Near an exceptional point $|\lambda_n - \lambda_m| = \epsilon$, the eigenmode overlap typically vanishes as $\langle \pi_n, \rho_n \rangle \sim \epsilon$

Vanishingly small eigenmode overlaps $\langle \pi_n, \rho_n \rangle = e^{-O(L^2)}$ found in the boundary dissipated 1D Bose-Hubbard model are ***not*** due to an exceptional point



$$s_{\mathrm{min}} = \min_{n,m:n\neq m} |\lambda_n - \lambda_m| = e^{-O(L)} \gg \langle \pi_n, \rho_n \rangle = e^{-O(L^2)}$$

comment 2: effect of non-Hermiticity

Hermitian case $\pi_n \propto \rho_n$ $\|\pi_n\|_{\mathrm{op}} = 1, \quad \|\rho_n\|_{\mathrm{tr}} = 1$

$$\langle \pi_n, \rho_n \rangle > \frac{1}{D} = e^{-O(L)}$$ D dimension of the Hilbert space

Gap discrepancy can happen even in Hermitian case,
but super-exponentially small overlap $\langle \pi_n, \rho_n \rangle = e^{-O(L^2)}$ is impossible

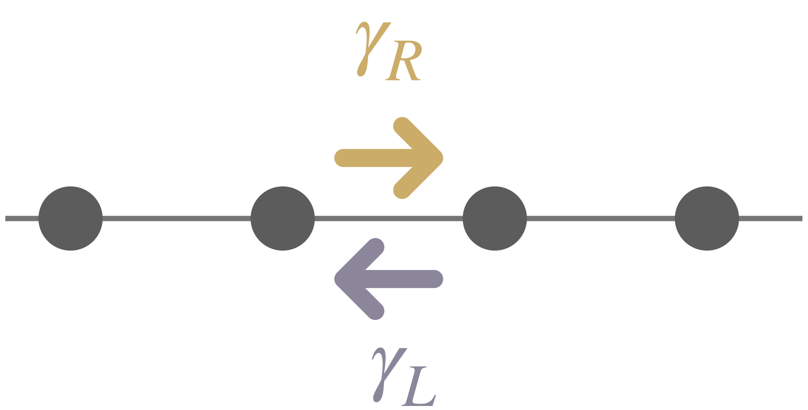
Non-Hermiticity of the Liouvillian is essential for diffusive relaxation time $\tau \sim L^2$ in boundary dissipated chaotic systems

comment 2: effect of non-Hermiticity

Liouvillian skin effect

T. Haga, M. Nakagawa, R. Hamazaki, and M. Ueda, Phys. Rev. Lett. (2021)

a simple model of single particle with asymmetric hopping



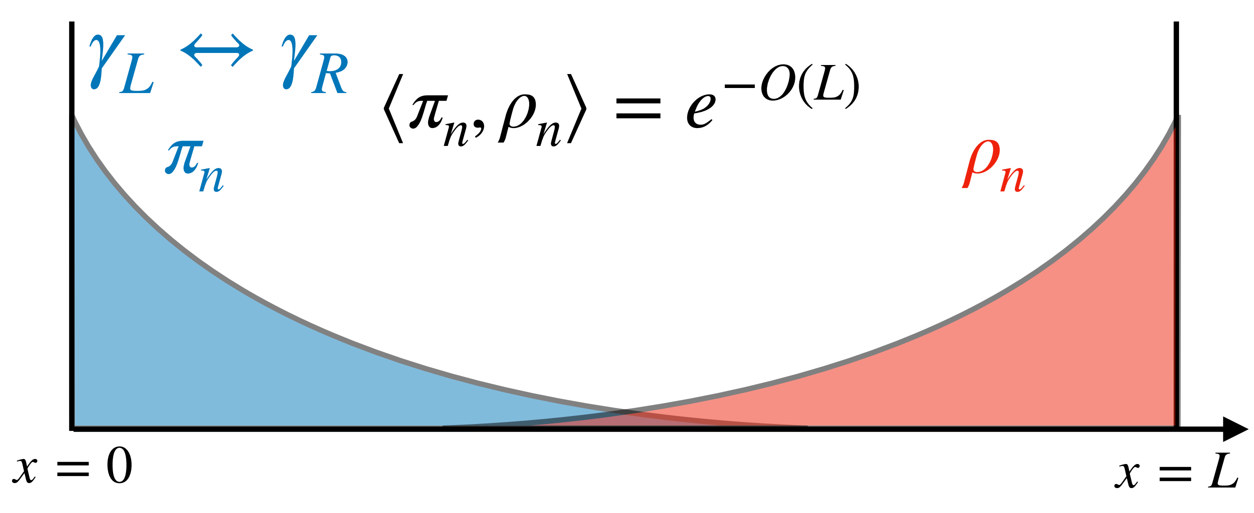
$$\frac{d\rho}{dt} = \sum_{i=1}^{L-1} \sum_{\alpha=\text{left,right}} \left(L_{i,\alpha}\rho L_{i,\alpha}^\dagger - \frac{1}{2}\{L_{i,\alpha}^\dagger L_{i,\alpha}, \rho\} \right)$$

$$L_{i,\text{left}} = \sqrt{\gamma_L} b_i^\dagger b_{i+1} \qquad L_{i,\text{right}} = \sqrt{\gamma_R} b_{i+1}^\dagger b_i$$

$\gamma_R > \gamma_L$

$\tau = O(L)$

$g = O(1)$



comment 3: classical Markov processes

Gap discrepancy problem also happens in classical Markov jump processes

$$\frac{d}{dt}P_n(t) = -\sum_m R_{nm}P_m(t)$$

spectral gap g of the transition rate matrix R

relaxation time τ

In some models, $\tau \gg g^{-1}$ (gap discrepancy)

Gap discrepancy problem is not a quantum effect

comment 4: metastability

So far, metastability has been characterized as a vanishingly small gap of Liouvillian or transition-rate matrix

$$0 < g = \mathrm{Re}\,\lambda_1 \ll \mathrm{Re}\,\lambda_2 \leq \mathrm{Re}\,\lambda_3 \leq \dots$$

A mathematical theory on how to construct metastable states was developed

B. Gaveau and L. S. Schulman, J. Phys. A (1987)
B. Gaveau and L. S. Schulman, J. Math. Phys. (1998)
G. Biroli and J. Kurchan, Phys. Rev. E (2001)

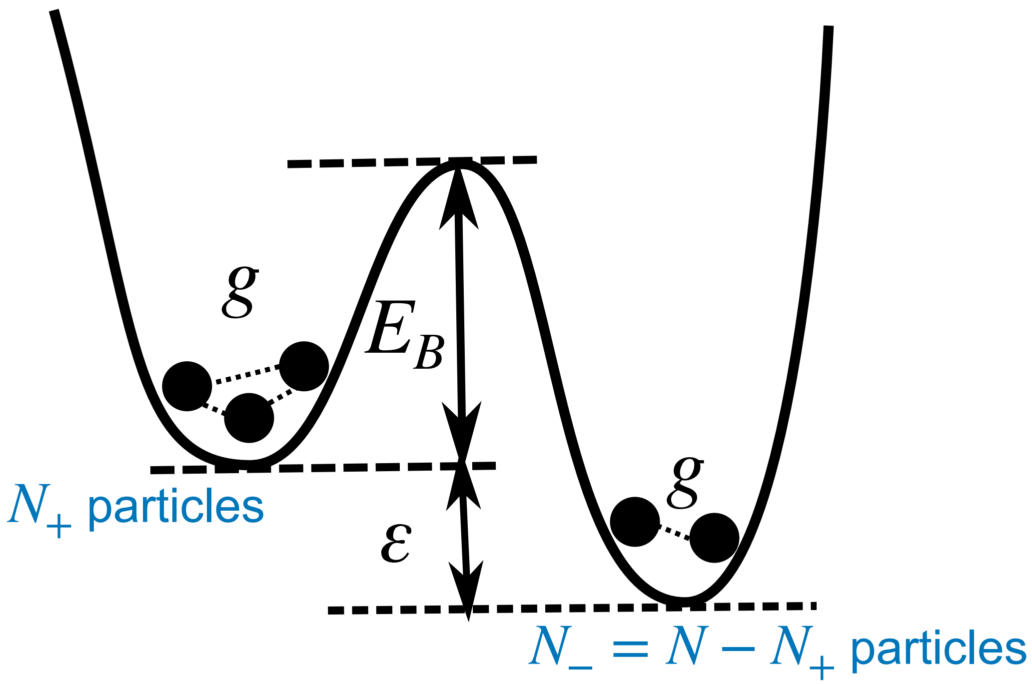
metastability in open quantum systems:

K. Macieszczak, M. Guta, I. Lesanovsky,and J. P. Garrahan, Phys. Rev. Lett. (2016)
D. C. Rose, K. Macieszczak, I. Lesanovsky, and J. P. Garrahan, Phys .Rev. E (2016)

Metastability associated with a small eigenmode overlap?

comment 4: metastability

TM, Phys. Rev. Res. (2021)



double-well potential with barrier E_B

and energy difference ε

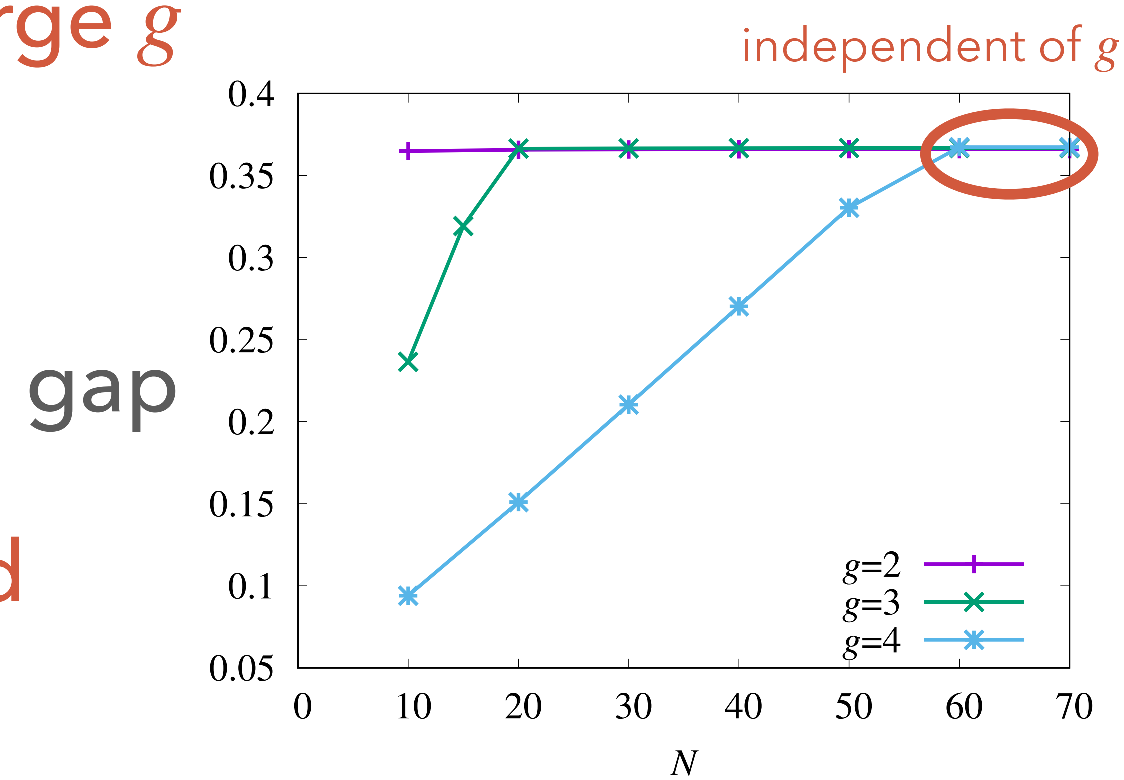
g : interaction between particles in the same well

Hamiltonian $H = N_+\varepsilon - \frac{g}{N}\left[\frac{N_+(N_+ - 1)}{2} + \frac{N_-(N_- - 1)}{2}\right]$

Markov jump with detailed balance $\frac{d}{dt}\vec{P}(t) = -R\vec{P}$

metastability induced by large E_B or large g

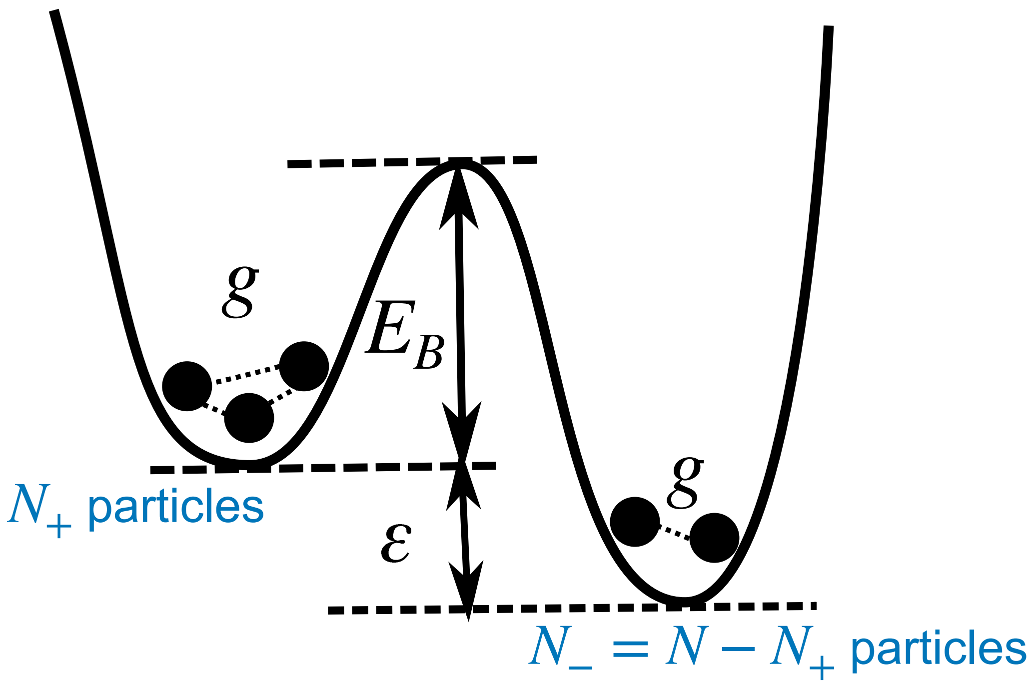
For sufficiently large N, the gap of the transition rate matrix R is ***independent of*** g



metastability at strong g is not captured by the spectral gap

comment 4: metastability

TM, Phys. Rev. Res. (2021)



double-well potential with barrier E_B

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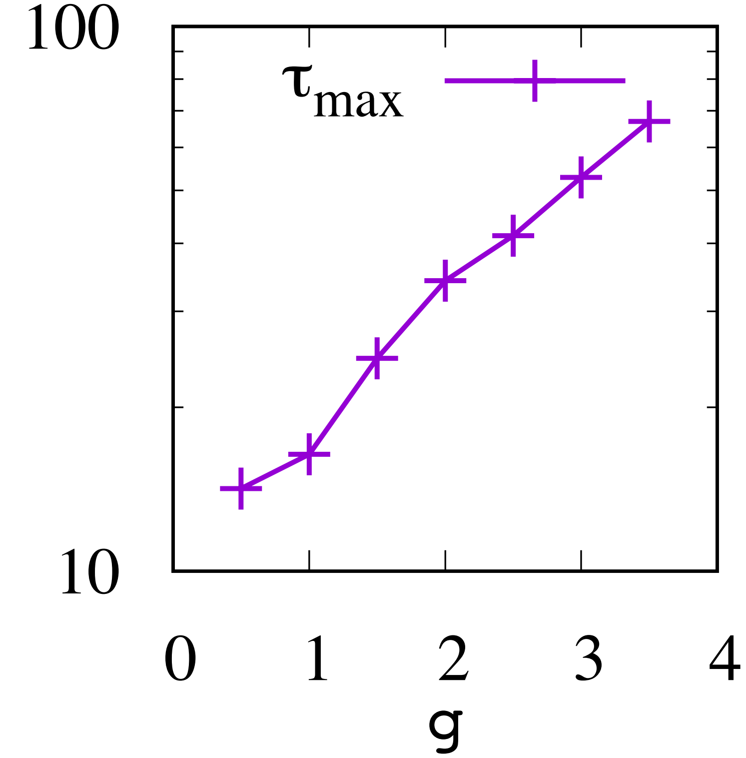
Hamiltonian $H = N_+\varepsilon - \frac{g}{N}\left[\frac{N_+(N_+-1)}{2} + \frac{N_-(N_--1)}{2}\right]$

Markov jump with detailed balance $\frac{d}{dt}\vec{P}(t) = -R\vec{P}$

metastability induced by large E_B or large g

For sufficiently large N, the gap of the transition rate matrix R is ***independent of*** g

metastability with large g is explained by vanishing eigenmode overlaps: $\tau_{\max} \sim e^{\beta g}$



Outline

relaxation time τ in open quantum systems

$$\frac{d\rho_S(t)}{dt} = -\mathscr{L}\rho_S(t) \qquad \overset{\text{Liouvillian}}{\mathscr{L}} \to \tau?$$

- Gap discrepancy problem and its resolution

  TM and T. Shirai, Phys. Rev. Lett. 125, 230604 (2020)

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- general upper bound on the auto-correlation function in the steady state: "instantaneous decay rate"

  TM and T. Shirai, to appear soon on arXiv

asymptotic and instantaneous decay rates

autocorrelation in the steady state

$$C_{AA}(t) = \mathrm{Tr}[\hat{A}(t)\hat{A}\rho_{\mathrm{ss}}] \quad \mathrm{Tr}[\hat{A}\rho_{\mathrm{ss}}] = 0$$

$$\hat{A}(t) = e^{-\tilde{\mathscr{L}}t}\hat{A}$$

$\tilde{\mathscr{L}}$ is an adjoint of $\mathscr{L}$ $\quad \mathrm{Tr}[\hat{A}\mathscr{L}\rho] = \mathrm{Tr}[(\tilde{\mathscr{L}}\hat{A})\rho]$

$$-\mathscr{L}\rho = -i[\hat{H},\rho] + \sum_k \left(\hat{L}_k \rho \hat{L}_k^\dagger - \frac{1}{2}\{\hat{L}_k^\dagger \hat{L}_k, \rho\} \right)$$

$$-\tilde{\mathscr{L}}\hat{A} = i[\hat{H},\hat{A}] + \sum_k \left(\hat{L}_k^\dagger \hat{A} \hat{L}_k - \frac{1}{2}\{\hat{L}_k^\dagger \hat{L}_k, \hat{A}\} \right)$$

asymptotic decay rate: Liouvillian gap

$$C_{AA}(t) \sim e^{-gt} \quad (t \to \infty)$$

but crossover time to the asymptotic regime might be too long

instantaneous decay rate in a transient regime?

characterization of the instantaneous decay rate

define an inner product $\langle \hat{A}, \hat{B} \rangle_{\mathrm{ss}} = \mathrm{Tr}[\hat{A}^\dagger \hat{B} \rho_{\mathrm{ss}}] \;\leftrightarrow\; \langle \hat{A}, \hat{B} \rangle = \mathrm{Tr}[\hat{A}^\dagger \hat{B}]$

R. Alicki, Rep. Math. Phys. (1976)

$C_{AA}(t) = \langle \hat{A}(t), \hat{A} \rangle_{\mathrm{ss}} \quad (\hat{A} = \hat{A}^\dagger)$

$\tilde{\mathscr{L}}^*$ is defined by $\langle \hat{A}, \tilde{\mathscr{L}} \hat{B} \rangle_{\mathrm{ss}} = \langle \tilde{\mathscr{L}}^* \hat{A}, \hat{B} \rangle_{\mathrm{ss}}$

symmetrized Liouvillian $\mathscr{K} = \dfrac{\tilde{\mathscr{L}} + \tilde{\mathscr{L}}^*}{2}$

Hermitian in the Hilbert space associated with $\langle \cdot , \cdot \rangle_{\mathrm{ss}}$

Theorem $|C_{AA}(t)| \le e^{-\int_0^t K_A(s) ds} C_{AA}(0)$

instantaneous decay rate $K_A(t) = \dfrac{\langle \hat{A}(t), \mathscr{K} \hat{A}(t) \rangle_{\mathrm{ss}}}{\langle \hat{A}(t), \hat{A}(t) \rangle_{\mathrm{ss}}}$

properties of the instantaneous decay rate

instantaneous decay rate $K_A(t) = \dfrac{\langle \hat{A}(t), \mathscr{K}\hat{A}(t)\rangle_{\mathrm{ss}}}{\langle \hat{A}(t), \hat{A}(t)\rangle_{\mathrm{ss}}}$ $\quad \mathscr{K} = \dfrac{\tilde{\mathscr{L}} + \tilde{\mathscr{L}}^*}{2}$

- it converges to the Liouvillian gap in the long-time limit

$$\lim_{t\to\infty} K_A(t) = g$$

- it is bounded from below by the spectral gap g_K of the symmetrized Liouvillian $\mathscr{K}$ ("symmetrized Liouvillian gap")

$K_A(t) \geq g_K$ $\quad |C_{AA}(t)| \leq e^{-g_K t} C_{AA}(0)$ $\quad$ We can show $0 \leq g_K \leq g$

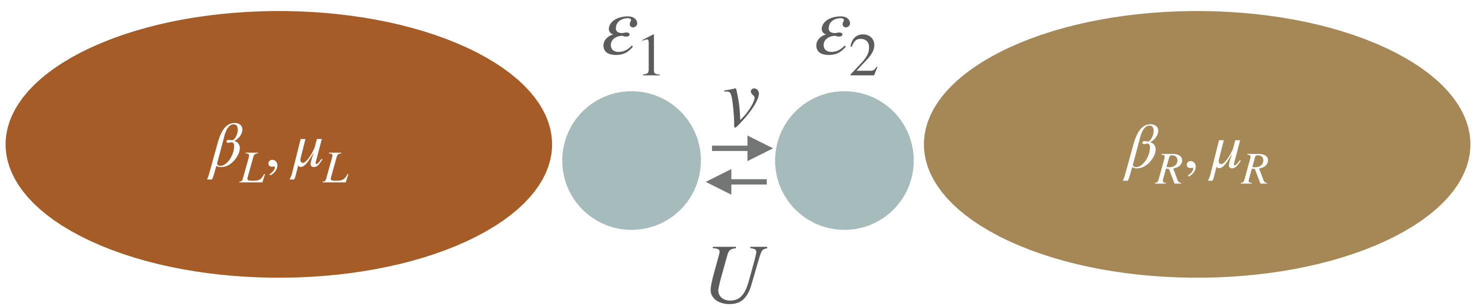
- The detailed balance condition implies $g = g_K$

R. Alicki, Rep. Math. Phys. (1976) $[\tilde{\mathscr{L}}, \tilde{\mathscr{L}}^*] = 0$

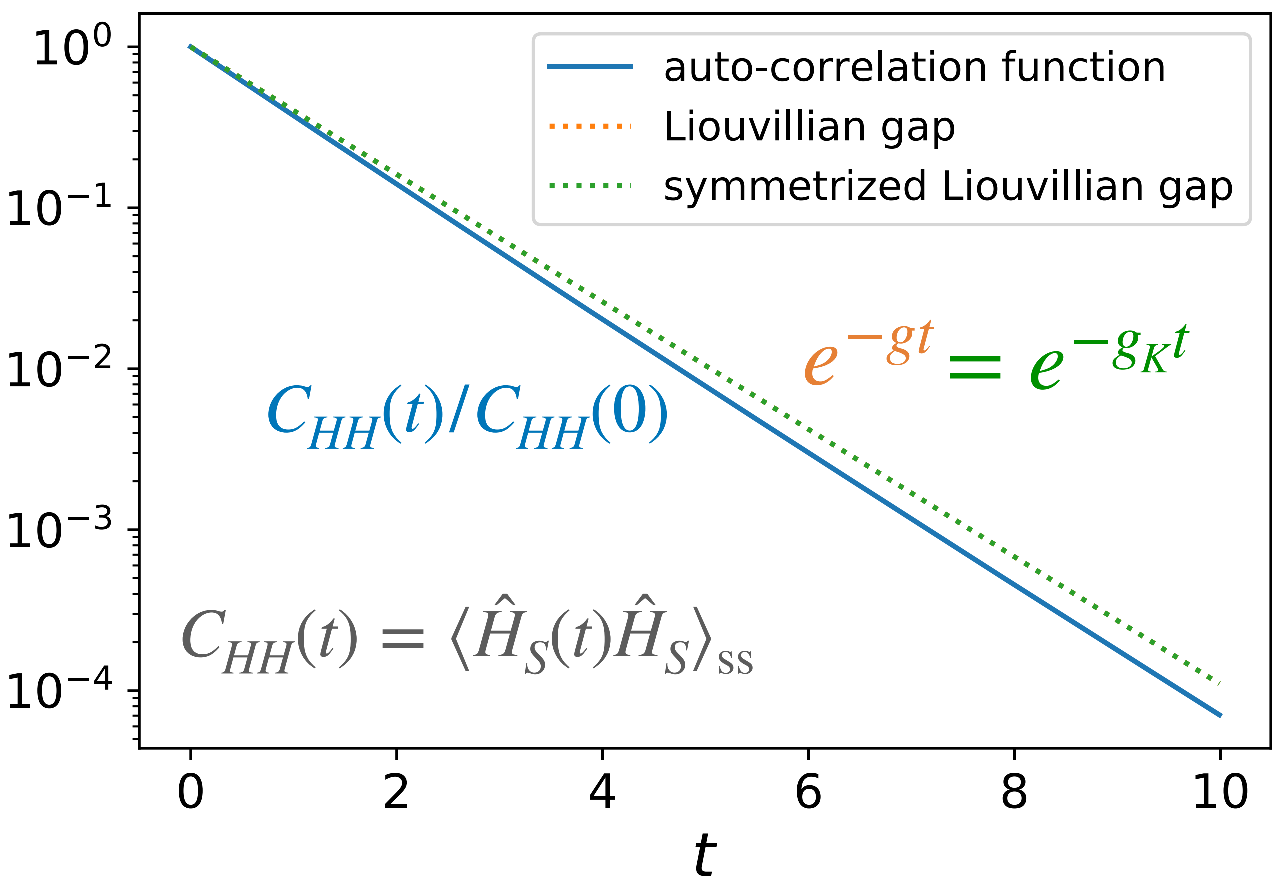
(eigenvalue of $\mathscr{K}$) = (real part of eigenvalue of $\tilde{\mathscr{L}}$)

Numerics in Born-Markov-secular Lindblad equation

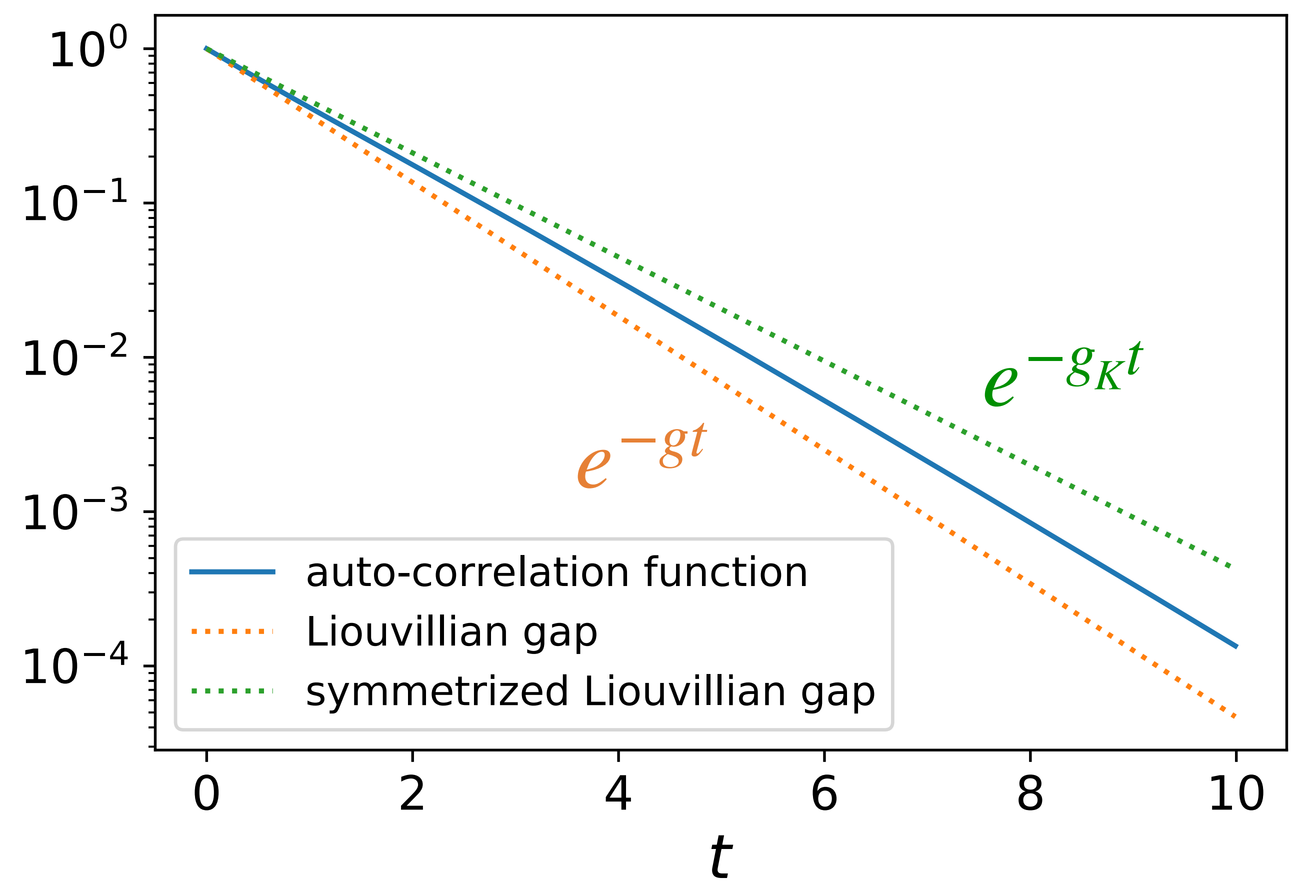
interacting double quantum dots



$$\beta_L = \beta_R, \quad \mu_L = \mu_R$$



$$\beta_L \neq \beta_R, \quad \mu_L \neq \mu_R$$



summary and outlook

TM and T. Shirai, Phys. Rev. Lett. 125, 230604 (2020)

- The Liouvillian gap does not necessarily give the maximum relaxation time (gap discrepancy problem)
- This problem is resolved by taking into account an explosive growth of expansion coefficients due to vanishingly small eigenmode overlaps

TM and T. Shirai, to appear soon on arXiv

- We give a bound on the autocorrelation function in the steady state by using the instantaneous decay rate $K_A(t)$
- The instantaneous decay rate is generally bounded from below by the spectral gap of the symmetrized Liouvillian
- Our numerical calculations show that the symmetrized Liouvillian gap gives a tight bound at short times