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東京大学
THE UNIVERSITY OF TOKYO

RIKEN

Exact eigenstates of multicomponent Hubbard models: SU(N) magnetic η pairing and weak ergodicity breaking

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MN, H. Katsura, and M. Ueda, arXiv:2205.07235

Outline

1. Introduction

   - Weak ergodicity breaking

   - η pairing

2. Exact eigenstates of the *N*-component Hubbard model

3. Weak ergodicity breaking in the *N*-component Hubbard model

4. Partial integrability

5. Summary

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Quantum thermalization

■ Thermalization in isolated quantum many-body systems

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle$$

$$\Rightarrow \quad \langle\psi(t)| A |\psi(t)\rangle \xrightarrow{?} \frac{1}{Z} \mathrm{Tr}[A e^{-\beta H}]$$

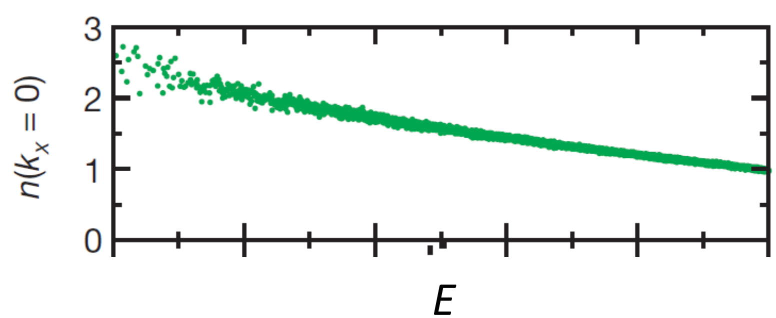
Thermodynamics from unitary time evolution?

■ (Strong) Eigenstate Thermalization Hypothesis (ETH)

[Deutsch, PRA 43, 2046 (1991); Srednicki, PRE 50, 888 (1994); Rigol *et al*., Nature 452, 854 (2008)]

$$\langle E| A |E\rangle \simeq \frac{1}{Z} \mathrm{Tr}[A e^{-\beta H}]$$

for **all** energy eigenstates



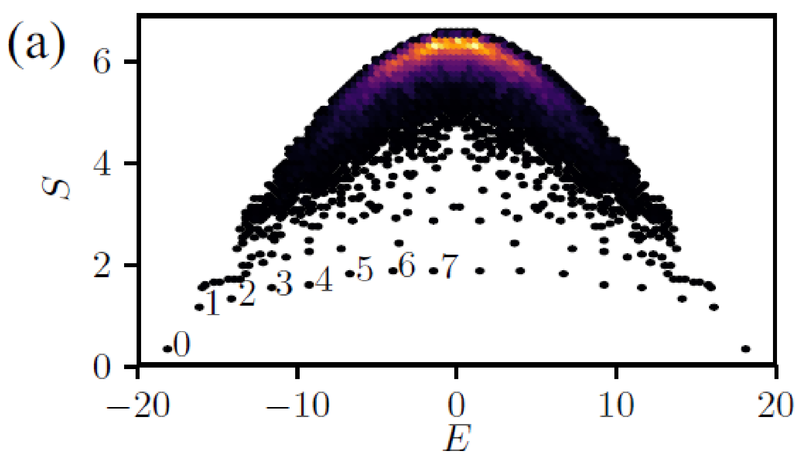
Non-integrable systems → strong ETH → thermalization?

Weak ergodicity breaking

- Weak ergodicity breaking in non-integrable systems

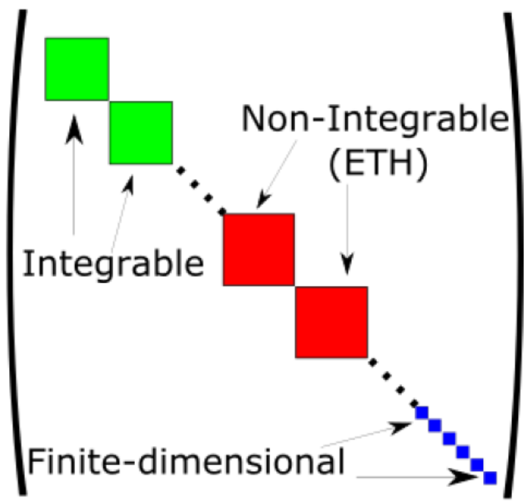
  → **Special initial states do not thermalize! Violation of strong ETH**

☐ Quantum many-body scar



[Turner *et al.*, PRB 98, 155134 (2018)]

☐ Hilbert space fragmentation



[Moudgalya *et al.*, Rep. Prog. Phys. 85, 086501 (2022)]

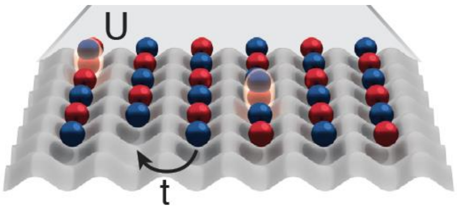
Non-integrability does not guarantee ergodicity!

Hubbard model

- Fermi-Hubbard model

$$H = -t \sum_{\langle i,j \rangle,\sigma} (c^{\dagger}_{i\sigma} c_{j\sigma} + \text{h.c.}) + U \sum_{j} n_{j\uparrow} n_{j\downarrow}$$

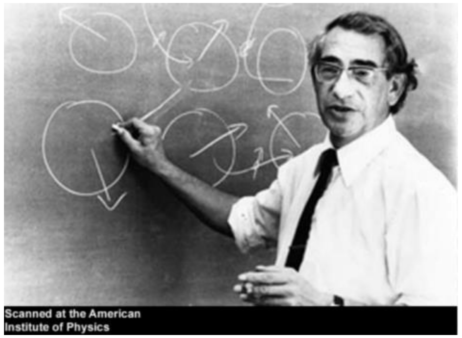
Hopping **Interaction**

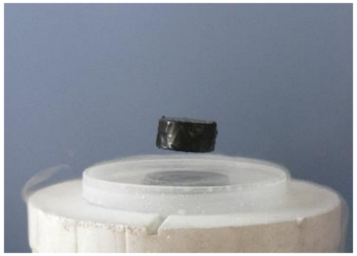


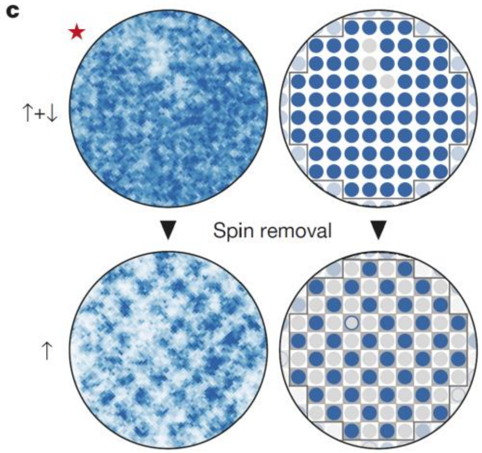
- Minimal model of interacting fermions

→ quantum magnetism, (high-T_c) superconductivity, …

- Quantum simulation with cold atoms







Antiferromagnetic long-range order
[Mazurenko *et al.*, Nature 545, 462 (2017)]

η Pairing and Off-Diagonal Long-Range Order in a Hubbard Model

Chen Ning Yang
State University of New York, Stony Brook, New York 11794-3840
and Chinese University of Hong Kong, Hong Kong
(Received 22 August 1989)

It is shown in a simple Hubbard model that through a mechanism called η pairing one can construct many eigenstates of the Hamiltonian possessing off-diagonal long-range order. The intrapair distance is small. It is shown that these eigenstates are metastable and possess an energy gap.

η pairing

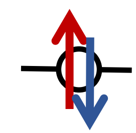
η pairing: exact eigenstate of the Hubbard model

(on hypercubic lattice, arbitrary dim.) [C. N. Yang, PRL 63, 2144 (1989)]

$$|\psi_M\rangle \equiv (\eta^\dagger)^M |0\rangle , \qquad \eta^\dagger = \sum_j e^{i\boldsymbol{Q}\cdot\boldsymbol{R}_j} c^\dagger_{j\uparrow} c^\dagger_{j\downarrow}, \quad \boldsymbol{Q} = (\pi, \cdots, \pi)$$

nonzero C.O.M. momentum

on-site Cooper pair ("doublon")



- ✓ **Hidden ("dynamical") η symmetry of the Hubbard model**

[C. N. Yang and S. C. Zhang, Mod. Phys. Lett. B 04, 759 (1990)]

$$\left[\eta^\dagger, H - \frac{U}{2} \sum_{j,\sigma} n_{j\sigma}\right] = 0$$

- ✓ **Off-diagonal long-range order: "BEC" of doublons with momentum Q**

- ✓ **Excited eigenstate**

η pairing

η pairing: exact eigenstate of the Hubbard model

(on hypercubic lattice, arbitrary dim.) [C. N. Yang, PRL 63, 2144 (1989)]

$$|\psi_M\rangle \equiv (\eta^\dagger)^M |0\rangle\,, \qquad \eta^\dagger = \sum_j e^{i\boldsymbol{Q}\cdot\boldsymbol{R}_j} c^\dagger_{j\uparrow} c^\dagger_{j\downarrow}, \quad \boldsymbol{Q} = (\pi, \cdots, \pi)$$

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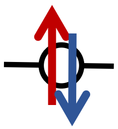

This talk:

- ☐ **Construction of exact eigenstates of the *N*-component Hubbard model**
- ☐ **Off-diagonal long-range order coexisting with SU(*N*) magnetism**
- ☐ **Weak ergodicity breaking in the *N*-component Hubbard model for *N* ≥ 3**

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Multicomponent Hubbard model

■ ***N*-component** Fermi-Hubbard model (on *d*-dim. hypercubic lattice)

$$H = -t \sum_{\langle i,j \rangle} \sum_{\alpha=1,\cdots,N} (c^{\dagger}_{i,\alpha} c_{j,\alpha} + \mathrm{H.c.}) + \sum_{j} \sum_{\alpha<\beta} U_{\alpha,\beta} n_{j,\alpha} n_{j,\beta}$$

hopping: SU(*N*) sym. **Interaction: not necessarily SU(*N*)**

e.g., *N* = 3: ^{6}Li atoms under a high magnetic field

[Ottenstein *et al*., PRL 101, 203202 (2008)]

■ SU(*N*) Hubbard model ($U_{\alpha,\beta} = U$)

→ alkaline-earth-like atoms (^{173}Yb: *N* = 6, ^{87}Sr: *N* = 10)

[Taie *et al*., Nat. Phys. 8, 825 (2012)]

■ SU(*M*) × SU(*N* - *M*) Hubbard model

→ ^{171}Yb-^{173}Yb mixture (SU(2) × SU(6)) [Taie *et al*., PRL 105, 190401 (2010)]

Generalized η pairing

Generalized η-pairing state

$$|\psi_{M_2,M_3,\cdots,M_N}\rangle \equiv (\eta^\dagger_{2,1})^{M_2}(\eta^\dagger_{3,1})^{M_3}\cdots(\eta^\dagger_{N,1})^{M_N}|0\rangle$$

$$\eta^\dagger_{\alpha,\beta} \equiv \sum_j e^{i\boldsymbol{Q}\cdot\boldsymbol{R}_j} c^\dagger_{j,\alpha} c^\dagger_{j,\beta}, \quad \boldsymbol{Q} = (\pi,\cdots,\pi)$$

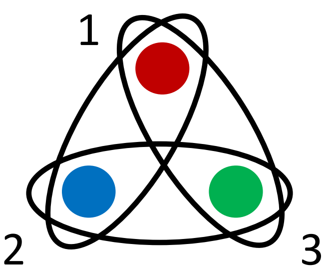
η pair of "color" α & β

All η pairs must contain the same "color" β = 1

Three-component case is special

$$|\psi^{(3)}_{l,m,n}\rangle \equiv (\eta^\dagger_{1,2})^{l}(\eta^\dagger_{2,3})^{m}(\eta^\dagger_{3,1})^{n}|0\rangle$$

Arbitrary η pairs are allowed!



Main result:

These states are exact eigenstates of the *N*-component Hubbard model

Generalized η-pairing states are exact eigenstates

- **Proof**
  - Kinetic term $-t\sum_{\langle i,j\rangle,\alpha}(c^\dagger_{i,\alpha}c_{j,\alpha}+\mathrm{H.c.})=\sum_{\boldsymbol{k},\alpha}\epsilon(\boldsymbol{k})c^\dagger_{\boldsymbol{k},\alpha}c_{\boldsymbol{k},\alpha}$

$$\eta^\dagger_{\alpha,\beta}=\sum_{\boldsymbol{k}}c^\dagger_{\boldsymbol{k},\alpha}c^\dagger_{\boldsymbol{Q}-\boldsymbol{k},\beta}$$

η pair = pair of fermions with momenta *k* & *Q* - *k*

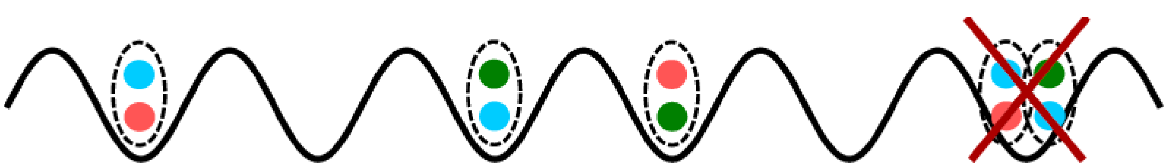
$$\epsilon(\boldsymbol{k})+\epsilon(\boldsymbol{Q}-\boldsymbol{k})=-2t\sum_{\mu}(\cos k_\mu+\cos(\pi-k_\mu))=0$$

Vanish!
→ eigenstate of kinetic term

  - Interaction term $\sum_j\sum_{\alpha<\beta}U_{\alpha,\beta}n_{j,\alpha}n_{j,\beta}$

→ multiple η pairs cannot occupy the same site **(Pauli exclusion)**

***N*-component case:**
Pauli exclu. via "red" particles
→ constant interaction energy



3-component case:
Pauli exclusion between arbitrary pairs!

SU(N) magnetic & off-diagonal long-range order

- Off-diagonal long-range order: (N - 1) types of pair correlations

$$\frac{\langle \psi_{M_2,\cdots,M_N} | \, c^\dagger_{i,\alpha} c^\dagger_{i,1} c_{j,1} c_{j,\alpha} \, | \psi_{M_2,\cdots,M_N} \rangle}{\langle \psi_{M_2,\cdots,M_N} | \psi_{M_2,\cdots,M_N} \rangle} = \frac{M_\alpha (N_\mathrm{s} - N_\mathrm{f}/2)}{N_\mathrm{s}(N_\mathrm{s}-1)} e^{i\boldsymbol{Q}\cdot(\boldsymbol{R}_i - \boldsymbol{R}_j)}$$

$(\alpha = 2, \cdots, N)$

Site-indep. magnitude **= ± 1**

(N_s: # of sites, N_f: # of particles)

- Simultaneous condensation of multiple pairs $\eta^\dagger_{\alpha,1} \ (\alpha = 2, \cdots, N)$

  → **fragmented condensate** [Mueller *et al.*, PRA 74, 033612 (2006)]

- Long-range magnetic correlation $F_{j,\alpha,\beta} \equiv c^\dagger_{j,\alpha} c_{j,\beta}$ **SU(N) spin operator**

$$\langle F_{i,\alpha,\beta} F_{j,\beta,\alpha} \rangle = \frac{M_\alpha M_\beta}{N_\mathrm{s}(N_\mathrm{s}-1)} \quad (\alpha, \beta \neq 1)$$

Site-indep. correlation!
SU(N) magnetic order

Coexistence of SU(N) magnetism & off-diagonal long-range order!

Spectrum generating algebra

- Original spin-1/2 Hubbard model → hidden *η* symmetry
- **No *η* symmetry** in the *N*-component Hubbard model for *N* ≥ 3

$$[\eta^\dagger_{\alpha,\beta}, H] = -U_{\alpha,\beta}\eta^\dagger_{\alpha,\beta} - \sum_j \sum_{\gamma(\neq\alpha,\beta)} (U_{\alpha,\gamma} + U_{\beta,\gamma}) e^{i\boldsymbol{Q}\cdot\boldsymbol{R}_j} c^\dagger_{j,\alpha} c^\dagger_{j,\beta} n_{j,\gamma}$$

$$\equiv R_{\alpha\beta}$$

$$R_{\alpha,1} \left|\psi_{M_2,\cdots,M_N}\right\rangle = 0 \quad (\alpha = 2, \cdots, N)$$

Generalized *η*-pairing states do not "feel" the residual term!

- "Symmetry" in subspace: (restricted) **spectrum generating algebra**

[Moudgalya *et al*., PRB 102, 085140 (2020); Mark *et al*., PRB 101, 195131 (2020)]

$$([\eta^\dagger_{\alpha,1}, H] + U_{\alpha,1}\eta^\dagger_{\alpha,1})W = 0 \quad (\alpha = 2, \cdots, N)$$

***W*: Hilbert subspace** spanned by $|0\rangle, c^\dagger_{j,1}|0\rangle, c^\dagger_{j,1}c^\dagger_{j,2}|0\rangle, \cdots, c^\dagger_{j,1}c^\dagger_{j,N}|0\rangle$ at each site

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Sub-volume law entanglement entropy

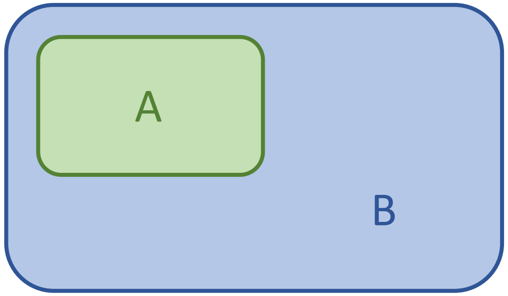
- Entanglement entropy of a generalized η-pairing state

[Two-component case: Vefek *et al*., SciPost Phys. 3, 043 (2017)]

$$S_A = -\mathrm{Tr}_A[\rho_A \log \rho_A] = \frac{N-1}{2} \log N_{\mathrm{s},A} + \mathrm{const.}$$

Sub-volume law!

($N_{s,A}$: number of sites in subsystem A)



- Violation of the eigenstate thermalization hypothesis (ETH)

ETH: $\rho_A = \mathrm{Tr}_B[|E\rangle\langle E|] \simeq \mathrm{Tr}_B[\rho_{\mathrm{MC},E}] \Rightarrow S_A \propto N_{\mathrm{s},A}$

microcan. ensemble

volume law
(extensive entropy)

Generalized η-pairing states violate the ETH!

Non-thermalizing dynamics

- Violation of the ETH (failure of thermalization)
  - Integrable system, many-body localization (many conserved quantities)
  - **Quantum many-body scar (weak ergodicity breaking)**

    → no conserved quantity, but exceptional eigenstates violating ETH

[Turner *et al*., Nat. Phys. 14, 745 (2018)]

Note: η-pairing state of the original Hubbard model is NOT a scar (∵ η symmetry)

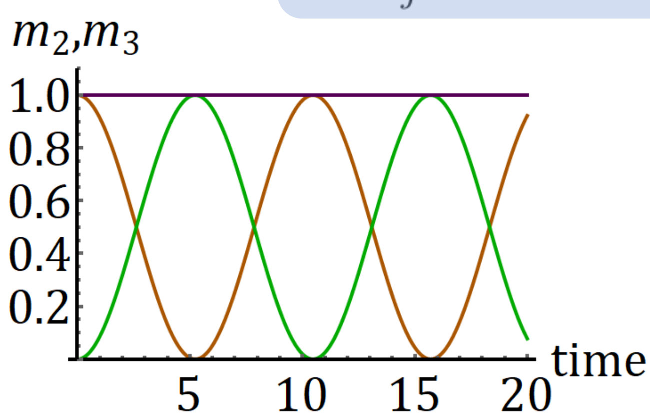
[See also Moudgalya *et al*., PRB 102, 085140 (2020); Mark *et al*., PRB 102, 075132 (2020)]

- Non-thermalizing dynamics of generalized η-pairing states

$$H_{\rm dyn} = H + \Omega_{23}\sum_j (c^\dagger_{j,2}c_{j,3} + {\rm H.c.})$$

Raman coupling between hyperfine states

[cf. ^{173}Yb: Tusi *et al*., Nat. Phys. 18, 1201 (2022)]



$$m_2 = \frac{1}{M_2+M_3}\sum_j \langle n_{j,1}n_{j,2}\rangle$$ **# of doublons**

Persistent oscillation

→ quantum many-body scar dynamics!

Construction of quantum many-body scars

- A systematic construction of quantum many-body scar states

[Mark *et al.*, PRB 101, 195131 (2020); O'Dea *et al.*, PRR 2, 043305 (2020) etc.]

$$H = H_0 + H'$$

Symmetry-breaking term that satisfies $H' |\phi_0\rangle = E' |\phi_0\rangle$

Symmetric Hamiltonian that has a "simple" eigenstate $|\phi_0\rangle$ **(e.g., ferro., *η* pairing, …)**

- Generalized *η* pairing in the *N*-component Hubbard model

$$H_0 = -t \sum_{\langle i,j\rangle} \sum_{\alpha=1,\cdots,N} (c^\dagger_{i,\alpha} c_{j,\alpha} + \mathrm{H.c.})$$

free fermion part
→ generalized *η* symmetry
(⊂ SO(2*N*) symmetry)

$$H' = \sum_j \sum_{\alpha<\beta} U_{\alpha,\beta} n_{j,\alpha} n_{j,\beta}$$

Interaction term breaks the *η* symmetry if *N* ≥ 3 !

Generalized *η* pairing in *N*(≥ 3)-component systems is a natural quantum many-body scar!

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Partial integrability

$$H = H_0 + H_{\mathrm{int}}$$

H_0 : SU(N) symmetric one-body term

H_{int} : (arbitrary) on-site two-body interactions

- Spin-polarized sector: **equivalent to free fermions (integrable)**

$$c^\dagger_{n_1,1} c^\dagger_{n_2,1} \ldots c^\dagger_{n_r,1} |0\rangle$$

Pauli exclusion → no on-site interaction!

- SU(N) ferromagnetic state ($F_{\alpha,\beta} \equiv \sum_j c^\dagger_{j,\alpha} c_{j,\beta}$: SU(N) spin operator)

$$|\Phi_{\mathrm{FM}}\rangle \equiv (F_{2,1})^{M_2} (F_{3,1})^{M_3} \cdots (F_{N,1})^{M_N} c^\dagger_{n_1,1} c^\dagger_{n_2,1} \cdots c^\dagger_{n_r,1} |0\rangle$$

Spin flips **Spin-polarized eigenstate**

$$H_{\mathrm{int}} |\Phi_{\mathrm{FM}}\rangle = 0$$

No double occupancy!

$$H |\Phi_{\mathrm{FM}}\rangle = H_0 |\Phi_{\mathrm{FM}}\rangle = (\epsilon_{n_1} + \cdots + \epsilon_{n_r}) |\Phi_{\mathrm{FM}}\rangle \quad (\because [F_{\alpha,\beta}, H_0] = 0)$$

SU(N) sym. of H_0 → exact eigenstate

***N*-component Fermi-Hubbard model has an integrable sector!**

Partial integrability

Non-thermalization due to partial integrability

- Dynamics from an initial state in the integrable sector

$$|\psi(0)\rangle \propto (F_{2,1})^{M_2}(F_{3,1})^{M_3}\cdots(F_{N,1})^{M_N}|\phi_1\rangle$$

Arbitrary spin-polarized state

For simplicity, assume translation invariance

$O_{\boldsymbol{k},\sigma} \equiv c^\dagger_{\boldsymbol{k},\sigma}c_{\boldsymbol{k},\sigma}$ momentum distribution

$$\begin{aligned}\langle O_{\boldsymbol{k},\sigma}(t)\rangle &= \langle\psi(0)|\, e^{iHt}O_{\boldsymbol{k},\sigma}e^{-iHt}\,|\psi(0)\rangle \\ &= \langle\psi(0)|\, e^{iH_0t}O_{\boldsymbol{k},\sigma}e^{-iH_0t}\,|\psi(0)\rangle \quad (\because H_{\mathrm{int}}\,|\psi(0)\rangle = 0) \\ &= \langle\psi(0)|\, O_{\boldsymbol{k},\sigma}\,|\psi(0)\rangle \quad (\because [H_0, O_{\boldsymbol{k},\sigma}] = 0) \\ &= \langle O_{\boldsymbol{k},\sigma}(0)\rangle\end{aligned}$$

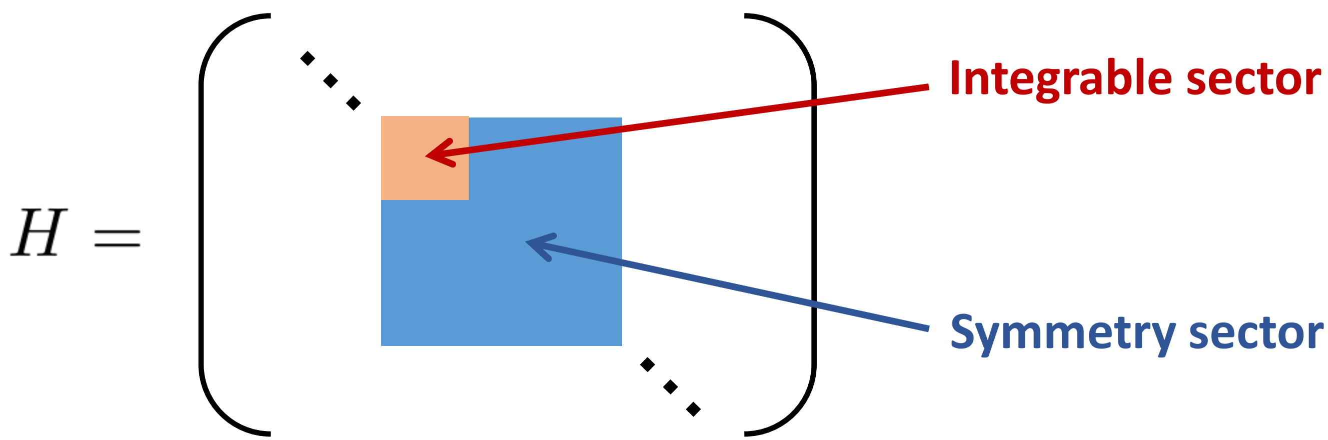
Conserved!

Non-thermalization in the integrable sector!

Weak ergodicity breaking due to partial integrability

- Integrable systems do not satisfy the strong ETH
- ETH → should be tested within each symmetry sector
- **No SU(*N*) symmetry in the interaction term**

  → The integrable sector cannot be distinguished from non-integrable one by symmetry eigenvalues!



Weak ergodicity breaking due to partial integrability!

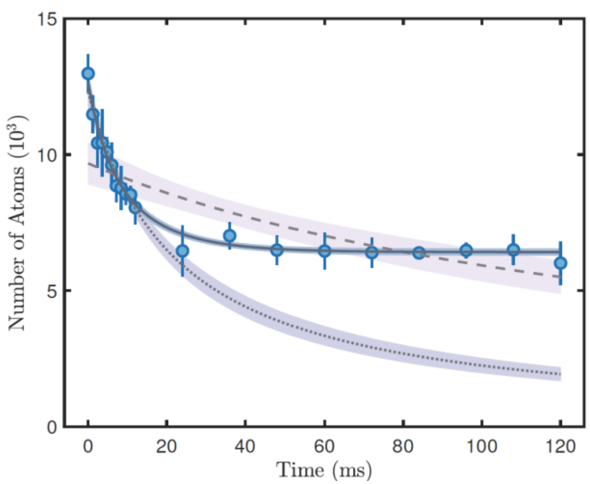
Dissipation-induced non-thermalization

- Initial states in the integrable sector do not thermalize
- How to prepare such initial states? → **Control of dissipation!**

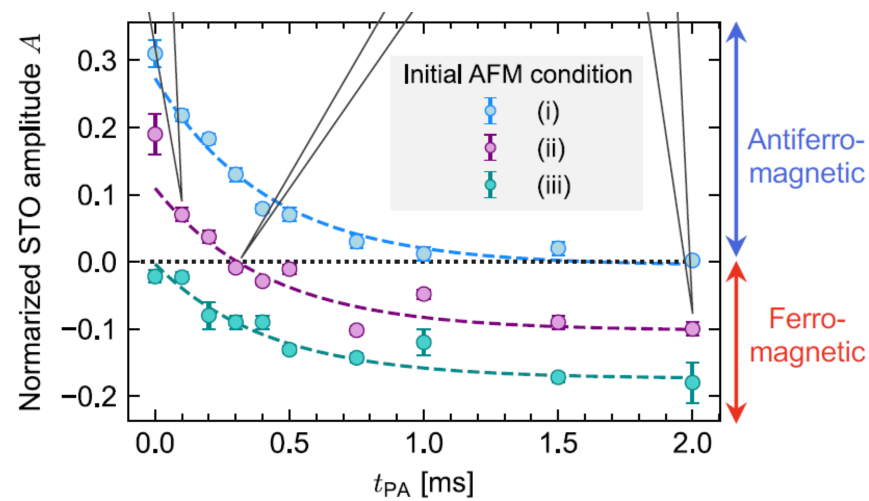
$$\frac{d\rho}{d\tau} = -i[H,\rho] + \sum_{j,\alpha<\beta}\left(L_{j,\alpha,\beta}\rho L^{\dagger}_{j,\alpha,\beta} - \frac{1}{2}\{L^{\dagger}_{j,\alpha,\beta}L_{j,\alpha,\beta},\rho\}\right)$$ **Lindblad eq.**

$$L_{j,\alpha,\beta} = \sqrt{\gamma}c_{j,\alpha}c_{j,\beta}$$ **On-site two-body loss → realized in cold atoms**

$$L_{j,\alpha,\beta}\left|\Phi_{\mathrm{FM}}\right\rangle = 0$$ **SU(*N*) FM states are dark (steady) states! No double occ.**



Experiment: SU(*N*) Hubbard + two-body loss
[Sponselee *et al.*, Quant. Sci. Tech. 4, 014002 (2018)]



Ferromagnetic spin correlation
[Honda, … , MN, and Takahashi, arXiv:2205.13162]

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Summary

■ Generalized η-pairing eigenstates of the N-component Hubbard model

- Simultaneous condensation of multicomponent η pairs

- Coexistence of SU(N) magnetism & off-diagonal long-range order

■ Weak ergodicity breaking in the N-component Hubbard model for $N \geq 3$

- No η symmetry → quantum many-body scar

- Partial integrability

- Dissipation-induced non-thermalization

Reference:

MN, H. Katsura, and M. Ueda, arXiv:2205.07235